Small Investment, Big Reward

Exponential Functions

12.1

LEARNING GOALS

In this lesson, you will:

- Construct an exponential function from a geometric sequence.
- Classify functions as exponential growth or decay.
- Compare tables, graphs, and equations of exponential functions.

KEY TERM

- half-life

Have you ever seen a funny picture or video online, and then it suddenly seems like everyone is talking about it? Social media and the internet have made it really easy to pass things along from person to person. You can pin, post, and share anything that you find interesting, thought-provoking, or funny with your friends all over the world. Your friends can in turn share it with their own friends, who share it with their friends, and before you know it, it seems like everyone in the world is exposed to it.

When something becomes extremely popular on the internet in a very short amount of time, it's known as "going viral." Trends can spread across the country or around the world in a matter of days. Some "viral" videos and pictures have produced overnight celebrities and inspired spin-offs in the form of books or TV shows.

What is your favorite "viral" video or picture that you've seen online?

PROBLEM 1 Big Things Come to Those Who Wait!

Allison and Beth each receive $10 per week for doing chores for their neighbor. One day, Allison decides to try and increase her income using her knowledge of exponential growth. She proposes that her payment be changed to a penny, and then doubled each week thereafter.

1. Complete the table to represent the amount that Allison and Beth will earn each week.

Week	Allison's Income (dollars)	Beth's Income (dollars)
1	0.01	10.00
2	0.02	10.00
3	0.04	10.00
4	0.08	10.00
5	0.16	10.00
6	0.32	10.00
7	0.64	10.00
8	1.28	10.00

2. How does Allison's income change as the number of weeks increases?

   As the number of weeks increases by 1, the amount that Allison earns is multiplied by 2.

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3. Does Allison's income represent an arithmetic or geometric sequence or series? Explain your reasoning and state the general formula.

   Allison's income represents a geometric sequence because the dollar amount is increasing by a common ratio. It is a sequence because the income increases in a pattern, but the amount is not cumulative.

   The general formula for a geometric sequence is $a_n = a_1 \cdot r^{(n-1)}$.

4. Write an equation to represent Allison's income after *n* weeks.

   $a_n = a_1 \cdot r^{(n-1)}$

   $= 0.01 \cdot 2^{(n-1)}$

5. What is the value of a_n for $n = 0$? Does this value make sense in this problem situation?

For $n = 0$, the value of a_n is 0.005.

Yes this value does make sense. Half of a penny is not real currency and would therefore be zero. Allison starts with no money before receiving her first income.

$$a_0 = 0.01 \cdot 2^{(0-1)}$$
$$= 0.01 \cdot 2^{-1}$$
$$= 0.005$$

6. If the pattern were to continue, how many weeks would it take for Allison to have a larger weekly income than Beth? Complete the table to show your answer.

Allison's weekly income amount would be larger than Beth's after 11 weeks.

$a^x \cdot a^y = a^{x+y}$

Week	Allison's Income (dollars)	Beth's Income (dollars)
9	2.56	10.00
10	5.12	10.00
11	10.24	10.00

You can write the explicit formula for the geometric sequence $a_n = 0.01 \cdot 2^{(n-1)}$ in function notation using the properties of powers.

Statement	Reason
$a_n = 0.01 \cdot 2^{(n-1)}$	Explicit formula for a geometric sequence
$f(n) = 0.01 \cdot 2^{(n-1)}$	Rewrite in function notation
$f(n) = 0.01 \cdot 2^n \cdot 2^{-1}$	Product Rule
$f(n) = 0.01 \cdot 2^n \cdot \frac{1}{2}$	Definition of negative exponents
$f(n) = 0.01 \cdot \frac{1}{2} \cdot 2^n$	Commutative Property of Multiplication
$f(n) = 0.005 \cdot 2^n$	Associative Property of Multiplication

So, $a_n = 0.01 \cdot 2^{(n-1)}$ written in function notation is $f(n) = 0.005 \cdot 2^n$.

Beth

Recall that a geometric sequence, when written in function notation, is called an exponential function. The function gets its name from the variable in the exponent.

7. Calculate the income that Allison would earn per week in the:

a. 15th week.

In the 15th week, Allison would earn $163.84.

$$f(15) = 0.005 \cdot 2^{15}$$
$$= 163.84$$

b. 20th week.

In the 20th week, Allison would earn $5242.88.

$$f(20) = 0.005 \cdot 2^{20}$$
$$= 5242.88$$

c. 24th week.

In the 24th week, Allison would earn $83,886.08.

$$f(24) = 0.005 \cdot 2^{24}$$
$$= 83,886.08$$

8. Predict the shape and characteristics of the graph that will model Allison's income as a function of the number of weeks.

Answers will vary.

The graph will be a smooth curve which increases sharply from left to right.
The y-intercept of the graph will be at (0, 0.005).

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PROBLEM 2 The Tortoise and the Hare

1. Beth is amazed at how quickly Allison was able to make a lot of money and decides that she wants in on the action. She asks her two friends, Quinton and Alisha, to help her come up with a plan.

Quinton

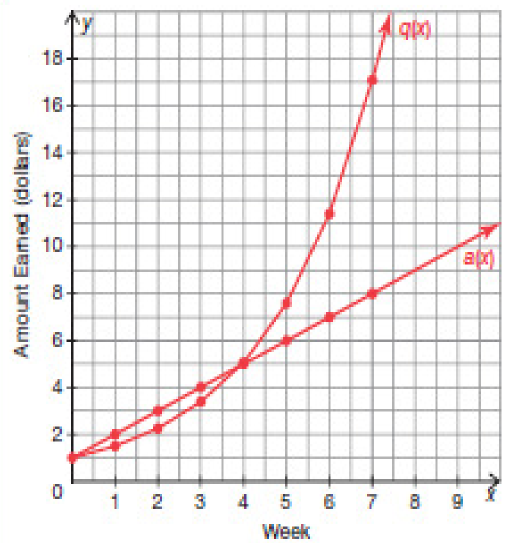
You could start with a dollar and ask for 50% more each week.

Alisha

You could start with a dollar and add another dollar each week.

Whose plan should Beth choose? Complete the table and graph to justify your reasoning. Round to the nearest hundredth.

Week	Quinton's Plan (dollars)	Alisha's Plan (dollars)
0	1.00	1.00
1	1.50	2.00
2	2.25	3.00
3	3.38	4.00
4	5.06	5.00
5	7.59	6.00
6	11.39	7.00
7	17.09	8.00



Beth should choose Quinton's plan.

Quinton's plan represents an exponential function, which means it will grow more quickly over time. Alisha's plan is linear because it is increasing at a constant rate, and will therefore not grow as quickly over time.

2. Write functions to represent Quinton's plan, $q(x)$, and Alisha's plan, $a(x)$.

$q(x) = 1.00 \cdot (1.5)^x$

$a(x) = 1.00 + 1x$

3. Use your choice from Question 1 to determine how much Beth will earn in Week 10.

Beth will earn \$57.67 in Week 10.

$q(10) = 1.00 \cdot (1.5)^{10}$

≈ 57.67

4. If Beth and Allison both start using their exponential model to earn income at the same time, who will earn a higher income in Week 12?

Beth will earn a higher income in Week 12.

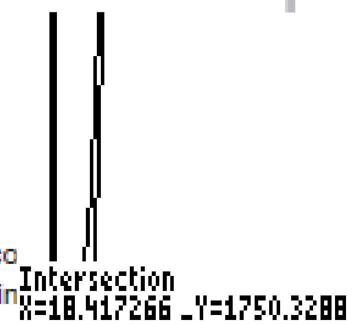
Beth: $q(12) = 1.00 \cdot (1.5)^{12}$

≈ 129.75

Allison: $f(12) = 0.005 \cdot (2)^{12}$

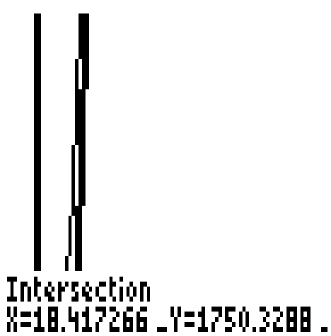
$= 20.48$

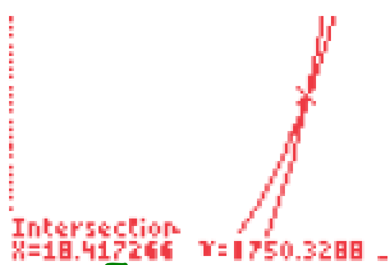
5. Use a graphing calculator to determine when Allison's and Beth's inco
Does this make sense in this problem situation? Explain your reasonin



Using my graphing calculator, I graphed $f(x) = 0.005 \cdot 2^x$ and $q(x) = 1.00 \cdot 1.5^x$.

The functions intersect at approximately $x = 18.4$.





This does not make sense in this problem situation.

Because x represents the number of weeks, and Allison's and Beth's incomes increase once each week, the models are not continuous functions. Therefore, 18.4 does not make sense because it is not a whole number.

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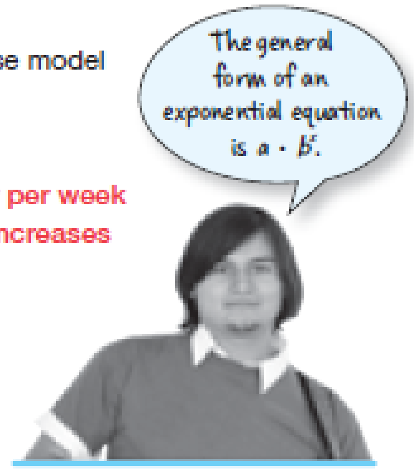
6. Compare Allison's and Beth's function models.

a. As the number of weeks continues to increase, whose model will earn them more per week?

Allison's model is better over time.

Even though Beth's model earned her more money per week in the beginning, after 19 weeks, Allison's income increases at a much faster rate than Beth's.



b. Consider the *a*- and *b*-values of the exponential functions if $y = ab^x$. How do they further support your claim?

Allison's *b*-value, or her income's rate of growth, is larger than Beth's, so eventually Allison's income surpasses Beth's.

PROBLEM 3 Half-Life of Caffeine

Simeon is studying for a big test and is trying to stay awake. He drank a 12-ounce can of Big Buzz Energy Drink that contains 80 milligrams of caffeine. He is wondering how long the caffeine will stay in his system if the caffeine has a *half-life* of 5 hours.

A **half-life** is the amount of time it takes a substance to decay to half of its original amount.

1. How much caffeine remains in Simeon's system after 5 hours? After 10 hours? Explain your reasoning.

After 5 hours, 40 milligrams of caffeine is left in Simeon's system, because it takes 5 hours to cut the amount in half.

After 10 hours, 20 milligrams of caffeine is left, because after another 5 hours, half of the previous amount is left.

Growth
Decay

2. Complete the table to determine the amount of caffeine in Simeon's system at each time interval.

Convergent → decay
divergent → growth

Time Elapsed (hours)	0	5	10	15	20
Caffeine in System (mg)	80	40	20	10	5
Number of Half-Life Cycles	0	1	2	3	4

3. What is the initial amount of caffeine in Simeon's system? What is the rate of decay?

The initial amount of caffeine is 80 milligrams.

The rate of decay is $\frac{1}{2}$, or 0.5, for every 5 hours.

4. Emily, Tyler, and Renee were asked to write an exponential function $A(t)$ to represent the amount of caffeine remaining in Simeon's system after t hours.

Emily

$A(t) = 80\left(\frac{1}{2}\right)^{\frac{t}{5}}$

The variable t represents the number of hours, and the half-life occurs in 5 hour cycles, so I divided my exponent by 5.

Tyler

$A(t) = 80\left(\frac{1}{2}\right)^{-\frac{t}{5}}$

The variable t represents the number of hours and since it's a decay function, I made my exponent negative.

Renee

$A(t) = 80\left(\frac{1}{2}\right)^{5t}$

The variable t represents the number of hours and I multiplied it by 5 to represent the half-life cycle of 5 hours.

a. Why is Tyler's reasoning incorrect?

Even though the function represents exponential decay, the negative exponent is incorrect because the base, $\frac{1}{2}$, is equivalent to 2^{-1}. Therefore, if written this way, the exponent would actually become positive.

It may be helpful to substitute the values from the table to check each student's function.

b. Why is Renee's reasoning incorrect?

The half-life cycle consists of 5 hours, so the exponent should be divided into 5 parts, not multiplied by 5.

5. How much caffeine remains in Simeon's system after 2 hours?

$A(2) = 80\left(\frac{1}{2}\right)^{\frac{2}{5}}$

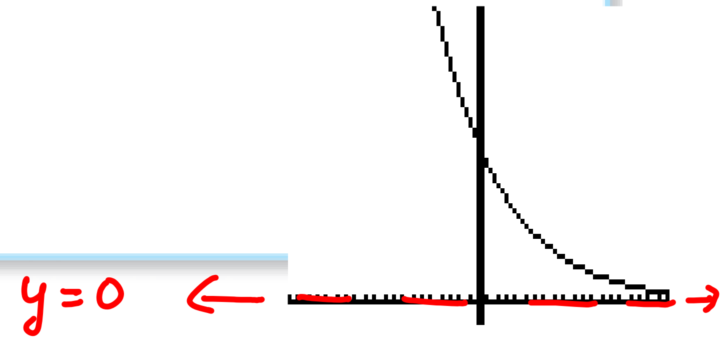
$A(2) \approx 60.63$

There is approximately 61 milligrams of caffeine remaining in Simeon's system after 2 hours.

6. Kendra suggests that she can calculate the amount of caffeine remaining by rewriting the equation as $A(t) = 40^{\frac{t}{5}}$. Is Kendra correct? Explain your reasoning.

Kendra is not correct.

Kendra did not follow the order of operations and multiplied $80 \cdot \left(\frac{1}{2}\right)$ before using the exponent. Because the variable is in the exponent, this is not possible.



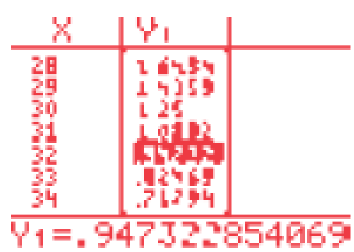
7. Use the table function of a graphing calculator to predict when the caffeine will be completely out of Simeon's system. Does this make sense, given what you know about exponential functions? Explain your reasoning.

The caffeine will never be completely out of Simeon's system because the amount will continue to be divided in half until there is just a trace amount left.

Mathematically, exponential functions always approach a horizontal asymptote. In this case, the horizontal asymptote is 0, and the function will never equal 0.

8. Approximately when will the amount of caffeine remaining in Simeon's system be less than 1 milligram?

The amount of caffeine remaining in Simeon's system will be less than 1 milligram between Hours 31 and 32.



X	Y_1
28	[illegible]
29	[illegible]
30	1.25
31	[illegible]
32	[illegible]
33	[illegible]
34	.71794

Y_1=.947322854069

9. Use the properties of exponents to rewrite your function so that only the variable t is in the exponent. What percentage of caffeine remains after each hour?

$A(t) = 80\left(\frac{1}{2}\right)^{\frac{1}{5} \cdot t}$

$= 80\left(\left(\frac{1}{2}\right)^{\frac{1}{5}}\right)^t$

$A(t) \approx 80(0.87055)^t$

There is approximately 87% of the previous amount of caffeine remaining after every hour.

10. Suppose Simeon is taking an antibiotic that extends the half-life of caffeine to 8 hours. Write a function $B(t)$ that models the amount of caffeine remaining under these new conditions.

$$B(t) = 80\left(\frac{1}{2}\right)^{\frac{t}{8}}$$

or

$$B(t) \approx 80(0.917)^t$$

11. Complete the table for the new half-life. Round to the nearest hundredth.

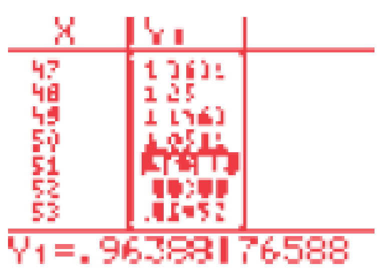
Time Elapsed (hours)	0	5	10	15	20
Caffeine in System (mg)	80	51.87	33.64	21.81	14.14
Number of Half-Life Cycles	0	0.625 $\frac{5}{8}$	1.25 $\frac{10}{8}$	1.875 $\frac{15}{8}$	2.5 $\frac{20}{8}$

12. How does the medication affect the amount of caffeine remaining in Simeon's system?

The 80 milligrams of caffeine takes longer to leave Simeon's system.

13. Under these new conditions, approximately when will the amount of caffeine remaining in Simeon's system be less than 1 milligram?

Using the table feature of a graphing calculator, I calculated that the amount of caffeine in Simeon's system will be less than 1 milligram between Hours 50 and 51.



14. What generalization can you make about the effect of larger or smaller half-lifes on substances?

The larger the half-life time, the longer it will take a substance to decay.

Be prepared to share your solutions and methods.

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