12.2

We Have Liftoff!

Properties of Exponential Graphs

LEARNING GOALS

In this lesson, you will:

- Identify the domain and range of exponential functions.
- Investigate graphs of exponential functions through intercepts, asymptotes, intervals of increase and decrease, and end behavior.
- Explore the irrational number *e*.

KEY TERM

- natural base *e*

Have you ever tried to remember a long list of things and ended up getting mixed up along the way?

There are lots of tried-and-true ways of memorizing things, and it all depends on what you're trying to memorize. Some people like to make mnemonic devices, where the first letter in each word corresponds to something in the list they're trying to memorize. You may have used one of these when you were learning the order of operations—Please Excuse My Dear Aunt Sally is a great way to help you remember parentheses, exponents, multiplication, division, addition, and subtraction. Some people try doing some sort of movement as they recite their list, so that they can use their muscle memory to help them. Some people like to use rhymes, some people use visualization, and some people rely on good old-fashioned repetition.

But there are some people who are just naturally skilled at remembering things. In fact, there are competitions held around the world to see who can memorize the most digits of pi. In 2005, Chao Lu of China set a world record by memorizing an incredible 67,890 digits of pi! It took him 24 hours and 4 minutes to accurately recite the digits, with no more than 15 seconds between each digit.

Do you have any memory tricks to help you remember things?

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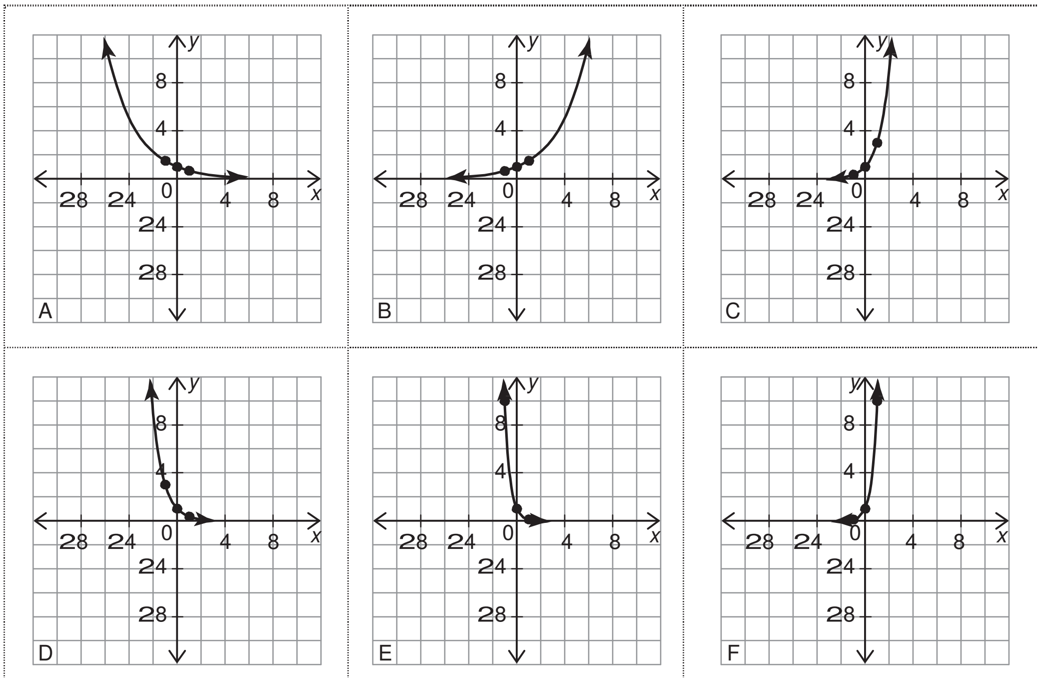
PROBLEM 1 I've Got the Power

1. Cut out the exponential graphs and equations and match them. Sort them into "growth" or "decay" functions and tape them onto the graphic organizer in this lesson. Finally, complete each table.



$f(x) = \left(\frac{2}{3}\right)^x$	$f(x) = 10^x$	$f(x) = 1.5^x$
$f(x) = 0.1^x$	$f(x) = 3^x$	$f(x) = \left(\frac{1}{3}\right)^x$

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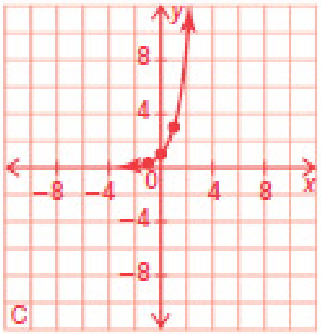
Growth

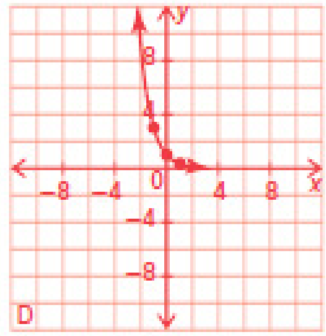
$f(x) = 3^x$



x	$f(x)$
-1	$\frac{1}{3}$
0	1
1	3

Decay

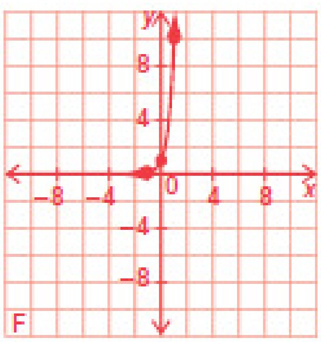
$f(x) = \left(\frac{1}{3}\right)^x$



x	$f(x)$
-1	3
0	1
1	$\frac{1}{3}$

Growth

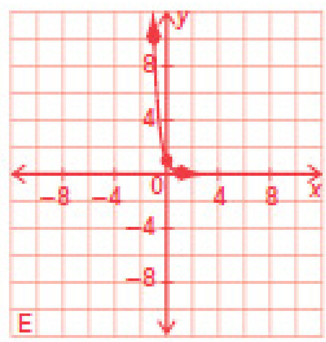
$f(x) = 10^x$



x	$f(x)$
-1	$\frac{1}{10}$
0	1
1	10

Decay

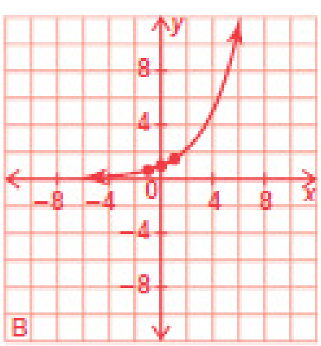
$f(x) = 0.1^x$



x	$f(x)$
-1	10
0	1
1	0.1

Growth

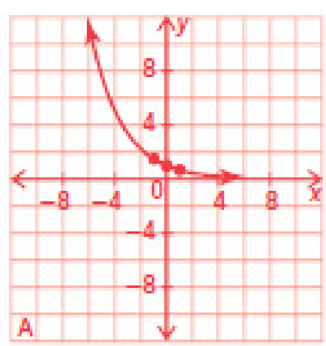
$f(x) = 1.5^x$



x	$f(x)$
-1	$\frac{1}{1.5} = \frac{2}{3}$
0	1
1	$\frac{3}{2}$

Decay

$f(x) = \left(\frac{2}{3}\right)^x$



x	$f(x)$
-1	$\frac{3}{2}$
0	1
1	$\frac{2}{3}$

2. Analyze the exponential growth and decay functions.

a. What point do the graphs have in common? Why?

Every basic exponential function has (0, 1) in common. The x-value represents the exponent, and any base raised to the power of 0 will equal 1, no matter what the base is.

b. Compare the equations of the six functions you just sorted. What differentiates an exponential growth from an exponential decay?

In the three exponential growth functions, the b-values are greater than 1.

In the three exponential decay functions, the b-values are less than 1 but greater than 0.

3. Sarah and Scott's teacher asked them to each write a rule that would determine whether a function was exponential growth or decay, based on its equation.

Sarah

For exponential growth functions, b is a value greater than 1, but for exponential decay functions, b is a fraction or decimal between 0 and 1.

Scott

For exponential growth functions, b is greater than 1. For exponential decay functions, b is less than 1.

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Why is Scott's reasoning incorrect? Provide a counterexample that would disprove his claim and explain your reasoning.

Answers will vary.

- $g(x) = (-5)^x$ is not exponential decay because the graph oscillates, and is therefore neither growth nor decay.
- $g(x) = 0^x$ is not exponential decay because even though 0 is less than 1, for every x-value greater than 0, $g(x)$ is equal to 0. Thus, it would be a constant function for $x > 0$.

4. What b-values in exponential functions produce neither growth nor decay? Provide examples to support your answer.

The b-values of 0 and 1 produce exponential functions that are neither grow nor decay functions.

$g(x) = 0^x$ is a constant function for $x > 0$. For every x-value greater than 0, $g(x) = 0$.

$g(x) = 1^x$ is a constant function. For every x-value, $g(x) = 1$.

5. Write an exponential function with the given characteristics.

a. Increasing over $(-\infty, \infty)$
Reference point (1, 6)

$f(x) = 6^x$

b. Decreasing over $(-\infty, \infty)$
Reference point (−1, 4)

$f(x) = \left(\frac{1}{4}\right)^x$

c. End behavior: $\lim_{x\to-\infty,} f(x) = 0$ and $\lim_{x\to\infty,} f(x) = \infty$
Reference point (2, 6.25)

$f(x) = 2.5^x$

6. Summarize the characteristics for the basic exponential growth and exponential decay functions.

	Basic Exponential Growth	Basic Exponential Decay
Domain	$(-\infty, \infty)$	$(-\infty, \infty)$
Range	$(0, \infty)$	$(0, \infty)$
Asymptote	$y = 0$	$y = 0$
Intercepts	(0, 1)	(0, 1)
End Behavior	$\lim_{x\to-\infty,} f(x) = 0$ and $\lim_{x\to\infty,} f(x) = \infty$	$\lim_{x\to-\infty,} f(x) = \infty$ and $\lim_{x\to\infty,} f(x) = 0$
Intervals of Increase or Decrease	Increasing over $(-\infty, \infty)$	Decreasing over $(-\infty, \infty)$

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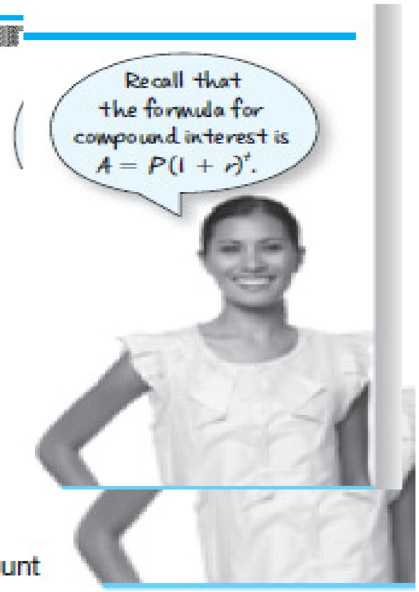
PROBLEM 2 Let's Compound Some Dough

Helen is opening her first savings account and is depositing $500. Suppose she decides on a bank that offers 6% annual interest to be calculated at the end of each year.



1. Write a function $A(t)$ to model the amount of money in Helen's savings account after t years.

   $A(t) = 500(1 + 0.06)^t$

2. Calculate the amount of money in Helen's savings account at the end of 1 year?

   After 1 year, Helen will have $530 in her savings account.

   $A(1) = 500(1 + 0.06)^1$

   $= 530$

3. How much money will be in Helen's savings account at the end of 5 years?

   After five years, Helen will have $669.11 in her savings account.

   $A(5) = 500(1 + 0.06)^5$

   $A(5) = 669.1127888$

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4. Suppose that the bank decides to start compounding interest at the end of every 6 months. If they still want to offer 6% per year, how much interest would they offer per 6-month period?

   If they offer 6% per year, and are compounding it twice a year, that would mean that the bank offers 3% interest per 6-month period.

5. John, Betty Jo, and Lizzie were each asked to calculate the amount of money Helen would have in her savings at the end of the year if interest was compounded twice a year. Who's correct? Explain your reasoning.

John	Betty Jo	Lizzie
1st 6 months: $A(t) = 500(1 + 0.03)$ $A(t) = 515$ 2nd 6 months: $A(t) = 515(1 + 0.03)$ $A(t) = 530.45$	$A(t) = 500\left(1 + \frac{0.06}{2}\right)^2$ $A(t) = 530.45$	$A(t) = 2(500(1 + 0.03))$ $A(t) = 1030$

John and Betty Jo are both right.

Lizzie calculated the first six months correctly, but then doubled it rather than compounding her interest.

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6. Write a function to model the amount of money in Helen's savings account at the end of t years, compounded n times during the year.

$$A(t) = 500\left(1 + \frac{0.06}{n}\right)^{nt}$$

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

7. Determine the amount of money in Helen's account at the end of 3 years if it is compounded:

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

a. twice a year.

Helen will have $597.03 at the end of 3 years if her interest is compounded twice a year.

Since interest is compounded twice a year, $n = 2$.

$$A(3) = 500\left(1 + \frac{0.06}{2}\right)^{(2 \cdot 3)}$$
$$= 500(1.03)^6$$
$$= 597.0261482645$$

b. monthly.

Helen will have $598.34 at the end of 3 years if her interest is compounded monthly.

Since interest is compounded monthly, $n = 12$.

$$A(3) = 500\left(1 + \frac{0.06}{12}\right)^{(12 \cdot 3)}$$
$$= 500(1.005)^{36}$$
$$= 598.3402624117\ldots$$

c. daily.

Helen will have $598.60 at the end of 3 years if her interest is compounded daily.

Since interest is compounded daily, $n = 365$.

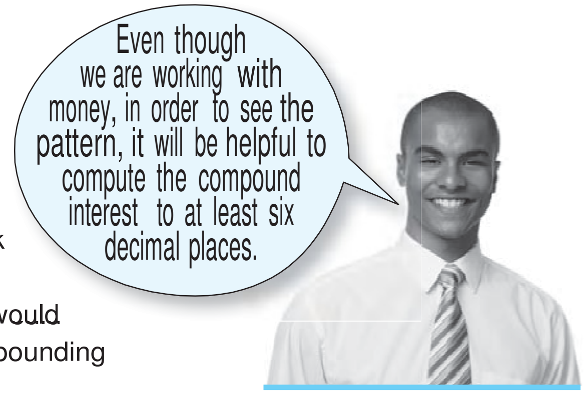
$$A(3) = 500\left(1 + \frac{0.06}{365}\right)^{(365 \cdot 3)}$$
$$= 500(1.000164383\ldots)^{1095}$$
$$= 598.599826468\ldots$$

8. What effect does the frequency of compounding have on the amount of money in her savings account?

The more often the account is compounded, the quicker her money grows.

PROBLEM 3 Easy "e"

Recall that in Problem 2 the variable n represented the number of compound periods per year. Let's examine what happens as the interest becomes compounded more frequently.



1. Imagine that Helen finds a different bank that offers her 100% interest. Complete the table to calculate how much Helen would accrue in 1 year for each period of compounding if she starts with $1.

Period of Compounding	$n =$	Formula	Amount
Yearly	1	$1\left(1 + \frac{1}{1}\right)^{1 \cdot 1}$	2.00
Semi-Annually	2	$1\left(1 + \frac{1}{2}\right)^{2 \cdot 1}$	2.25
Quarterly	4	$1\left(1 + \frac{1}{4}\right)^{4 \cdot 1}$	2.44140625 . . .
Monthly	12	$1\left(1 + \frac{1}{12}\right)^{12 \cdot 1}$	2.61303529 . . .
Weekly	52	$1\left(1 + \frac{1}{52}\right)^{52 \cdot 1}$	2.692596954 . . .
Daily	365	$1\left(1 + \frac{1}{365}\right)^{365 \cdot 1}$	2.714567482 . . .
Hourly	8760	$1\left(1 + \frac{1}{8760}\right)^{1 \cdot 8760}$	2.718126692 . . .
Every Minute	525600	$1\left(1 + \frac{1}{525600}\right)^{525600 \cdot 1}$	2.718279243 . . .
Every Second	31536000	$1\left(1 + \frac{1}{31536000}\right)^{31536000 \cdot 1}$	2.718281781 . . .

2. Make an observation about the frequency of compounding and the amount that Helen earns. What is it approaching?

The more frequently that Helen's interest is compounded, the amount becomes closer to approximately $2.72.

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The amount that Helen's earnings approach is actually an irrational number called e.

$$e \approx 2.718281828459045\ldots$$

It is often referred to as the **natural base *e***.

In geometry, you worked with π, an irrational number that was approximated as 3.14159265 . . . and so on. Pi is an incredibly important part of many geometric formulas and occurs so frequently that, rather than write out "3.14159265 . . . " each time, we use the symbol π.

Similarly, the symbol e is used to represent the constant 2.718281 . . . It is often used in models of population changes as well as radioactive decay of substances, and it is vital in physics and calculus.

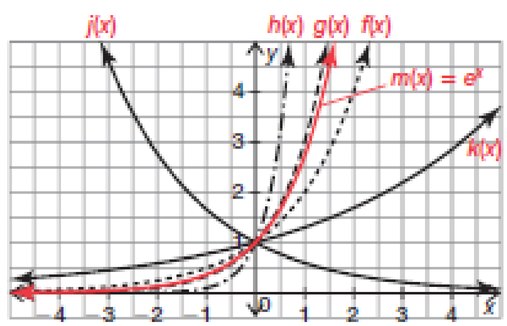
The symbol for the natural base e was first used by Swiss mathematician Leonhard Euler in 1727 as part of a research manuscript he wrote at age 21. In fact, he used it so much, e became known as Euler's number.

The constant e represents continuous growth and has many other mathematical properties that make it unique, which you will study further in calculus.

3. The following graphs are sketched on the coordinate plane shown.

   $f(x) = 2^x$, $g(x) = 3^x$, $h(x) = 10^x$, $j(x) = \left(\frac{3}{5}\right)^x$, $k(x) = 1.3^x$.

   a. Label each function.



   b. Consider the function $m(x) = e^x$. Use your knowledge of the approximate value of e to sketch its graph. Explain your reasoning.

   See graph.

   Because $e \approx 2.72$, and 2.72 is closer to 3 than 2, the graph of $m(x) = e^x$ should be closer to $h(x) = 3^x$ than $g(x) = 2^x$.

PROBLEM 4 It Keeps Growing and Growing and Growing…

1. The formula for population growth is $N(t) = N_0e^{rt}$. Complete the table to identify the contextual meaning of each quantity.

Quantity	Contextual Meaning
N_0	initial amount of population
r	rate of growth
t	time
$N(t)$	population after t years

2. Why is e used as the base? The constant e is used as the base because the population is continuously growing. The population doesn't increase at set intervals, such as every minute or once a week. It changes continuously.

3. How could this formula be used to represent a decline in population?

   I could change the rate of growth to a negative value to represent a decline in population.

4. The population of the city of Fredericksburg, Virginia, was approximately 19,360 in 2000 and has been continuously growing at a rate of 2.9% each year.

   a. Use the formula for population growth to write a function to model this growth.

   $$N(t) = 19{,}360e^{0.029t}$$

   b. Use your function model to predict the population of Fredericksburg in 2013.

   In the model, t represents the number of years since 2000. So, in 2013, $t = 13$.

   $N(13) = 19{,}360e^{(0.029 \cdot 13)}$

   $= 28225.0274294$

   According to the model, in 2013, Fredericksburg would have approximately 28,225 people.

   c. What value does your function model give for the population of Fredericksburg in the year 1980?

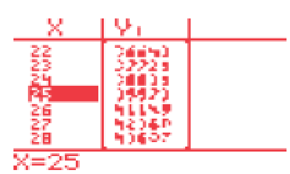
   In the model, t represents the number of years since 2000, so in 1980, $t = -20$.

   $N(-20) = 19{,}360e^{(0.029 \cdot (-20))}$

   $= 10839.6323767$

   According to the model, in 1980, Fredericksburg had approximately 10,340 people.

5. Use a graphing calculator to estimate the number of years it would take Fredericksburg to grow to 40,000 people, assuming that the population trend continues. The population of Fredericksburg would grow to 40,000 between the years 2025 and 2026.



Be prepared to share your solutions and methods.

Check for Students' Understanding

Jan invested $5000 in a savings account for 6 years. The bank pays 2% compounded monthly. At the end of 6 years, how much would Jan's investment be worth?

At the end of 6 years, Jan's investment would be worth $5636.92.

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$A(6) = 5000\left(1 + \frac{0.02}{12}\right)^{12 \cdot 6}$$

$$A(6) \approx 5000(1.00166)^{72}$$

$$A(6) \approx 5636.921$$