12.3

I Like to Move It

Transformations of Exponential Functions

LEARNING GOALS

In this lesson, you will:

- Dilate, reflect, and translate exponential functions using reference points and transformational function form.
- Investigate graphs of exponential functions through intercepts, asymptotes, intervals of increase and decrease, and end behavior.
- Describe how transformations of exponential functions affect their key characteristics.

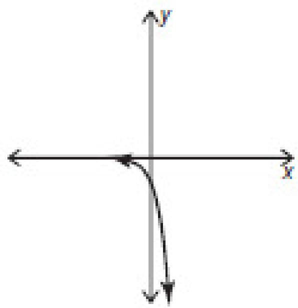
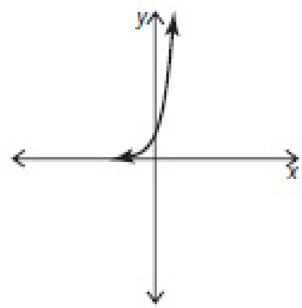
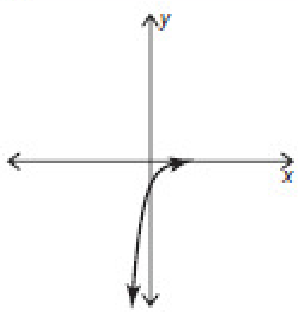
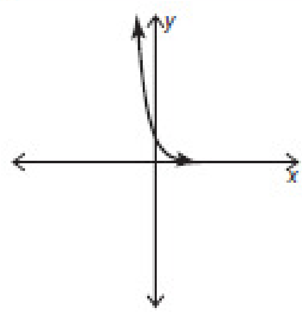
Andy Warhol was an American pop artist whose work explored the relationship between artistic expression, celebrity culture, and advertisement. A recurring theme throughout Warhol's art is the transformation of the mundane and commonplace into art. His most renowned images are silk-screened reproductions of Campbell's soup cans and publicity photographs of pop culture icons like Marilyn Monroe and Elvis Presley.

Have you ever seen any of Andy Warhol's work?

PROBLEM 1 It's the Same . . . But Different!

1. The two tables show four exponential functions and four exponential graphs.
   a. Match the exponential function to its corresponding graph, and write the function under the graph it represents.
   b. Explain the method(s) you used to match the functions with their graphs.

Exponential Functions	
$f(x) = 10^x$	$g(x) = 10^{-x}$
$h(x) = -10^x$	$j(x) = -10^{-x}$

Exponential Graphs	
 $h(x) = -10^x$	 $f(x) = 10^x$
 $j(x) = -10^{-x}$	 $g(x) = 10^{-x}$

Answers will vary.

The graph of $f(x) = 10^x$ is an exponential growth function.

The graph of $g(x) = 10^{-x}$ is a reflection of the graph of $f(x)$ across the y-axis.

The graph of $h(x) = -10^x$ is a reflection of the graph of $f(x)$ across the x-axis.

The graph of $j(x) = -10^{-x}$ is a reflection of the graph of $f(x)$ across the x- and y-axes.

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2. Analyze the graphs.

a. Write an equation for $h(x)$ in terms of $f(x)$. Describe the transformation on $f(x)$.

$h(x) = -f(x)$

The transformation on $f(x)$ is a reflection across the x-axis.

b. Write an equation for $g(x)$ in terms of $f(x)$. Describe the transformation on $f(x)$.

$g(x) = f(-x)$

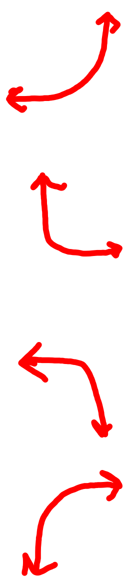
The transformation on $f(x)$ is a reflection across the y-axis.

c. Write an equation for $j(x)$ in terms of $f(x)$. Describe the transformation on $f(x)$.

$j(x) = -f(-x)$

The transformation on $f(x)$ is a reflection of the graph of $f(x)$ across the x- and y-axes.

3. Determine the asymptotes, intervals of increase and decrease, and end behavior for each exponential function.



Function	Asymptotes	Intervals of Increase and Decrease	End Behavior
$f(x) = 10^x$	$y = 0$	Increasing over $(-\infty, \infty)$	$\lim_{x\to-\infty} f(x) = 0$ $\lim_{x\to\infty} f(x) = \infty$
$g(x) = 10^{-x}$	$y = 0$	Decreasing over $(-\infty, \infty)$	$\lim_{x\to-\infty} g(x) = \infty$ $\lim_{x\to\infty} g(x) = 0$
$h(x) = -10^x$	$y = 0$	Decreasing over $(-\infty, \infty)$	$\lim_{x\to-\infty} h(x) = 0$ $\lim_{x\to\infty} h(x) = -\infty$
$j(x) = -10^{-x}$	$y = 0$	Increasing over $(-\infty, \infty)$	$\lim_{x\to-\infty} j(x) = -\infty$ $\lim_{x\to\infty} j(x) = 0$

4. How would the graph of $k(x) = \left(\frac{1}{10}\right)^x$ compare to the graph of $g(x) = 10^{-x}$?

The graphs of $g(x) = 10^{-x}$ and $k(x) = \left(\frac{1}{10}\right)^x$ would be equivalent. The functions $g(x)$ and $k(x)$ represent the same exponential decay function, which is the reflection of $f(x) = 10^x$ across the y-axis.

$g(x) = 10^{-x}$

$= (10^{-1})^x$

$= \left(\frac{1}{10}\right)^x$

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5. How do the transformations on $f(x)$ affect the asymptotes, intervals of increase and decrease, and end behavior?

The transformations on $f(x)$ do not affect the horizontal asymptote of the function.

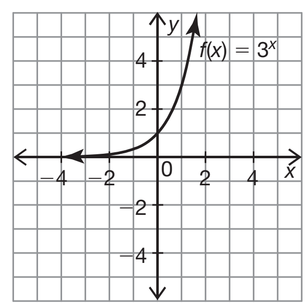
A single reflection over the x- or y-axis affects the intervals of increase and decrease. Reflections across both the x- and y-axes do not affect the intervals of increase and decrease.

The transformations on $f(x)$ all affect the end behavior of the function.

PROBLEM 2 Keep On Moving

Consider the functions $y = f(x)$ and $g(x) = Af(B(x - C)) + D$. Recall that the D-value translates $f(x)$ vertically, the C-value translates $f(x)$ horizontally, the A-value vertically stretches or compresses $f(x)$, and the B-value horizontally stretches or compresses $f(x)$. Exponential functions are transformed in the same manner.

The function $f(x) = 3^x$ is shown. Recall the key characteristics of basic exponential functions, including a domain of all real numbers, a range of positive numbers, and a horizontal asymptote at $y = 0$.



1. Suppose that $a(x) = f(x + 1)$.

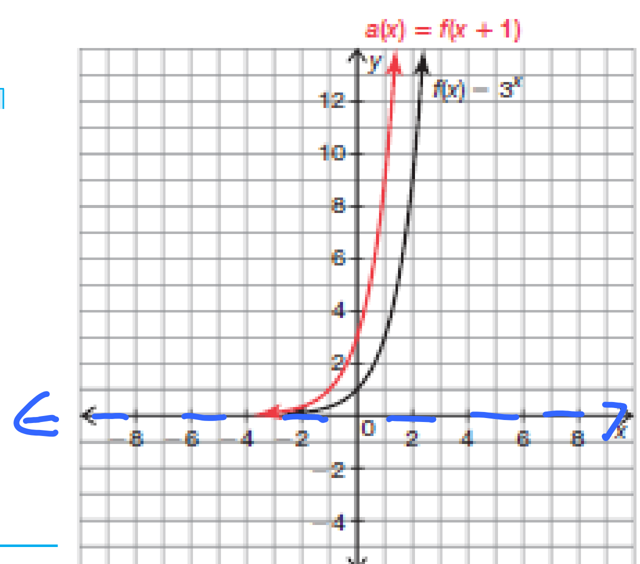
   a. Describe the transformation on the graph of $f(x)$ that produces $a(x)$.

   The graph of $f(x)$ is translated horizontally left 1 unit to produce $a(x)$.

   b. Complete the table to determine the corresponding points on $a(x)$, given reference points on $f(x)$. Then, graph and label $a(x)$.

(x, y) $(x-1, y)$

Reference Points on $f(x)$	Corresponding Points on $a(x)$
$\left(-1, \frac{1}{3}\right)$	$\left(-2, \frac{1}{3}\right)$
(0, 1)	(−1, 1)
(1, 3)	(0, 3)



c. Determine the domain, range, and asymptotes of $a(x)$.

c. Determine the domain, range, and asymptotes of $a(x)$.

Domain: All real numbers OR $(-\infty, \infty)$

Range: $y > 0$ OR $(0, \infty)$

Horizontal asymptote: $y = 0$

2. Suppose that $b(x) = f(x) + 1$.

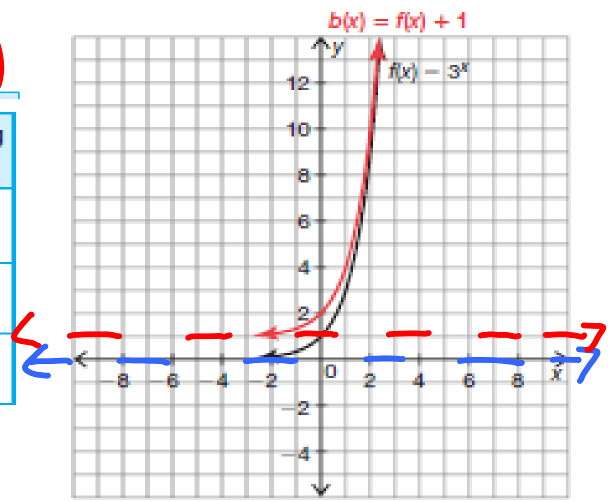
a. Describe the transformation on the graph of $f(x)$ that produces $b(x)$.

The graph of $f(x)$ is translated vertically up 1 unit to produce $b(x)$.

b. Complete the table to determine the corresponding points on $b(x)$, given reference points on $f(x)$. Then, graph and label $b(x)$.

$(x, y+1)$

Reference Points on $f(x)$	Corresponding Points on $b(x)$
$\left(-1, \frac{1}{3}\right)$	$\left(-1, 1\frac{1}{3}\right)$
(0, 1)	(0, 2)
(1, 3)	(1, 4)



c. Determine the domain, range, and asymptotes of $b(x)$.

Domain: All real numbers OR $(-\infty, \infty)$

Range: $y > 1$ or $(1, \infty)$

Horizontal asymptote: $y = 1$

3. Suppose that $c(x) = f(x) - 5$.

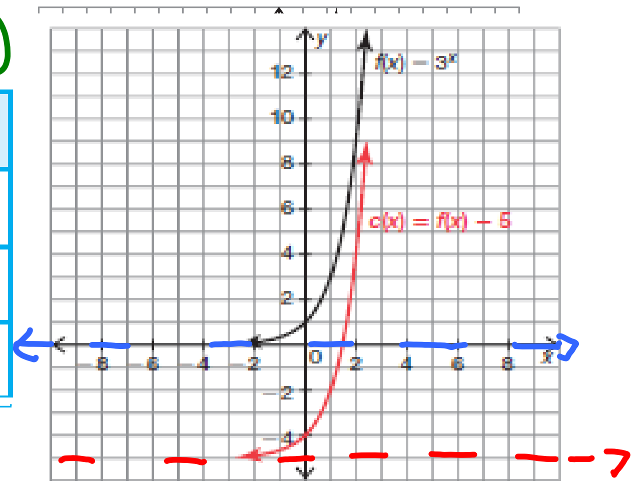
a. Describe the transformation on the graph of $f(x)$ that produces $c(x)$.

The graph of $f(x)$ is translated vertically down 5 units to produce $c(x)$.

b. Complete the table to determine the corresponding points on $c(x)$, given reference points on $f(x)$. Then, graph and label $c(x)$.

$(x, y-5)$

Reference Points on $f(x)$	Corresponding Points on $c(x)$
$\left(-1, \frac{1}{3}\right)$	$\left(-1, -4\frac{2}{3}\right)$
(0, 1)	(0, −4)
(1, 3)	(1, −2)



c. Determine the domain, range, and asymptotes of $c(x)$.

Domain: All real numbers OR $(-\infty, \infty)$

Range: $y > -5$ $(-5, \infty)$

Horizontal asymptote: $y = -5$

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4. Suppose that $d(x) = f(2x)$.

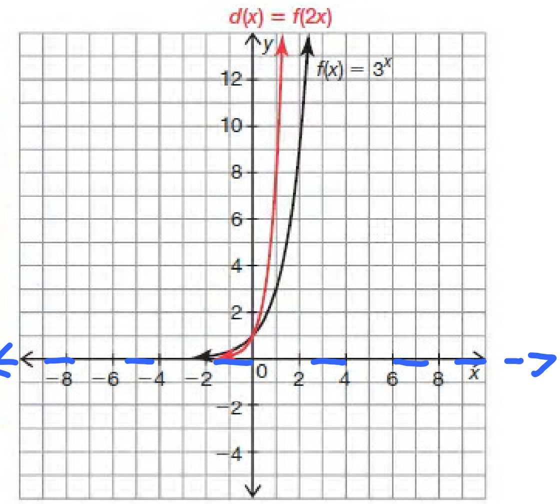
a. Describe the transformation on the graph of $f(x)$ that produces $d(x)$.

The graph of $f(x)$ is compressed horizontally by a factor of $\frac{1}{2}$ to produce $d(x)$.

b. Complete the table to determine the corresponding points on $d(x)$, given reference points on $f(x)$. Then, graph and label $d(x)$.

$\left(\frac{1}{2}x, y\right)$

Reference Points on $f(x)$	Corresponding Points on $d(x)$
$\left(-1, \frac{1}{3}\right)$	$\left(-\frac{1}{2}, \frac{1}{3}\right)$
$(0, 1)$	$(0, 1)$
$(1, 3)$	$\left(\frac{1}{2}, 3\right)$



c. Determine the domain, range, and asymptotes of $d(x)$.

Domain: All real numbers $(-\infty, \infty)$

Range: $y > 0$ $(0, \infty)$

Horizontal asymptote: $y = 0$

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5. Analyze the transformations performed on $f(x)$ in Questions 1 through 4.

a. Which, if any, of these transformations affected the domain, range, and asymptotes?

Vertical translations of $f(x)$, such as $b(x)$ and $c(x)$, affected the range and the horizontal asymptote.

The domain was not affected.

b. What generalizations can you make about the effects of transformations on the domain, range, and asymptotes of exponential functions?

The domain of exponential functions is not affected by vertical translations, horizontal translations, and horizontal dilations.

↓↑ Vertical translations affect the range and the horizontal asymptote of exponential functions.

dilation or ⇄ Horizontal translations and horizontal dilations do not affect the range and the horizontal asymptote of exponential functions.

6. Andres and Tomas each described the effects of transforming the graph of $f(x) = 3^x$, such that $p(x) = 3f(x)$.

Andres

$p(x) = 3f(x)$

The A-value is 3 so the graph is stretched vertically by a scale factor of 3.

Tomas

$p(x) = 3f(x)$

$p(x) = 3 \cdot 3^x$

$p(x) = 3^{1+x}$

$p(x) = f(x + 1)$

The C-value is 1 so the graph is horizontally translated 1 unit to the left.

a. Explain Andres' and Thomas' reasoning.

Andres used his knowledge of transformational function form to describe the graph of $p(x)$ in terms of $f(x)$.

Tomas used the properties of exponents to rewrite $p(x)$. Then, he used transformational function form to rewrite $p(x)$, and he described its graph in terms of $f(x)$.

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b. Determine the domain, range, and asymptotes of $p(x)$.

Domain: All real numbers or $(-\infty, \infty)$

Range: $y > 0$ or $(0, \infty)$

Horizontal asymptote: $y = 0$

c. What generalizations can you make about the effects of vertical dilations on the domain, range, and asymptotes of exponential functions?

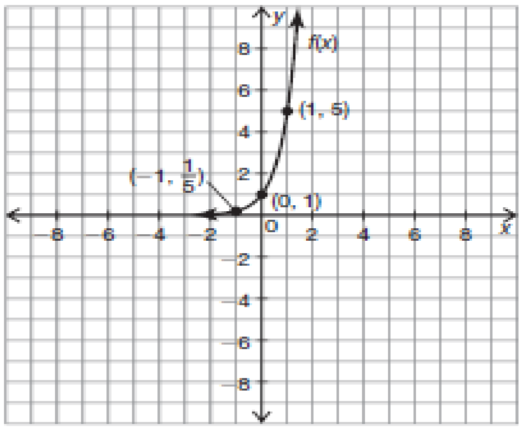
stretch/shrink

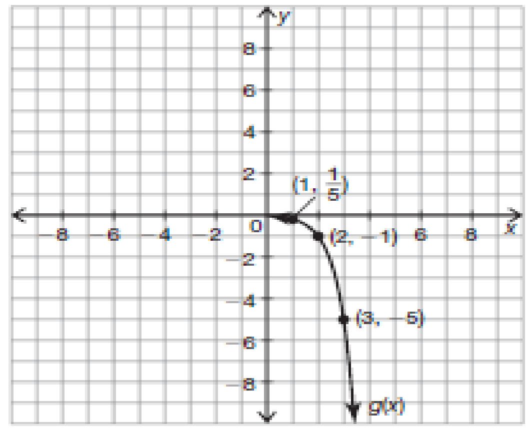
Vertical dilations do not affect the domain, range, or the horizontal asymptote of exponential functions.

PROBLEM 3 Multiple Transformations

1. Analyze the graphs of $f(x)$ and $g(x)$. Describe the transformations performed on $f(x)$ to create $g(x)$. Then, write an equation for $g(x)$ in terms of $f(x)$. For each set of points shown on $f(x)$, the corresponding points are shown on $g(x)$.

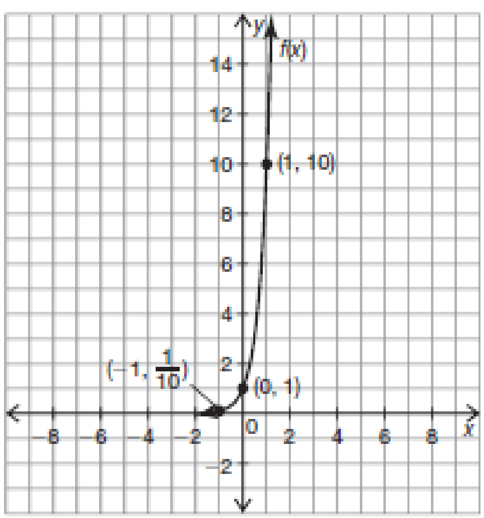
a. $g(x) =$ $-f(x - 2)$

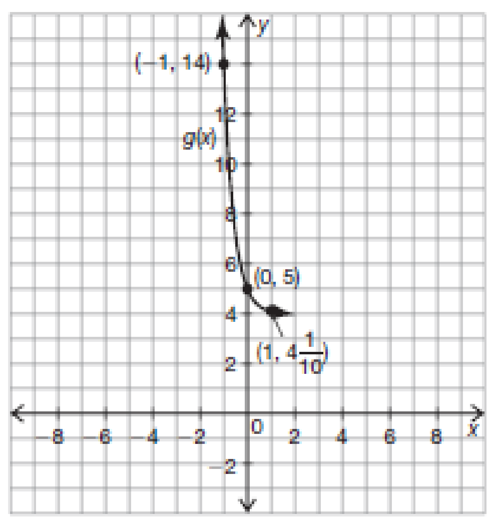




To create $g(x)$, the graph of $f(x)$ is horizontally translated right 2 units and reflected across the x-axis.

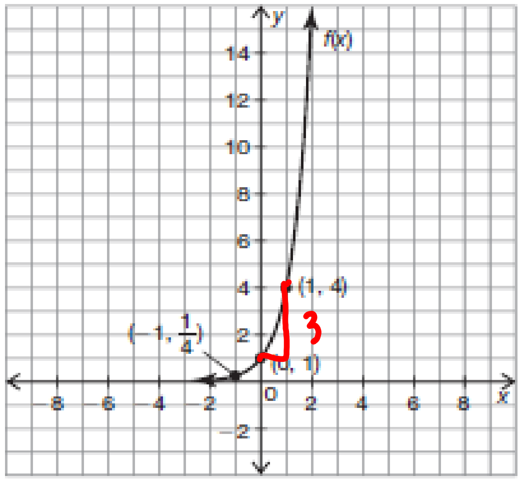
b. $g(x) =$ $f(-x) + 4$

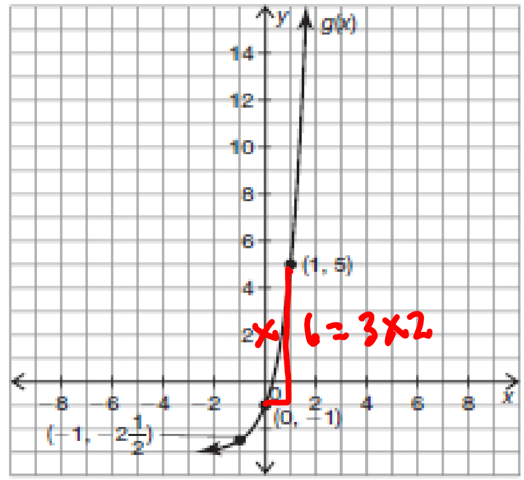




To create $g(x)$, the graph of $f(x)$ is reflected across the y-axis and vertically translated up 4 units.

c. $g(x) =$ $2f(x) - 3$





To create $g(x)$, the graph of $f(x)$ is stretched vertically by a factor of 2, and vertically translated down 3 units.

2. The equation for an exponential function $m(x)$ is given. The equation for the transformed function $t(x)$ in terms of $m(x)$ is also given. Describe the graphical transformation(s) on $m(x)$ that produce(s) $t(x)$. Then, write an exponential equation for $t(x)$.

a. $m(x) = 2^x$
$t(x) = 0.5m(x + 3)$
The graph of the function $m(x)$ is horizontally translated left 3 units and compressed vertically by a factor of 0.5 to produce $t(x)$.

$t(x) = 0.5 \cdot 2^{x+3}$

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b. $m(x) = e^x$
$t(x) = -m(x) - 1$
The graph of the function $m(x)$ is reflected across the x-axis and vertically translated down 1 unit to produce $t(x)$.

$t(x) = -e^x - 1$

c. $m(x) = 6^x$
$t(x) = 2m(-x)$
The graph of the function $m(x)$ is reflected across the y-axis and stretched vertically by a factor of 2 to produce $t(x)$.

$t(x) = 2 \cdot 6^{-x}$

Be prepared to share your solutions and methods.