12.4

I Feel the Earth Move

Logarithmic Functions

LEARNING GOALS

In this lesson, you will:

- Graph the inverses of exponential functions with bases of 2, 10, and *e*.
- Recognize the inverse of an exponential function as a logarithm.
- Identify the domain and range of logarithmic functions.
- Investigate graphs of logarithmic functions through intercepts, asymptotes, intervals of increase and decrease, and end behavior.

KEY TERMS

- logarithm
- logarithmic function
- common logarithm
- natural logarithm

You may have heard about the Richter scale rating. The Richter scale was developed in 1935 by Charles F. Richter of the California Institute of Technology. The Richter scale is used to rate the magnitude of an earthquake—the amount of energy it releases. This is calculated using information gathered by a seismograph.

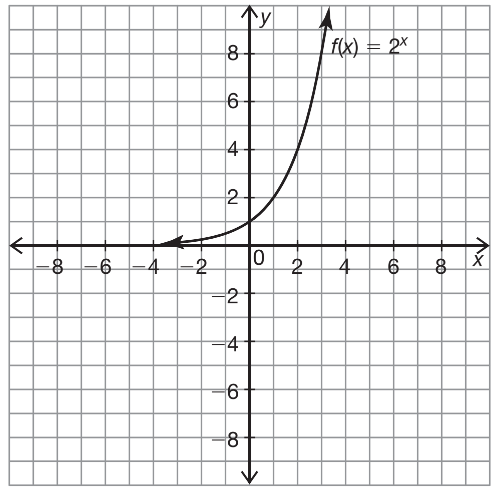
The Richter scale is logarithmic, meaning that whole-number jumps in the rating indicate a tenfold increase in the wave amplitude of the earthquake. For example, the wave amplitude in a Level 4 earthquake is ten times greater than the amplitude of a Level 5 earthquake, and the amplitude increases 100 times between a Level 6 earthquake and a Level 8 earthquake.

Most earthquakes are extremely small, with a majority registering less than 3 on the Richter scale. These tremors, called microquakes, aren't even felt by humans. Only a tiny portion, 15 or so of the 1.4 million quakes that register above 2 each year, register at 7 or above, which is the threshold for a quake to be considered major.

PROBLEM 1 Return of the Inverse

Consider the table and graph for the basic exponential function $f(x) = 2^x$.

x	$f(x) = 2^x$
−3	$\frac{1}{8}$
−2	$\frac{1}{4}$
−1	$\frac{1}{2}$
0	1
1	2
2	4
3	8



You learned that the key characteristics of basic exponential functions are:

- The domain is the set of all real numbers.
- The range is the set of all positive numbers.
- The y-intercept is (0, 1).
- There is no x-intercept.
- There is a horizontal asymptote at $y = 0$.
- The function increases over the entire domain.
- As x approaches negative infinity, $f(x)$ approaches 0.
- As x approaches positive infinity, $f(x)$ approaches positive infinity.

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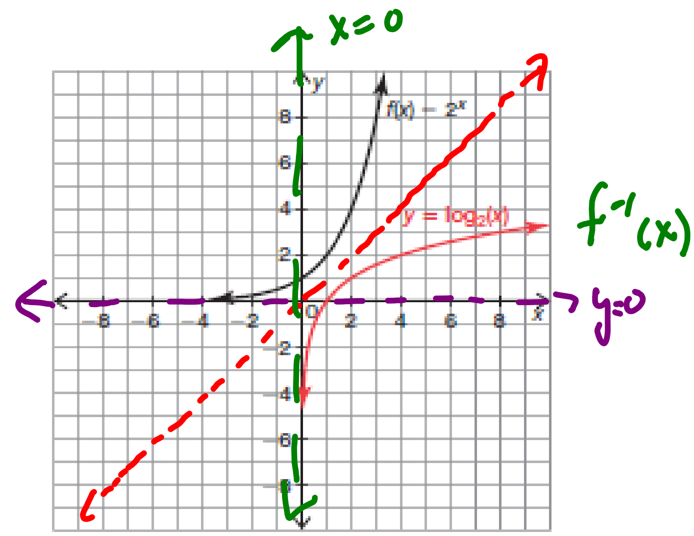
Recall that for any function f with ordered pairs (x, y), or $(x, f(x))$, the inverse of the function f is the set of all ordered pairs (y, x), or $(f(x), x)$.

1. Graph the inverse of $f(x) = 2^x$ on the same coordinate plane as $f(x)$. Complete the table of values for the inverse of $f(x)$.

x	y
$\frac{1}{8}$	-3
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3



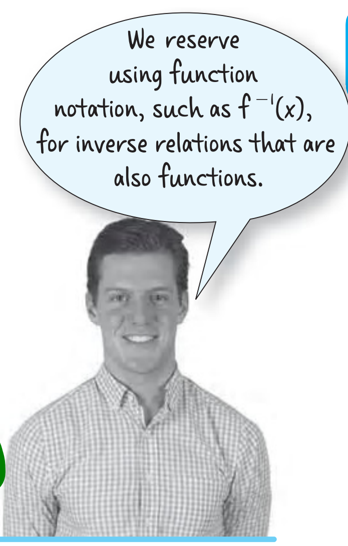
2. Analyze the key characteristics of the inverse of $f(x) = 2^x$.

a. Is the inverse of $f(x) = 2^x$ a function? Explain your reasoning.

Yes. The function $f(x) = 2^x$ is an invertible function.

The graph of the exponential function passes the Horizontal Line Test. The graph of the inverse passes the Vertical Line Test.

b. Identify the domain, range, intercepts, asymptotes, intervals of increase and decrease, and end behavior of $f^{-1}(x)$.



Domain: $x > 0$ $(0, \infty)$

Range: All real numbers $(-\infty, \infty)$

Intercepts: The x-intercept is $(1, 0)$. There is no y-intercept.

Asymptotes: Vertical asymptote at $x = 0$

Intervals of increase and decrease: Increasing over entire domain → $(0, \infty)$

End behavior: $\lim_{x \to \infty} f^{-1}(x) = \infty$

Asymptotic Behavior: $\lim_{x \to 0^+} f^{-1}(x) = -\infty$

c. What do you notice about the domain and range of the exponential function and its inverse?

The domain of the exponential function is the same as the range of the inverse function. The range of the exponential function is the same as the domain of the inverse function.

d. What do you notice about the asymptotes of the exponential function and its inverse?

For the exponential function, there is a horizontal asymptote of $y = 0$. For the inverse of the exponential function, there is a vertical asymptote of $x = 0$. The equations of the asymptotes are the same, except the variables are swapped.

e. What do you notice about the intervals of increase and decrease of the exponential function and its inverse?

Both the exponential function and its inverse are increasing over their domains.

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f. What do you notice about the end behavior of the exponential function and its inverse?

For the exponential function, as x approaches negative infinity, y approaches 0, whereas for the inverse of the exponential function, as x approaches 0 from the right, y approaches negative infinity. The variables are swapped.

For the exponential function and its inverse, as x approaches positive infinity, y approaches positive infinity.

g. Write the equation for the inverse of $y = 2^x$. Explain your reasoning.

$x = 2^y$

Because the inverse of a function is the relation formed when the independent variable is exchanged with the dependent variable, I can write the inverse as $x = 2^y$.

It is necessary to define a new function in order to write the equation for the inverse of an exponential function. The **logarithm** of a number for a given base is the exponent to which the base must be raised in order to produce the number. If $y = b^x$, then x is the logarithm and can be written as $x = \log_b(y)$. The value of the base of a logarithm is the same as the base in the exponential expression b^x.

For example, the number 3 is the logarithm to which base 2 must be raised to produce the argument 8. The base is written as the subscript 2. The logarithm, or exponent, is the output 3. The argument of the logarithm is 8.

$$3 = \log_2(8)$$

logarithm, or exponent → 3; argument → 8; base → 2

You can write any exponential equation as a logarithmic equation and vice versa.

Example	Exponential Form	⇔	Logarithmic Form
A	$y = b^x$	⇔	$x = \log_b(y)$
B	$16 = 4^2$	⇔	$2 = \log_4(16)$
C	$1000 = 10^3$	⇔	$3 = \log_{10}(1000)$
D	$32 = 16^{1.25}$	⇔	$1.25 = \log_{16}(32)$
E	$a = b^c$	⇔	$c = \log_b(a)$

3. Rewrite your equation in Question 2, part (g), in logarithmic form. Label the graph from Question 1 with your equation.

$x = 2^y \Leftrightarrow y = \log_2(x)$

See graph.

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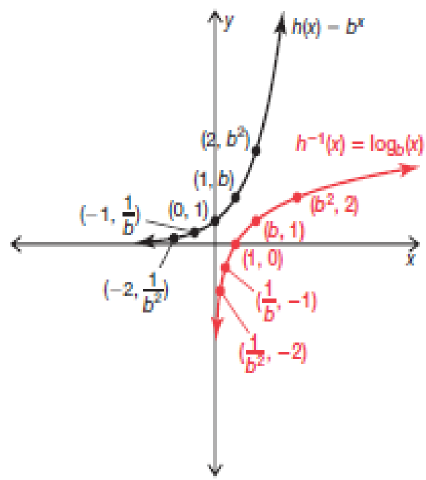
4. Analyze the exponential equation $y = b^x$ and its related logarithmic equation, $x = \log_b(y)$. State the restrictions, if any, on the variables. Explain your reasoning.

$y = b^x \Leftrightarrow x = \log_b(y)$		
Variable	Restrictions	Explanation
x	No restrictions	The exponent, or logarithm, can be any real number.
b	$b > 0$ and $b \neq 1$	For an exponential function, the base must be a positive number not equal to 1.
y	$y > 0$	The range of an exponential function is the set of all positive numbers.

PROBLEM 2 A Logarithm by Any Other Name . . .

1. The graph of $h(x) = b^x$ is shown. Sketch the graph of the inverse of $h(x)$ on the same coordinate plane. Label coordinates of points on the inverse of $h(x)$.



2. Write the equation for the inverse of $h(x) = b^x$. Label the graph.

$h^{-1}(x) = \log_b(x)$

See graph.

3. Do you think all exponential functions are invertible? If so, explain your reasoning. If not, provide a counterexample.

Yes. All exponential functions are invertible.

All exponential functions will pass the Horizontal Line Test.