More Than Meets the Eye

Transformations of Logarithmic Functions

12.5

LEARNING GOALS

In this lesson, you will:

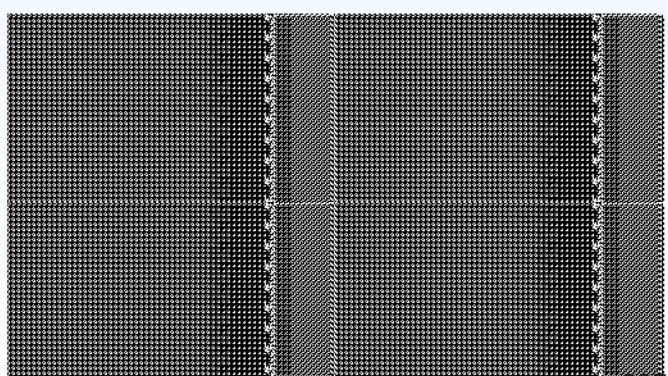
- Dilate, reflect, and translate logarithmic functions using reference points.
- Investigate graphs of logarithmic functions through intercepts, asymptotes, intervals of increase and decrease, and end behavior.

'Practice makes perfect'!

When you practice the same motion over and over, you are building up procedural memory in your brain that instructs your muscles to perform a task. The more often your muscles receive those same instructions, the more quickly and efficiently they are able to carry them out until they become like second nature to you. Athletes use this idea of "muscle memory" when conditioning their bodies to perform, and musicians use it to train their fingers to hit the correct keys or strings accurately.

The best way to train your body and mind when learning a new skill is to practice it slowly at first to be sure that your technique is perfect, and then repeat that same quality practice as often as possible. Break the skill or information up into pieces and work on it, a piece at a time, until it is committed to memory. Once you've learned all of the parts, continue to practice, practice, practice until your new skill becomes an ingrained habit.

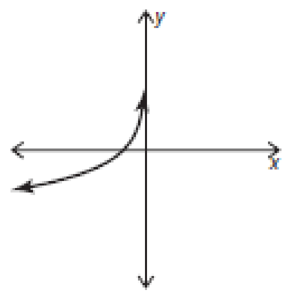
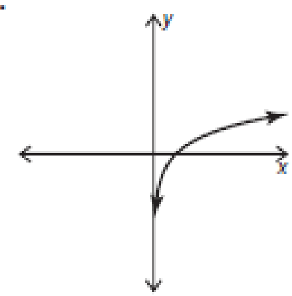
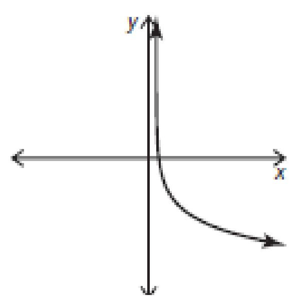
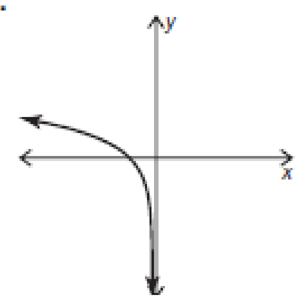
What skills do you like to practice?



PROBLEM 1 Don't Flip Out! It's Just a Reflection

1. The two tables show four logarithmic functions and four logarithmic graphs. Match the logarithmic function to its corresponding graph, and write the function under the graph it represents.

Logarithmic Functions	
$f(x) = \log_2(x)$	$g(x) = -\log_2(x)$
$h(x) = \log_2(-x)$	$j(x) = -\log_2(-x)$

Logarithmic Graphs	
A.  $j(x) = -\log_2(-x)$	B.  $f(x) = \log_2(x)$
C.  $g(x) = -\log_2(x)$	D.  $h(x) = \log_2(-x)$

2. Analyze the graphs of $f(x)$, $g(x)$, $h(x)$, and $j(x)$. Write an equation for each function $g(x)$, $h(x)$, and $j(x)$ in terms of $f(x)$. Describe each transformation on $f(x)$.

Logarithmic Function	Transformation on $f(x)$
$g(x) = -f(x)$	Reflection across the line $y = 0$
$h(x) = f(-x)$	Reflection across the line $x = 0$
$j(x) = -f(-x)$	Reflection across the lines $x = 0$ and $y = 0$

3. Analyze the key characteristics of each logarithmic function. What similarities exist among the four functions?

For the four logarithmic functions, the range is the set of all real numbers. Each function has a vertical asymptote at $x = 0$.

4. What generalizations can you make about the effects of these transformations on the domain and range of a logarithmic function?

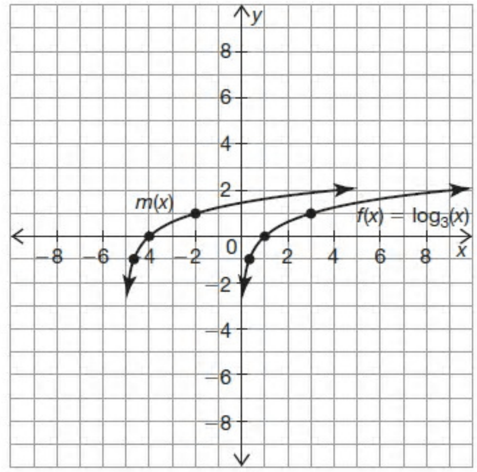
The domain is the set of all positive numbers for the logarithm of x. The domain for the logarithm of the opposite of x is the set of all negative numbers. The domain will never equal zero, regardless of how the graph of the logarithmic function is reflected, because there is a vertical asymptote at $x = 0$.

The range will always be the set of all real numbers for any basic logarithmic function, regardless of reflection.

PROBLEM 2 You've Got Some Moves!

1. Analyze the graphs of $f(x)$ and the transformed function. Describe the transformations produced on $f(x)$ to create the transformed function. Then, write an equation for the transformed function in terms of $f(x)$. For each set of points shown on $f(x)$, the corresponding points are shown on the transformed function.

a. Describe the transformation:

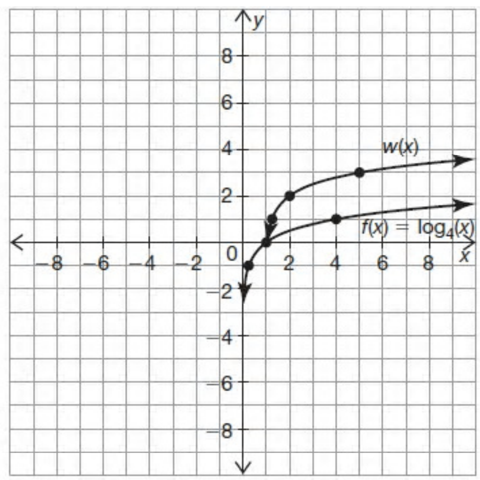


horizontally translate left 5 units

Transformed function form: $m(x) = \log_3(x+5)$

$m(x) =$ $f(x + 5)$

b. Describe the transformation:



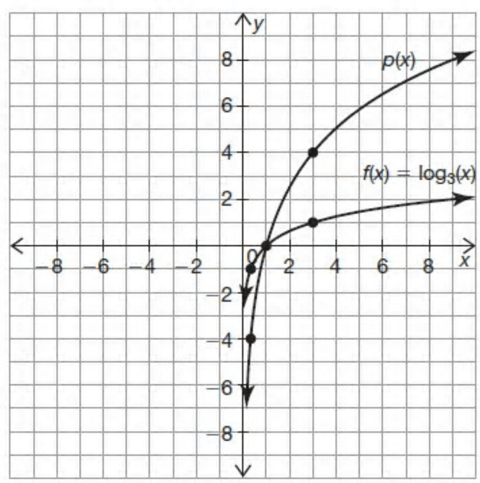
horizontally translate right 1 unit,

vertically translate up 2 units

Transformed function form: $W(x) = \log_4(x-1) + 2$

$w(x) =$ $f(x - 1) + 2$

c. Describe the transformation:



vertical dilation by a factor of 4

Stretch

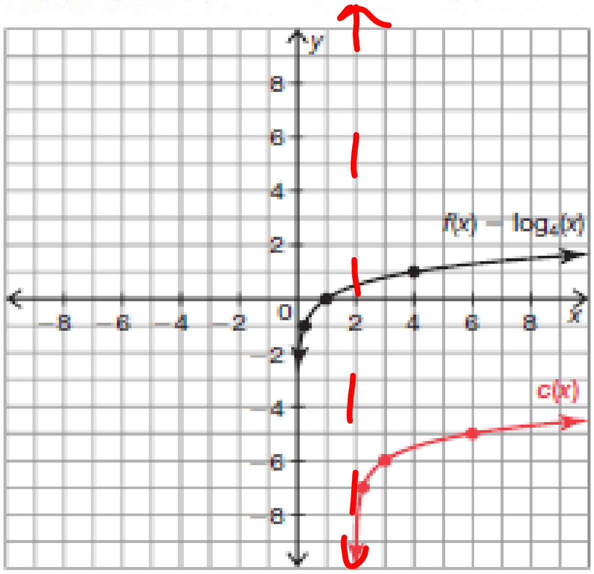
Transformed function form: $p(x) = 4\log_3(x)$

$p(x) =$ $4f(x)$

2. The graph of a basic logarithmic function $f(x)$ is shown. Graph each transformation of $f(x)$. State the domain, range, and asymptotes of your graph.

Write the logarithmic function described by the transformations below:

a. $c(x) = f(x - 2) - 6$



$c(x) =$ $\log_4(x-2) - 6$

Domain of $c(x)$: $(2, \infty)$

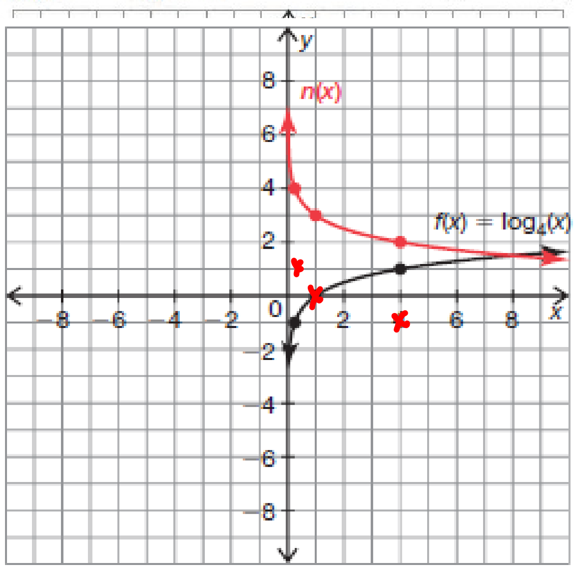
Range of $c(x)$: $(-\infty, \infty)$

Asymptote of $c(x)$: $x = 2$

End Behavior: $\lim_{x \to \infty} c(x) = \infty$

Asymptotic Behavior: $\lim_{x \to 2^+} c(x) = -\infty$

b. $n(x) = -f(x) + 3$



$n(x) =$ $-\log_4(x) + 3$

Domain of $n(x)$: $(0, \infty)$

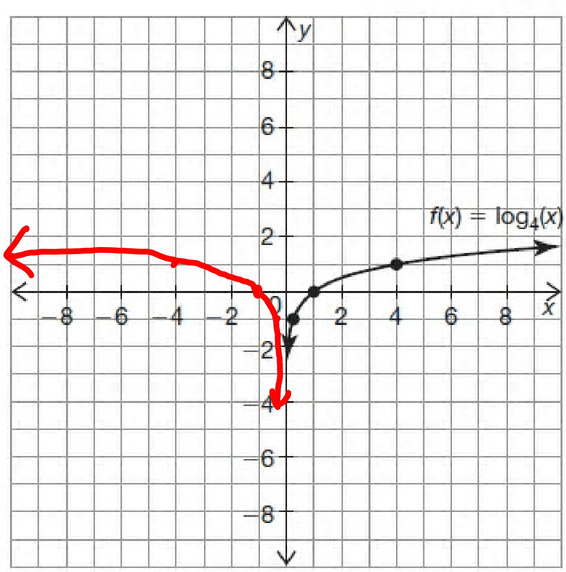
Range of $n(x)$: $(-\infty, \infty)$

Asymptote of $n(x)$: $x = 0$

End Behavior: $\lim_{x \to \infty} n(x) = -\infty$

Asymptotic Behavior: $\lim_{x \to 0^+} n(x) = +\infty$

c. $z(x) = f(-x)$



$z(x) =$ $\log_4(-x)$

Domain of $z(x)$: $(-\infty, 0)$

Range of $z(x)$: $(-\infty, \infty)$

Asymptote of $z(x)$: $x = 0$

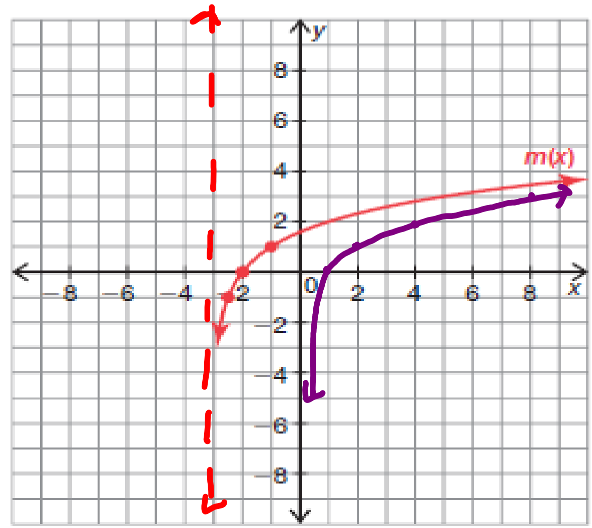
End Behavior: $\lim_{x \to -\infty} z(x) = +\infty$

Asymptotic Behavior: $\lim_{x \to 0^-} z(x) = -\infty$

3. Write a transformed logarithmic function in terms of $f(x)$ with the characteristic(s) given. Then, graph the transformed function.

a. $f(x) = \log_2(x)$

vertical asymptote at $x = -3$



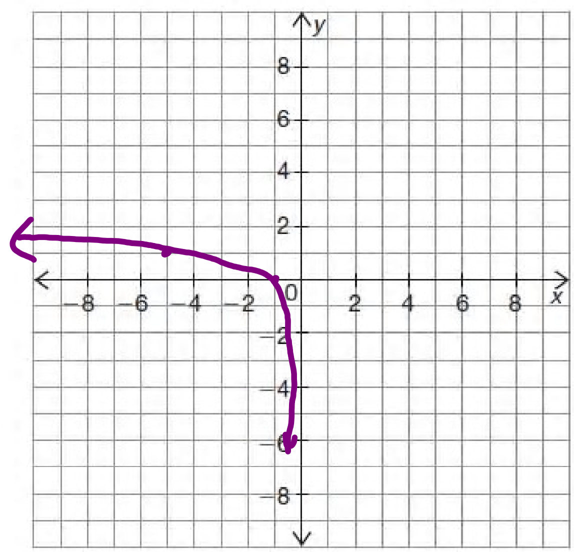
Transformed function form:

$m(x) = \log_2(x+3)$

$m(x) =$ $f(x+3)$

c. $f(x) = \log_5(x)$

Domain: $(-\infty, 0)$



Transformed function form:

$k(x) = \log_5(-x)$

$k(x) =$ $f(-x)$

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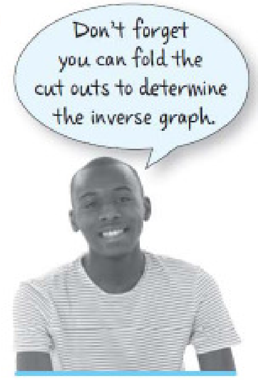
PROBLEM 3 Makin' Moves



1. Consider the functions $f(x) = 2^x$ and $g(x) = f^{-1}(x)$, or $\log_2(x)$. The graphs of $f(x)$ and $g(x)$ are shown.

a. Graph and label, $t(x)$, the transformed function for examples A, B, C, D. Describe in words the transformation(s) performed on $f(x)$.

b. Graph the inverse of $t(x)$, label the graph, $t^{-1}(x)$.

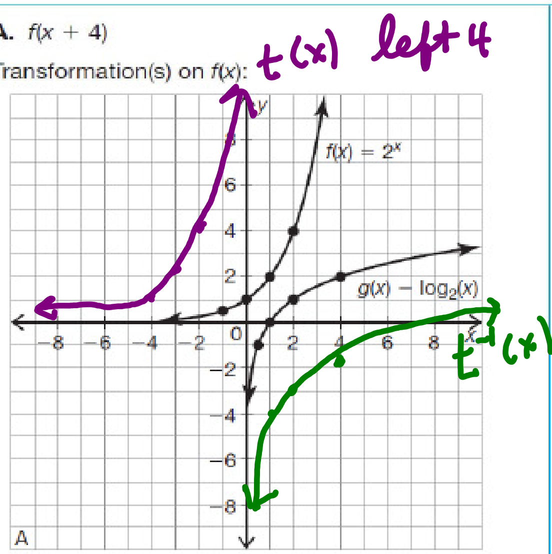
c. Describe the inverse, $t^{-1}(x)$, of the transformed function as a transformation on $g(x)$.

d. Compare the transformed function of $f(x)$ to the transformed function $g(x)$. What do you notice about the functions?

For example in problem A: $t(x) = f(x + 4)$

A. $f(x + 4)$

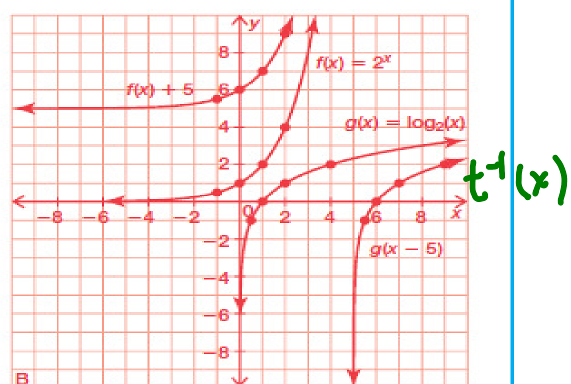
Transformation(s) on $f(x)$: t(x) left 4



Transformation(s) on $g(x)$: g(x) down 4 to get $t^{-1}(x)$

B. $f(x) + 5$

Transformation(s) on $f(x)$: vertical translation up 5 units



Transformation(s) on $g(x)$: horizontal translation right 5 units

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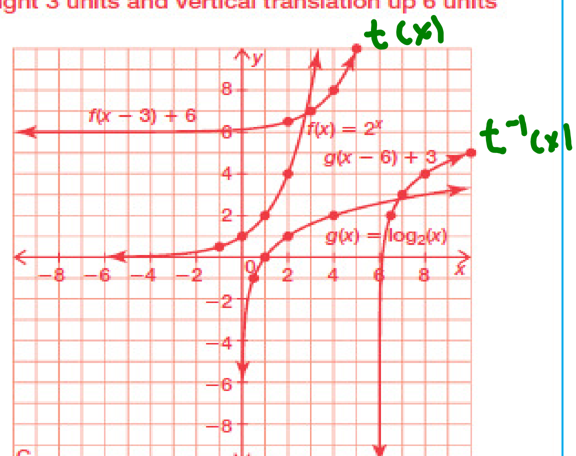
C. $f(x - 3) + 6$

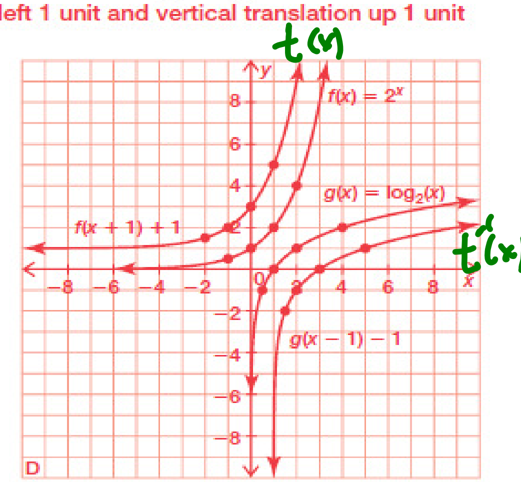
Transformation(s) on $f(x)$: horizontal translation right 3 units and vertical translation up 6 units



Transformation(s) on $g(x)$: horizontal translation left 6 units and vertical translation down 3 units

D. $f(x + 1) + 1$

Transformation(s) on $f(x)$: horizontal translation left 1 unit and vertical translation up 1 unit



Transformation(s) on $g(x)$: horizontal translation right 1 unit and vertical translation down 1 unit

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A. $f(x + 4)$ Transformation(s) on $f(x)$: Transformation(s) on $g(x)$:	**B.** $f(x) + 5$ Transformation(s) on $f(x)$: Transformation(s) on $g(x)$:
C. $f(x - 3) + 6$ Transformation(s) on $f(x)$: Transformation(s) on $g(x)$:	**D.** $f(x + 1) + 1$ Transformation(s) on $f(x)$: Transformation(s) on $g(x)$:

2. Generalize the effects of transformations on $f(x)$ and its inverse function, $f^{-1}(x)$. Complete the table to organize your results.

	Transformation on $f(x)$	Effect of Transformation on $f^{-1}(x)$
$f(x + C)$	Translate horizontally left C units	Translate vertically down C units
$f(x - C)$	Translate horizontally right C units	Translate vertically up C units
$f(x) + D$	Translate vertically up D units	Translate horizontally right D units
$f(x) - D$	Translate vertically down D units	Translate horizontally left D units

3. Consider the function $y = f(x)$ and the transformed function $g(x)$. Write an equation for $g^{-1}(x)$ in terms of $f^{-1}(x)$.

a. $y = f(x)$

$g(x) = f(x - 1)$

$g^{-1}(x) =$ $f^{-1}(x) + 1$

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b. $y = f(x)$

$g(x) = f(x) - 2$

$g^{-1}(x) =$ $f^{-1}(x + 2)$

c. $y = f(x)$

$g(x) = f(x + 5)$

$g^{-1}(x) =$ $f^{-1}(x) - 5$

d. $y = f(x)$

$g(x) = f(x - 4) + 3$

$g^{-1}(x) =$ $f^{-1}(x - 3) + 4$

Chapter 12 Summary

KEY TERMS

- half-life (12.1)
- natural base e (12.2)
- logarithm (12.4)
- logarithmic function (12.4)
- common logarithm (12.4)
- natural logarithm (12.4)

12.1 Constructing an Exponential Function from a Geometric Sequence

The general formula for a geometric sequence is $a_n = a_1 \cdot r^{(n-1)}$. This formula can be written as an exponential function by using properties of exponents and multiplication.

Example

$$a_n = 20 \cdot 5^{(n-1)}$$

$$f(n) = 20 \cdot 5^n \cdot 5^{-1}$$

$$f(n) = 20 \cdot \frac{1}{5} \cdot 5^n$$

$$f(n) = 4 \cdot 5^n$$

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12.1 Using an Exponential Function to Solve Half-Life Problems

Half-life refers to the amount of time it takes a substance to decay to half of its original amount. An exponential function can be used to solve problems about half-life.

Example

An exponential function $A(t)$ represents the amount of a drug in a person's system, where t represents elapsed time. If the half-life of the drug occurs in multiple-hour cycles—for example, every 3 hours—divide the exponent t by that amount: $\frac{t}{3}$.

Elapsed Time (Hours)	0	2	4	6	8	20
Drug in System (mg)	160	80	40	20	10	?
Number of Half-Life Cycles	0	1	2	3	4	10

$$A(t) = 160\left(\frac{1}{2}\right)^{\frac{t}{2}}$$

$$A(20) = 160\left(\frac{1}{2}\right)^{\frac{20}{2}}$$

$$A(20) = 160\left(\frac{1}{2}\right)^{10}$$

$$A(20) \approx 160(0.00098)$$

$$A(20) \approx 0.15625$$

After 20 hours, there will be about 0.15625 mg of the drug remaining in the person's system.

12.2 Investigating Exponential Growth and Decay

For exponential growth functions, b is a value greater than 1. For exponential decay functions, b is a fraction or decimal between 0 and 1.

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Example

$f(x) = 15^x$	$f(x) = \left(\frac{2}{3}\right)^x$
growth	decay

12.2 Investigating Graphs of Exponential Functions

Every basic exponential function has the point (0, 1) in common. The x-value represents the exponent, and any base raised to the power of 0 will equal 1. Basic functions of exponential growth and decay can be identified by their domain, range, asymptotes, and end behavior as described in the table below.

	Basic Exponential Growth	**Basic Exponential Decay**
Domain	$(-\infty, \infty)$	$(-\infty, \infty)$
Range	$(0, \infty)$	$(0, \infty)$
Asymptote	$y = 0$	$y = 0$
Intercepts	(0, 1)	(0, 1)
End Behavior	As $x \to -\infty$, $f(x) \to 0$ As $x \to \infty$, $f(x) \to \infty$	As $x \to -\infty$, $f(x) \to \infty$ As $x \to \infty$, $f(x) \to 0$
Intervals of Increase or Decrease	Increasing over $(-\infty, \infty)$	Decreasing over $(-\infty, \infty)$

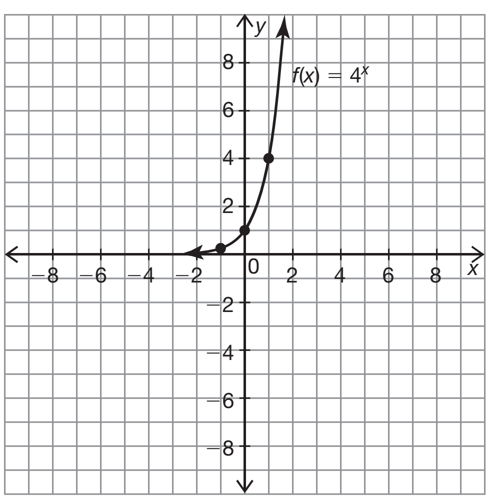
Examples

$f(x) = 4^x$

x	$f(x)$
-1	$\frac{1}{4}$
0	1
1	4

Increasing over $(-\infty, \infty)$.

End behavior: As $x \to -\infty$, $f(x) \to 0$
As $x \to \infty$, $f(x) \to \infty$

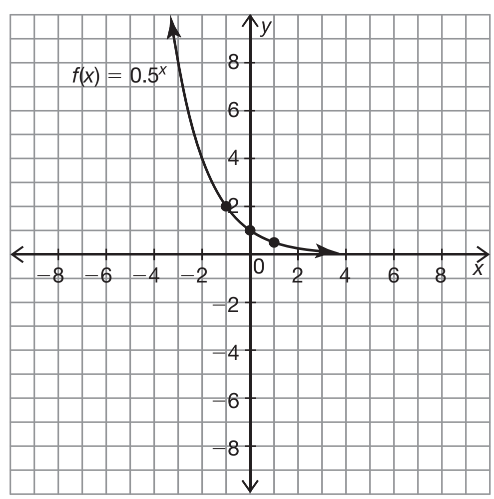


$f(x) = 0.5^x$

x	$f(x)$
-1	2
0	1
1	0.5

Decreasing over $(-\infty, \infty)$.

End behavior: As $x \to -\infty$, $f(x) \to \infty$
As $x \to \infty$, $f(x) \to 0$



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12.2 Using Exponential Equations to Solve Compound Interest Problems

The formula for compound interest is $A = P(1 + r)^t$, where A is the amount earned, P is the original amount, or principal, r is the rate, and t is the time in years. If interest is compounded more than once per year, then the formula is: $A(t) = \left(1 + \frac{1}{n}\right)^{n \cdot t}$.

Example

Sarah invests $500 in the bank. Her bank compounds interest 4 times a year at a rate of 4%. How much money will she have in her account after 10 years?

$$A(10) = 500\left(1 + \frac{0.04}{4}\right)^{4 \cdot 10}$$
$$= 500(1.01)^{40}$$
$$\approx 500(1.4889)$$
$$\approx 744.43$$

Sarah will have $744.43 in her account after 10 years.

12.2 Using the Natural Base, *e*

The constant e represents continuous growth and is often referred to as the natural base e. The symbol e is used to represent the constant 2.718281. . . and so on. The natural base e is used in the formula for population growth: $N(t) = N_0e^{rt}$.

Example

Miami's population in 2005 was 216,500 people and is growing at a rate of about 3%. According to the population growth model, what would be the approximate population of Miami in 2020?

$$N(15) = 216{,}500e^{(0.03 \cdot 15)}$$
$$= 216{,}500e^{0.45}$$
$$\approx 339{,}540$$

Miami's population in 2020 would be approximately 339,540 people.

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12.3 Dilating, Reflecting, and Translating Exponential Functions Using Reference Points

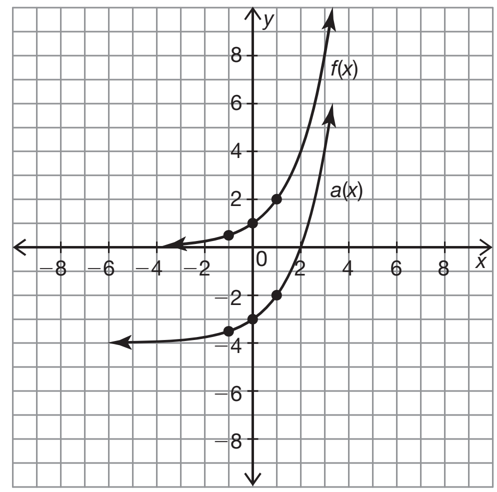
Consider the functions $y = f(x)$ and $g(x) = Af(B(x - C)) + D$. Recall that the D-value translates $f(x)$ vertically, the C-value translates $f(x)$ horizontally, the A-value vertically stretches or compresses $f(x)$, and the B-value horizontally stretches or compresses $f(x)$. Exponential functions are transformed in the same manner.

Example

$f(x) = 2^x$

$a(x) = f(x) - 4$

Reference Points on $f(x)$	Corresponding Points on $a(x)$
$\left(-1, \frac{1}{2}\right)$	$\left(-1, -\frac{7}{2}\right)$
(0, 1)	(0, −3)
(1, 2)	(1, −2)



Domain: All real numbers

Range: $y > -4$

Horizontal asymptote: $y = -4$

12.3 Describing Transformations Performed on Exponential Equations

Using the functions $y = f(x)$ and $g(x) = Af(B(x - C)) + D$, you can describe the transformations to the graph of an exponential function.

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Example

$m(x) = 4^x$

$t(x) = -m(x) + 3$

The graph of the function $m(x)$ is translated vertically up 3 units and is reflected across the x-axis to produce $t(x)$.

$t(x) = -4^x + 3$

12.4 Writing Exponential Equations as Logarithmic Equations

The inverse of an exponential equation can be written as a logarithmic equation. The logarithm of a number for a given base is the exponent to which the base must be raised in order to produce the number. If $y = b^x$, then the logarithm is x, and can be written as $x = \log_b y$. The value of the base of a logarithm is the same as the base in the exponential expression b^x.

Examples

$4^3 = 64$ $\qquad$ $\log_5\left(\frac{1}{625}\right) = -4$

$\log_4(64) = 3$ $\qquad$ $5^{-4} = \frac{1}{625}$

12.4 Graphing the Inverse of an Exponential Function

A logarithmic function is the inverse of an exponential function. It is a function involving a logarithm. Many real-world situations can be modeled using logarithmic functions. Two frequently used logarithms are logarithms with base 10 and base *e*. A logarithm with base 10 is called the common logarithm and is usually written *log* without a base specified. A logarithm with base *e* is called the natural logarithm and is usually written as *ln*.

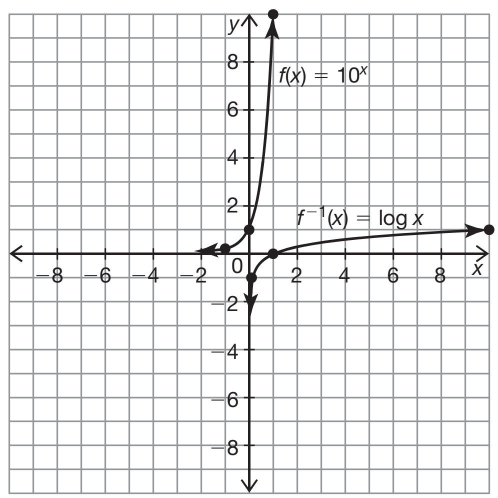
Example

$f(x) = 10^x$

$f^{-1}(x) = \log x$

x	$f(x)$
−1	0.1
0	1
1	10

x	$f^{-1}(x)$
0.1	−1
1	0
10	1



12.5 Describing Transformations Performed on Logarithmic Functions

Using the functions $y = f(x)$ and $g(x) = Af(B(x - C)) + D$, you can describe the transformations to a logarithmic function.

Example

$f(x) = \log_4(x)$

$g(x) = -f(x + 3)$

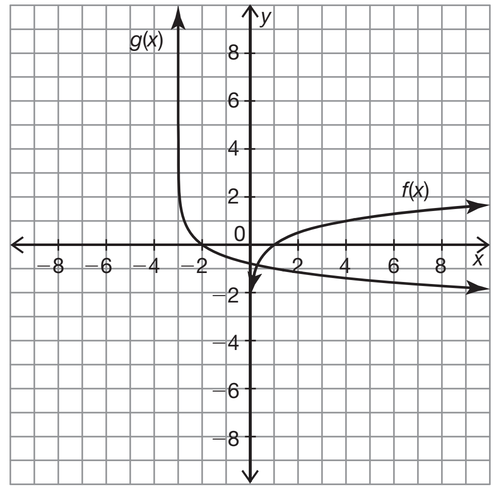
$g(x) = -\log_4(x + 3)$

horizontally translated left 3 units and reflected across the x-axis

Domain of $g(x)$: $(-3, \infty)$

Range of $g(x)$: $(-\infty, \infty)$

Asymptote of $g(x)$: $x = -3$



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12.5 Describing the Effects of Transformations on Inverse Functions

Transformations performed on a function and its inverse will have inverse effects.

	Transformation on $f(x)$	Effect of Transformation on $f^{-1}(x)$
$f(x + C)$	Translate horizontally left C units	Translate vertically down C units
$f(x - C)$	Translate horizontally right C units	Translate vertically up C units
$f(x) + D$	Translate vertically up D units	Translate horizontally right D units
$f(x) - D$	Translate vertically down D units	Translate horizontally left D units
$Af(x)$	Vertical dilation of 3	Horizontal dilation of 3
$f(Bx)$	Horizontal dilation of $\frac{1}{B}$	Vertical dilation of $\frac{1}{B}$

Example

Consider the transformation on the function $f(x) = 2^x$ and its inverse function $f^{-1}(x) = \log_2(x)$.

$m(x) = \frac{1}{3}f(x)$	$m^{-1}(x) = f^{-1}(3x)$
$\left(-1, \frac{1}{6}\right)$	$\left(\frac{1}{6}, -1\right)$
$\left(0, \frac{1}{3}\right)$	$\left(\frac{1}{3}, 0\right)$
$\left(1, \frac{2}{3}\right)$	$\left(\frac{2}{3}, 1\right)$

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Transformation on $f(x)$: vertical dilation of $\frac{1}{3}$

Effect on the inverse: horizontal dilation of $\frac{1}{3}$