13.2

Mad Props

Properties of Logarithms

LEARNING GOALS

In this lesson, you will:

- Derive the properties of logarithms.
- Expand logarithmic expressions using the properties of logarithms.
- Rewrite multiple logarithmic expressions as a single logarithmic expression.

KEY TERMS

- Zero Property of Logarithms
- Logarithm with Same Base and Argument
- Product Rule of Logarithms
- Quotient Rule of Logarithms
- Power Rule of Logarithms

Dimitri Mendeleev is best known for his work on the periodic table—arranging the 63 known elements into a periodic table based on atomic mass, which he published in *Principles of Chemistry* in 1869. His first periodic table was compiled on the basis of arranging the elements in ascending order of atomic weight and grouping them by similarity of properties. He predicted the existence and properties of new elements and pointed out accepted atomic weights that were in error. His table did not include any of the noble gases, however, which had not yet been discovered. Dmitri Mendeleev revolutionized our understanding of the properties of atoms and created a table that probably embellishes every chemistry classroom in the world.

Warm Up

Identify the property of exponents associated with each example.

1. $\left(\frac{2}{3}\right)^5 \cdot \left(\frac{2}{3}\right)^2 = \left(\frac{2}{3}\right)^7$ The Product Rule of Exponents

2. $\left(\frac{2}{3}\right)^1 = \left(\frac{2}{3}\right)$ Any number raised to the first power is equal to its base number.

3. $\left(\frac{2}{3}\right)^0 = 1$ Any number raised to the zero power is equal to one.

4. $\left(\frac{2^{14}}{2^5}\right) = 2^9$ The Quotient Rule of Exponents

Evaluate the following expressions using the Laws of Exponents.

5. $x^3 \cdot x^7 =$ x^{10}

6. $2^x \cdot 2^{7x} =$ 2^{8x}

7. $\frac{n^{18}}{n^6} =$ n^{12}

8. $\frac{2^{3a}}{2^a} =$ 2^{2a}

9. $(b^3)^5 =$ b^{15}

10. $\left(\frac{3a^5}{2b^3}\right)^4 =$ $\frac{81a^{20}}{16b^{12}}$

PROBLEM 1 Setting Ground Rules

Logarithms by definition are exponents, so they have properties that are similar to those of exponents and powers. In this lesson, you will develop logarithmic rules and properties that correspond to various exponential rules and properties you already know.

1. Let's consider the Zero Property of Powers to develop a corresponding logarithmic property.

   a. Write a sentence to summarize the Zero Property of Powers, $b^0 = 1$.

   Any base raised to the zero power is 1.

   b. Write the Zero Property of Powers in logarithmic form. This is a corresponding logarithmic property called the *Zero Property of Logarithms*.

   $\log_b(1) = 0$

   c. State the Zero Property of Logarithms in words.

   The logarithm of 1, with any base, is always equal to 0.

2. Let's consider the exponent rule that says that any number raised to the first power is equal to the base.

   a. Write an exponential equation to represent this rule. Use b as the base.

   $b^1 = b$

   b. Write your exponential equation from part (a) in logarithmic form.

   $\log_b(b) = 1$

   c. State this logarithmic relationship in words.

   When the base and argument are equal, the logarithm is always equal to 1.

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$2^x = 8$

BASE is 2

EXPONENT is x (input)

POWER is 8 (output)

$\log_2 x = 16$

BASE is 2

EXPONENT is 16 (output)

POWER is x (input)

Exponent Rule: *When multiplying powers with the same base, add the exponents* $x^3 \cdot x^4 = x^7$ $5^x \cdot 5^2 = 5^{x+2}$ $m^{10} = m^5 \cdot m^5$ $4^{x-3} = 4^x \cdot 4^{-3}$	**Logarithm Rule**: *When adding logarithms* (exponents) *with the same base,* MULTIPLY THE POWERS! $\log_3 6 + \log_3 5 = \log_3 30$ $\ln x + \ln 2 = \ln(2x)$ $\log 14 = \log 2 + \log 7$ $\log_9(xy) = \log_9 x + \log_9 y$
Exponent Rule: *When dividing powers with the same base, subtract the exponents* $\frac{x^{15}}{x^{10}} = x^5$ $\frac{3^x}{3^4} = 3^{x-4}$ $y^3 = \frac{y^8}{y^5}$ $5^{2-n} = \frac{5^2}{5^n} = \frac{25}{5^n}$	**Logarithm Rule:** *When subtracting logarithms* (exponents) *with the same base,* DIVIDE THE POWERS $\log 24 - \log 6 = \log 4$ $\log_3 8 - \log_3 7 = \log_3\left(\frac{8}{7}\right)$ $\ln\left(\frac{x}{5}\right) = \ln x - \ln 5$ $\log_5\left(\frac{4x}{9}\right) = \log_5(4x) - \log_5(9)$ $= \log_5 4 + \log_5 x - \log_5 9$
Exponent Rule: *Multiplying an exponent by another constant is the same as raising the power to that constant* $(x^3)^4 = x^{3\cdot 4} = x^{12}$ $(4^x)^5 = 4^{5x}$ $6^{8x} = (6^8)^x = (6^x)^8$ $\sqrt{x^5} = (x^5)^{\frac{1}{2}} = (x^{\frac{1}{2}})^5 = x^{\frac{5}{2}}$	**Logarithm Rule:** *When multiplying a logarithm by a constant, that constant is an exponent that may be applied to the power (argument of the log function)* $4 \cdot \log x = \log x^4$ $3\log_4 2 = \log_4 2^3 = \log_4 8$ $\ln 25 = \ln 5^2 = 2\ln 5$ $\log \sqrt[3]{x} = \log x^{\frac{1}{3}} = \frac{1}{3}\log x$

Change of base formula for logarithms:

Change of base formula for logarithms:

base b = change to base c

$\log_b x = \dfrac{\log_c x}{\log_c b}$

Consider $\boxed{\log_9 16}$.

Change to base 7: $\dfrac{\log_7 16}{\log_7 9}$

Change to COMMON LOG: $\dfrac{\log 16}{\log 9}$

Change to NATURAL LOG: $\dfrac{\ln 16}{\ln 9}$

$\log_b 1 = 0$ $\quad$ $\log_b b = 1$ $\quad$ $\log_b\left(\frac{1}{b}\right) = -1$

* Assuming x and y are positive, write each expression as a sum/ difference of logarithms.

1.) $\log(xy^7)$

$= \log x + \log y^7$

$= \log x + 7\log y$

2.) $\ln\left(\frac{9}{y}\right)$

$= \ln 9 - \ln y$

$= \ln 3^2 - \ln y$

$= 2\ln 3 - \ln y$

3.) $\log_4\left(\frac{16\sqrt{x}}{3y^5}\right)$

$= \log_4(16\sqrt{x}) - \log_4(3y^5)$

$= \log_4 16 + \log_4 x^{1/2} - (\log_4 3 + \log_4 y^5)$

$= 2 + \frac{1}{2}\log_4 x - \log_4 3 - 5\log_4 y$

* Assuming x, y, and z are positive, write each expression as a single logarithm.

4.) $3\ln(x^4y^2) - 2\ln(yz^6)$

$= \ln(x^4y^2)^3 - \ln(yz^6)^2$

$= \ln(x^{12}y^6) - \ln(y^2z^{12})$

$= \ln\left(\frac{x^{12}y^6}{y^2z^{12}}\right)$

$= \ln\left(\frac{x^{12}y^4}{z^{12}}\right)$

5.) $\frac{1}{3}\log(8) + 2\log(6x) - \frac{1}{2}\log(4y^{16}) - \log 1000$

$= \log 8^{1/3} + \log(6x)^2 - \log(4y^{16})^{\frac{1}{2}} - 3$

$= \log 2 + \log(36x^2) - \log(2y^8) - 3$

$= \log(72x^2) - \log(2y^8) - 3$

$= \log\left(\frac{72x^2}{2y^8}\right) - 3$

$= \log\left(\frac{36x^2}{y^8}\right) - 3$

6. In this problem, you derived different properties of logarithms. Complete the tables to define each exponential and logarithmic property verbally and symbolically. Provide examples for each property.

Exponential Property	Logarithmic Property
Zero Property of Powers	**Zero Property of Logarithms**
Verbal: Any base raised to the zero power is 1.	**Verbal:** The logarithm of 1, with any base, is always equal to 0.
Symbolic: $b^0 = 1$	**Symbolic:** $\log_b(1) = 0$
Examples: $m^0 = 1$; $10^0 = 1$	**Examples:** $\log_a(1) = 0$; $\log_7(1) = 0$
Base Raised to First Power	**Logarithm with Same Base and Argument**
Verbal: Any base raised to the first power is the base itself.	**Verbal:** The logarithm of a number, with the base equal to the same number, is always equal to 1.
Symbolic: $b^1 = b$	**Symbolic:** $\log_b(b) = 1$
Examples: $p^1 = p$; $6^1 = 6$	**Examples:** $\log_d(d) = 1$; $\log_2(2) = 1$
Product Rule of Powers	**Product Rule of Logarithms**
Verbal: To multiply powers with the same base, you add the exponents.	**Verbal:** The logarithm of a product is equal to the sum of the logarithms of the factors.
Symbolic: $b^m \cdot b^n = b^{m+n}$	**Symbolic:** $\log_b(xy) = \log_b(x) + \log_b(y)$
Examples: $z^4 \cdot z^7 = z^{11}$; $3^m \cdot 3^n \cdot 3^p = 3^{m+n+p}$	**Examples:** $\log_a(mnp) = \log_a(m) + \log_a(n) + \log_a(p)$; $\log_2(50) = \log_2(5) + \log_2(10)$

Exponential Property	Logarithmic Property
Quotient Rule of Powers	**Quotient Rule of Logarithms**
Verbal: To divide powers with the same base, you subtract the exponents.	**Verbal:** The logarithm of a quotient is equal to the difference of the logarithms of the dividend and the divisor.
Symbolic: $\frac{b^m}{b^n} = b^{m-n}$, if $n \neq 0$	**Symbolic:** $\log_b \left(\frac{x}{y}\right) = \log_b (x) - \log_b (y)$
Examples: $\frac{r^8}{r^5} = r^3$ $\frac{5^d}{5^f} = 5^{d-f}$	**Examples:** $\log_a \left(\frac{m}{n}\right) = \log_a (m) - \log_a (n)$ $\log_2 (5) = \log_2 (45) - \log_2 (9)$
Power to a Power Rule	**Power Rule of Logarithms**
Verbal: To simplify a power to a power, keep the base and multiply the exponents.	**Verbal:** The logarithm of a power is equal to the product of the exponent and the logarithm of the base of the power.
Symbolic: $(b^m)^n = b^{mn}$	**Symbolic:** $\log_b (x^n) = n \cdot \log_b (x)$
Examples: $(a^3)^4 = b^{12}$ $(7^w)^v = 7^{wv}$	**Examples:** $\log_a (r^s) = s \log_a (r)$ $\log_2 (27) = 3 \log_2 (3)$

PROBLEM 2 Don't Break the Rules!

1. Use the properties of logarithms to rewrite each logarithmic expression in expanded form.

   a. $\log_4 (6x^5)$

   $\log_4 (6x^5) = \log_4 (6) + 5 \log_4 (x)$

   b. $\log_7 \left(\frac{3y^4}{x^3}\right)$

   $\log_7 \left(\frac{3y^4}{x^3}\right) = \log_7 (3) + 4 \log_7 (y) - 3 \log_7 (x)$

   c. $\ln (3xy^3)$

   $\ln (3xy^3) = \ln (3) + \ln (x) + 3 \ln (y)$

2. Use the properties of logarithms to rewrite each logarithmic expression as a single logarithm.

   a. $\log_2 (10) + 3 \log_2 (x)$

   $\log_2 (10) + 3 \log_2 (x) = \log_2 (10x^3)$

   b. $4 \log (12) - 4 \log (2)$

   $4 \log (12) - 4 \log (2) = \log \left(\frac{12^4}{2^4}\right) = \log (1296) \approx 3.1126$

   c. $3(\ln 3 - \ln x) + (\ln x - \ln 9)$

   $3(\ln 3 - \ln x) + (\ln x - \ln 9) = \ln \left(\frac{3}{x}\right)^3 \left(\frac{x}{9}\right) = \ln \left(\frac{3}{x^2}\right)$

3. Suppose $\log_a (5) = p$, $\log_a (3) = q$, and $\log_a (2) = r$. Write an algebraic expression for each logarithmic expression.

   a. $\log_a (50)$

   $\log_a (50) = \log_a (5^2 \cdot 2)$
   $= 2 \log_a (5) + \log_a (2)$
   $= 2p + r$

   b. $\log_a (0.3)$

   $\log_a (0.3) = \log_a \left(\frac{3}{10}\right)$
   $= \log_a (3) - (\log_a (2) + \log_a (5))$
   $= q - r - p$

   c. $\log_a \left(\frac{1}{27}\right)$

   $\log_a \left(\frac{1}{27}\right) = \log_a \left(\frac{1}{3^3}\right)$
   $= \log_a (1) - 3 \log_a (3)$
   $= -3q$

Be prepared to share your solutions and methods.