What's Your Strategy?
Solving Exponential Equations

13.3

LEARNING GOALS

In this lesson, you will:

- Solve exponential equations using the Change of Base Formula.
- Solve exponential equations by taking the log of both sides.
- Analyze different solution strategies to solve exponential equations.

PROBLEM 1 Don't Burst My Bubble

The newest online game is Bubblez Burst, a highly addictive game that runs on social media. On the first day of its release, 50 people subscribe. The creators estimate that everyone who subscribes will then send 3 more people to subscribe.

1. Write a function using the creator's estimate to model the total number of subscribers P who will be playing Bubblez Burst after t number of days.

$P(t) = 50 \cdot 3^t$

2. How many days will it take for Bubblez Burst to have 4050 subscribers?

It will take 3 days for Bubblez Burst to have 4050 subscribers.

$P(t) = 50 \cdot 3^t$

$4050 = 50 \cdot 3^t$

$81 = 3^t$

$3^4 = 3^t$

$4 = t$

3. How many days will it take for Bubblez Burst to reach 20,000 subscribers?

It will take approximately 5.4 days for Bubblez Burst to reach 20,000 subscribers.

$P(t) = 50 \cdot 3^t$

$20{,}000 = 50 \cdot 3^t$

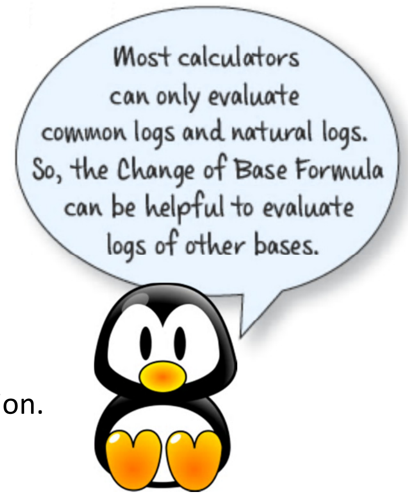
$400 = 3^t$

I know that $3^5 = 243$ and $3^6 = 729$, so $5 < t < 6$. I would estimate t to be about 5.4.

4. How did your methods in Question 2 and Question 3 differ?

In Question 2, I was able to use common bases to solve for t because 81 is a power of 3. In Question 3, I had to use estimation because 300 is not a power of 3.

So far, you have used estimation to determine the value of logarithms whose values were not integers. The *Change of Base Formula* allows you to calculate an exact value for a logarithm by rewriting it in terms of a different base. First, you will use the Change of Base Formula and then you will derive it.



The **Change of Base Formula** states:

$\log_b(c) = \frac{\log_a(c)}{\log_a(b)}$, where $a, b, c > 0$ and $a, b \neq 1$.

5. Rewrite the exponential equation you wrote in #3 as a logarithmic equation.

$400 = 3^t$

$\log_3(400) = t$

6. Use the Change of Base Formula to evaluate the logarithmic expression using common logs. Round to four decimal places.

$\log_3(400) = x$

$\frac{\log 400}{\log 3} = x$

$5.454 \approx x$

The calculated value of $\log_3(400) \approx 5.454$ is more accurate.

The estimated value of $\log_3(400)$ was 5.4, which yields 377.10 when checked in the equivalent exponential equation.

$3^{5.4} \stackrel{?}{=} 400$

$377.10 \approx 400$

The calculated value for $\log_3(400)$ was approximately 5.454, which yields 400.15, or approximately 400, when checked in the equivalent exponential equation.

$3^{5.454} \stackrel{?}{=} 400$

$400.15 \approx 400$

****7.** Compare your estimate in #3 with the calculated value in #6 by substituting each value back into the original equation. What do you notice?

8. Use a calculator to determine how many days it will take for Bubblez Burst to reach one million subscribers.

It will take approximately 9 days to reach one million subscribers.

$P = 50 \cdot 3^x$

$1{,}000{,}000 = 50 \cdot 3^x$

$20{,}000 = 3^x$

$\log_3(20{,}000) = x$

$\frac{\log 20{,}000}{\log 3} = x$

$9.0145428 \approx x$

9. Tammy was asked to approximate how many days it would take Bubblez Burst to reach 30,000 subscribers. Describe the calculation error Tammy made. Then, use your knowledge of estimation to explain why x could not equal 2.3.

Tammy

$30{,}000 = 50 \cdot 3^x$

$600 = 3^x$

$\log_3(600) = x$

$\frac{\log 600}{\log 3} = x$

~~$\log 200 = x$~~

5.823 $\quad 2.3 \approx x$

Tammy's reasoning was incorrect because she performed the order of operations incorrectly. She divided the arguments before taking the common log, instead of taking the common log of each argument first.

Tammy's solution does not make sense. The exponent $x \approx 2.3$ would produce an argument between 9 and 27 because $3^2 = 9$ and $3^3 = 27$, but the argument of 600 does not fall between 9 and 27.

I know the logarithm should be between 5 and 6 because $3^5 = 243$ and $3^6 = 729$, and the argument 600 is between 243 and 729.

10. In 2012, there were approximately 314 million people in the United State. In that year, how long would it take for everyone in the country to subscribe to E

It will take approximately 14.2 days for everyone in the United States to subscribe to Bubblez Burst.

$P = 50 \cdot 3^x$

$314{,}000{,}000 = 50 \cdot 3^x$

$6{,}280{,}000 = 3^x$

$\log_3(6{,}280{,}000) = x$

$\frac{\log 6{,}280{,}000}{\log 3} = x$

$14.24786588 \approx x$

Solve each exponential equation.

11) $2^x = 64$

$2^x = 2^6$

$x = 6$

12) $y^3 = 125$

$y^3 = 5^3$

$y = 5$

Solve each logarithmic equation.

13) $log_3(x) = log_3(20)$

$w = 20$

14) $log_m(9) = log_4(9)$

$m = 4$

**Explain the strategy used to solve each exponential and logarithmic equation above.

For $\log_3(w) = \log_3(20)$, because the bases are equal and the logarithms are equal, I know that the arguments must also be equal.

For $\log_m(9) = \log_4(9)$, because the logarithms are equal and the arguments are equal, I know that the bases must also be equal.

You just derived the relationship that if $\log_b(a) = \log_b(c)$, then $a = c$. The converse is also true. If $a = c$, then $\log_b(a) = \log_b(c)$. You can use this knowledge to now derive the Change of Base Formula.

15) Todd and Danielle each solved the exponential equation $4^{x-1} = 50$.

Todd solved for x by first rewriting the equation as a logarithmic equation. Then he used the Change of Base Formula.

Danielle solved for x by first taking the log of both sides. Then she used the properties of logarithms to solve.

Describe how Todd's and Danielle's methods are different....

Why do both methods work?

Todd

$4^{x-1} = 50$

$x - 1 = \log_4(50)$

$x - 1 = \dfrac{\log 50}{\log 4}$

$x - 1 \approx 2.822$

$x \approx 3.822$

Danielle

$4^{x-1} = 50$

$\log(4^{x-1}) = \log 50$

$(x-1)\log 4 = \log 50$

$x - 1 = \dfrac{\log 50}{\log 4}$

$x = \dfrac{\log 50}{\log 4} + 1$

$x \approx 3.822$

Solve the following:

$5^{4n+5} = 5^{n-7}$

4n + 5 = n - 7

3n = -12

n = -4

$2 \cdot 3^{5x} + 1 = 55$

$2 \cdot 3^{5x} = 54$

$3^{5x} = 27$

$3^{5x} = 3^3$

x = 3/5

$4^{3x} = 8^{x-1}$

$2^{2(3x)} = 2^{3(x-1)}$

2(3x) = 3(x - 1)

6x = 3x - 3

3x = -3

x = -1

$8^{m-7} = 92$

$\log_8 92 = m - 7$

$\frac{\log 92}{\log 8} = m - 7$

2.1745 = m - 7

9.1745 = m

$12^{x+4} = 65$

$\log_{12} 65 = x + 4$

1.6799 = x + 4

-2.3201 = m

What method did you choose?

16) Consider the exponential equation $a^x = c$, where x is the unknown in the exponent and a and c are constants.

a. Solve the exponential equation for x by rewriting it in logarithmic form.

$$x = \log_a (c)$$

b. Solve the exponential equation for x by taking the log of both sides.

$$\log (a^x) = \log c$$

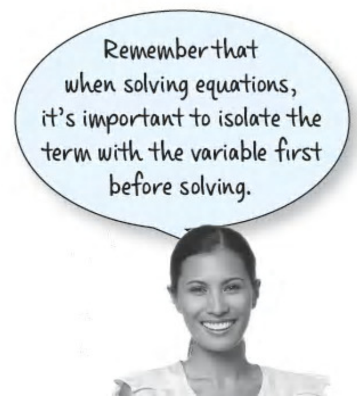
$$x \log a = \log c$$

$$x = \frac{\log c}{\log a}$$

Both methods yield the same result.

I know that $x = \log_a (c)$ and $x = \frac{\log c}{\log a}$, therefore $\log_a (c) = \frac{\log c}{\log a}$.

17) Solve the exponential equation $8^x = 38.96$ using both Todd's and Danielle's methods. Round to the nearest thousandth and check your work.



$8^x = 38.96$

$\log (8^x) = \log 38.96$

$x \log 8 = \log 38.96$

$x = \frac{\log 38.96}{\log 8}$

$x \approx 1.761$

$8^x = 38.96$

$x = \log_8 (38.96)$

$x = \frac{\log 38.96}{\log 8}$

$x \approx 1.761$

Check: $8^{1.761} \stackrel{?}{=} 38.96$

$38.94 \approx 38.96$

The first few, I tried to get a common base and then set exponents equal to solve for x. If I could not get the bases the same then I moved equation into log form and used change of base.

18) John, Bobbi, and Randy each solved the equation $9^{x+4} = 27$.

Randy

$$9^{x+4} = 27$$
$$(3^2)^{x+4} = 3^3$$
$$3^{2(x+4)} = 3^3$$
$$2(x+4) = 3$$
$$2x + 8 = 3$$
$$2x = -5$$
$$x = -2.5$$

John

$$9^{x+4} = 27$$
$$x + 4 = \log_9(27)$$
$$x + 4 = \frac{\log 27}{\log 9}$$
$$x + 4 = 1.5$$
$$x = -2.5$$

Bobbi

$$9^{x+4} = 27$$
$$\log(9^{x+4}) = \log 27$$
$$(x+4)\log 9 = \log 27$$
$$x + 4 = \frac{\log 27}{\log 9}$$
$$x + 4 = 1.5$$
$$x = -2.5$$

a. Describe each method used.

John converted the exponential equation into a logarithmic equation and then used the Change of Base Formula.

Bobbi solved the equation by taking the log of both sides.

Randy solved the equation by using the idea of equivalent bases and setting the exponents equal to each other.

b. Will each method work for every logarithmic equation? Describe any limitations of each method.

John's method and Bobbi's method will each work for every exponential equation.

Randy's method will work only if the bases are equal.

19) Ameet and Neha each took the logarithm of both sides to solve $24^x = 5$.

Ameet

$$24^x = 5$$
$$\ln(24^x) = \ln 5$$
$$x \ln 24 = \ln 5$$
$$x = \frac{\ln 5}{\ln 24}$$
$$x \approx 0.506$$

Check:

$$24^x \stackrel{?}{=} 5$$
$$24^{0.506} \stackrel{?}{=} 5$$
$$4.99 \approx 5$$

Neha

$$24^x = 5$$
$$\log(24^x) = \log 5$$
$$x \log 24 = \log 5$$
$$x = \frac{\log 5}{\log 24}$$
$$x \approx 0.506$$

Check:

$$24^x \stackrel{?}{=} 5$$
$$24^{0.506} \stackrel{?}{=} 5$$
$$4.99 \approx 5$$

Describe the similarities and differences in their methods. Explain why each student's method is correct.

Ameet and Neha's methods are similar in that they both rewrote $24^x = 5$ as a logarithmic equation with a base which can be evaluated using a calculator.

Their methods are different in that Ameet wrote an equivalent logarithmic expression with base e, whereas Neha wrote an equivalent logarithmic expression with base 10.

Each student's method is correct because they evaluated $\log_{24} 5$ using the Change of Base Formula and determined an approximate value of 0.506.

20) Describe the error in Ashley's reasoning. Then solve the equation correctly.

Ashley

$$9\left(\frac{1}{3}\right)^{2x} = 15$$
$$3^{2x} = 15$$
$$\log(3^{2x}) = \log 15$$
$$2x \log 3 = \log 15$$
$$2x = \frac{\log 15}{\log 3}$$
$$2x = 2.464973521\ldots$$
$$x \approx 1.232$$

Ashley did not perform the order of operations correctly and multiplied 9 and $\frac{1}{3}$ together before working with the exponent on $\frac{1}{3}$.

$$3^2\left(\frac{1}{3}\right)^{2x} = 15$$
$$3^2(3)^{-2x} = 15$$
$$3^{2-2x} = 15$$
$$2 - 2x = \log_3(15)$$
$$-2x = \log_3(15) - 2$$
$$x = \frac{\frac{\log 15}{\log 3} - 2}{-2}$$
$$x \approx -0.2325$$

Choose a method to solve the following equations. Remember to isolate the exponential term first.

21) $11^{x-4} + 8 = 59$

$11^{x-4} = 51$

$\log_{11} 51 = x - 4$

$5.6397 = x$

22) $9 \cdot \left(\frac{3}{5}\right)^{2x} = 999$

$\left(\frac{3}{5}\right)^{2x} = 111$

$\log_{\frac{3}{5}} 111 = 2x$

$-9.2194 = 2x$

$-4.6097 = x$

23) $-8(2)^{x-9} - 5 = -77$

$-8(2)^{x-9} = -72$

$2^{x-9} = 9$

$\log_2 9 = x - 9$

$3.1699 = x - 9$

$12.1699 = x$

24) $2(5)^{2x+1} + 4 = 18$

$2(5)^{2x+1} + 4 = 18$

$2(5)^{2x+1} = 14$

$5^{2x+1} = 7$

$\log_5 7 = 2x + 1$

$1.2091 = 2x + 1$

$.2091 = 2x$

25) $4^{x-3} - 5 = 16$

$$4^{x-3} = 21$$
$$x - 3 = \log_4(21)$$
$$x - 3 = \frac{\log 21}{\log 4}$$
$$x = \frac{\log 21}{\log 4} + 3$$
$$x \approx 5.20$$

I used the Change of Base Formula because 21 cannot be written as a power of 4.

26) $10 \cdot \frac{3}{2}^{2x} = 360$

$$\left(\frac{3}{2}\right)^{2x} = 36$$
$$2x = \log_{\frac{3}{2}}(36)$$
$$2x = \frac{\log 36}{\log\left(\frac{3}{2}\right)}$$
$$x = \left(\frac{1}{2}\right)\frac{\log 36}{\log\left(\frac{3}{2}\right)}$$
$$x \approx 4.42$$

I took the log of both sides because 36 cannot be written as a power of $\frac{3}{2}$.