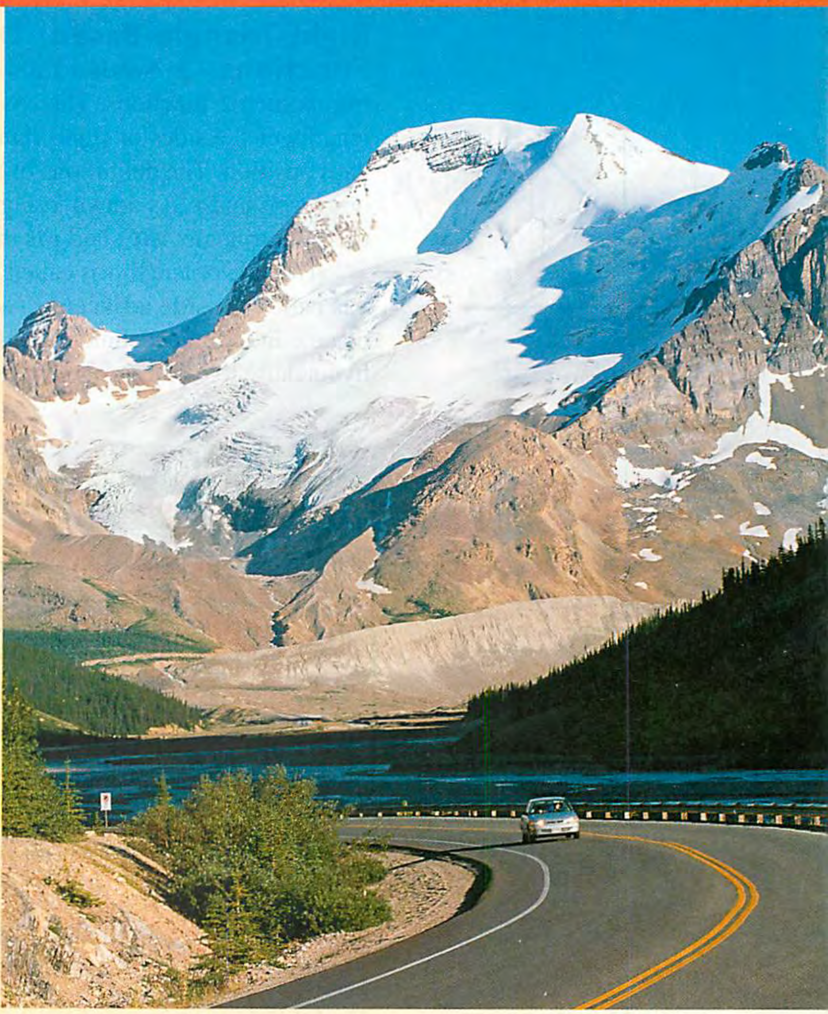
2 Acute Angles and Right Triangles



Highway transportation is critical to the economy of the United States. In 1970 there were 1150 billion miles traveled, and by the year 2000 this increased to approximately 2500 billion miles. When an automobile travels around a curve, objects like trees, buildings, and fences situated on the curve may obstruct a driver's vision. Trigonometry is used to determine how far inside the curve land must be cleared to provide visibility for a safe stopping distance. (*Source:* Mannering, F. and W. Kilareski, *Principles of Highway Engineering and Traffic Analysis,* Second Edition, John Wiley and Sons, 1998.)

A problem like this is presented in Exercise 39 of Section 2.5.

2.1 Trigonometric Functions of Acute Angles

Right-Triangle-Based Definitions of the Trigonometric Functions ▪ Cofunctions ▪ Trigonometric Function Values of Special Angles

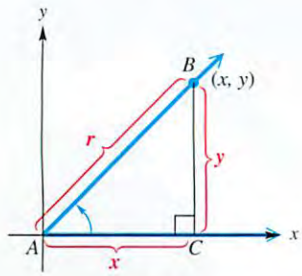


Figure 1

Right-Triangle-Based Definitions of the Trigonometric Functions In **Section 1.3** we used angles in standard position to define the trigonometric functions. There is another way to approach them: as ratios of the lengths of the sides of right triangles. Figure 1 shows an acute angle A in standard position. The definitions of the trigonometric function values of angle A require x, y, and r. As drawn in Figure 1, x and y are the lengths of the two legs of the right triangle ABC, and r is the length of the hypotenuse.

The side of length y is called the **side opposite** angle A, and the side of length x is called the **side adjacent** to angle A. We use the lengths of these sides to replace x and y in the definitions of the trigonometric functions, and the length of the hypotenuse to replace r, to get the following right-triangle-based definitions.

RIGHT-TRIANGLE-BASED DEFINITIONS OF TRIGONOMETRIC FUNCTIONS

For any acute angle A in standard position,

$$\sin A = \frac{y}{r} = \frac{\text{side opposite}}{\text{hypotenuse}} \qquad \csc A = \frac{r}{y} = \frac{\text{hypotenuse}}{\text{side opposite}}$$

$$\cos A = \frac{x}{r} = \frac{\text{side adjacent}}{\text{hypotenuse}} \qquad \sec A = \frac{r}{x} = \frac{\text{hypotenuse}}{\text{side adjacent}}$$

$$\tan A = \frac{y}{x} = \frac{\text{side opposite}}{\text{side adjacent}} \qquad \cot A = \frac{x}{y} = \frac{\text{side adjacent}}{\text{side opposite}}.$$

▶ EXAMPLE 1 FINDING TRIGONOMETRIC FUNCTION VALUES OF AN ACUTE ANGLE

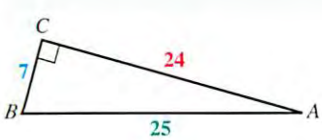


Figure 2

Find the sine, cosine, and tangent values for angles A and B in the right triangle in Figure 2.

Solution The length of the side opposite angle A is 7, the length of the side adjacent to angle A is 24, and the length of the hypotenuse is 25. Use the relationships given in the box.

$$\sin A = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{7}{25} \qquad \cos A = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{24}{25} \qquad \tan A = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{7}{24}$$

The length of the side opposite angle B is 24, while the length of the side adjacent to B is 7, so

$$\sin B = \frac{24}{25} \qquad \cos B = \frac{7}{25} \qquad \tan B = \frac{24}{7}.$$

NOW TRY EXERCISE 1. ◀

▶ **Note** Because the cosecant, secant, and cotangent ratios are the reciprocals of the sine, cosine, and tangent values, respectively, in Example 1 we can conclude that $\csc A = \frac{25}{7}$, $\sec A = \frac{25}{24}$, $\cot A = \frac{24}{7}$, $\csc B = \frac{25}{24}$, $\sec B = \frac{25}{7}$, and $\cot B = \frac{7}{24}$.

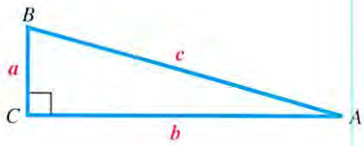


Figure 3

Cofunctions

In Example 1, you may have noticed that $\sin A = \cos B$ and $\cos A = \sin B$. Such relationships are always true for the two acute angles of a right triangle. Figure 3 shows a right triangle with acute angles A and B and a right angle at C. (Whenever we use A, B, and C to name angles in a right triangle, C will be the right angle.) The length of the side opposite A is a, and the length of the side opposite angle B is b. The length of the hypotenuse is c.

By the preceding definitions, $\sin A = \frac{a}{c}$. Since $\cos B$ is also equal to $\frac{a}{c}$,

$$\sin A = \frac{a}{c} = \cos B.$$

Similarly, $\tan A = \frac{a}{b} = \cot B$ and $\sec A = \frac{c}{b} = \csc B$.

Since the sum of the three angles in any triangle is 180° and angle C equals 90°, angles A and B must have a sum of $180° - 90° = 90°$. As mentioned in **Section 1.1,** angles with a sum of 90° are complementary angles. Since angles A and B are complementary and $\sin A = \cos B$, the functions sine and cosine are called **cofunctions.** Tangent and cotangent are also cofunctions, as are secant and cosecant. And since the angles A and B are complementary, $A + B = 90°$, or $B = 90° - A$, giving

$$\sin A = \cos B = \cos(90° - A).$$

Similar results, called the **cofunction identities,** are true for the other trigonometric functions.

COFUNCTION IDENTITIES

For any acute angle A,

$$\sin A = \cos(90° - A) \qquad \sec A = \csc(90° - A) \qquad \tan A = \cot(90° - A)$$

$$\cos A = \sin(90° - A) \qquad \csc A = \sec(90° - A) \qquad \cot A = \tan(90° - A).$$

▶ EXAMPLE 2 WRITING FUNCTIONS IN TERMS OF COFUNCTIONS

Write each function in terms of its cofunction.

(a) cos 52° **(b)** tan 71° **(c)** sec 24°

Solution

(a) Since $\cos A = \sin(90° - A)$,

$$\cos 52° = \sin(90° - 52°) = \sin 38°.$$

(b) $\tan 71° = \cot(90° - 71°) = \cot 19°$ **(c)** $\sec 24° = \csc 66°$

NOW TRY EXERCISES 19 AND 25. ◀

▶ EXAMPLE 3 **SOLVING EQUATIONS USING THE COFUNCTION IDENTITIES**

Find one solution for each equation. Assume all angles involved are acute angles.

(a) $\cos(\theta + 4°) = \sin(3\theta + 2°)$ **(b)** $\tan(2\theta - 18°) = \cot(\theta + 18°)$

Solution

(a) Since sine and cosine are cofunctions, $\cos(\theta + 4°) = \sin(3\theta + 2°)$ is true if the sum of the angles is 90°.

$$(\theta + 4°) + (3\theta + 2°) = 90°$$

$$4\theta + 6° = 90° \quad \text{Combine terms.}$$

$$4\theta = 84° \quad \text{Subtract } 6°. \text{ (Appendix A)}$$

$$\theta = 21° \quad \text{Divide by 4.}$$

(b) Tangent and cotangent are cofunctions, so

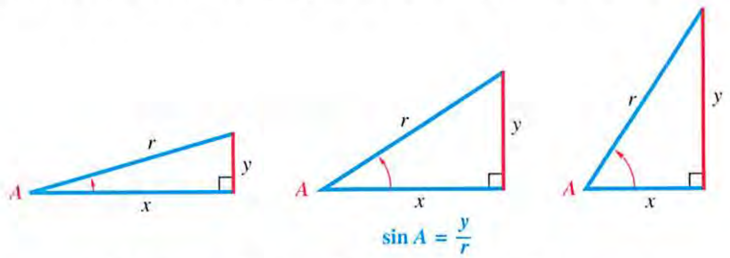
$$(2\theta - 18°) + (\theta + 18°) = 90°$$

$$3\theta = 90°$$

$$\theta = 30°.$$

NOW TRY EXERCISES 27 AND 29. ◀

Figure 4 shows three right triangles. From left to right, the length of each hypotenuse is the same, but angle A increases in measure. As angle A increases in measure from 0° to 90°, the length of the side opposite angle A also increases.



As A increases, y increases. Since r is fixed, $\sin A$ increases.

Figure 4

Since $$\sin A = \frac{\text{side opposite}}{\text{hypotenuse}},$$

as angle A increases, the numerator of this fraction also increases, while the denominator is fixed. This means that $\sin A$ *increases* as A increases from 0° to 90°.

As angle A increases from 0° to 90°, the length of the side adjacent to A decreases. Since r is fixed, the ratio $\frac{x}{r}$ will decrease. This ratio gives $\cos A$, showing that the values of cosine *decrease* as the angle measure changes from 0° to 90°. Finally, increasing A from 0° to 90° causes y to increase and x to decrease, making the values of $\frac{y}{x} = \tan A$ increase.

A similar discussion shows that as A increases from 0° to 90°, the values of $\sec A$ increase, while the values of $\cot A$ and $\csc A$ decrease.

EXAMPLE 4 COMPARING FUNCTION VALUES OF ACUTE ANGLES

Determine whether each statement is *true* or *false*.

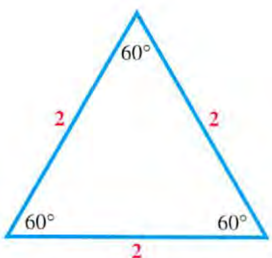
(a) $\sin 21° > \sin 18°$ **(b)** $\cos 49° \leq \cos 56°$

Solution

(a) In the interval from 0° to 90°, as the angle increases, so does the sine of the angle, which makes $\sin 21° > \sin 18°$ a true statement.

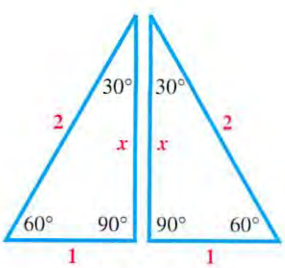
(b) In the interval from 0° to 90°, as the angle increases, the cosine of the angle decreases. The given statement $\cos 49° \leq \cos 56°$ is false.

NOW TRY EXERCISE 37.



Equilateral triangle

(a)



30°–60° right triangle

(b)

Figure 5

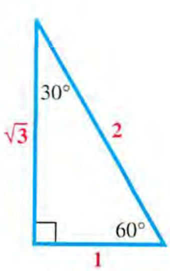


Figure 6

Trigonometric Function Values of Special Angles Certain special angles, such as 30°, 45°, and 60°, occur so often in trigonometry and in more advanced mathematics that they deserve special study. We start with an equilateral triangle, a triangle with all sides of equal length. Each angle of such a triangle measures 60°. While the results we will obtain are independent of the length, for convenience we choose the length of each side to be 2 units. See Figure 5(a).

Bisecting one angle of this equilateral triangle leads to two right triangles, each of which has angles of 30°, 60°, and 90°, as shown in Figure 5(b). Since the hypotenuse of each right triangle has length 2, the shorter leg will have length 1. (Why?) If x represents the length of the longer leg then,

$$2^2 = 1^2 + x^2 \quad \text{Pythagorean theorem (Appendix B)}$$
$$4 = 1 + x^2$$
$$3 = x^2 \quad \text{Subtract 1.}$$
$$\sqrt{3} = x. \quad \text{Square root property (Appendix A); choose the positive square root.}$$

Figure 6 summarizes our results using a 30°–60° right triangle. As shown in the figure, the side opposite the 30° angle has length 1; that is, for the 30° angle,

$$\text{hypotenuse} = 2, \quad \text{side opposite} = 1, \quad \text{side adjacent} = \sqrt{3}.$$

Now we use the definitions of the trigonometric functions.

$$\sin 30° = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{1}{2}$$

$$\cos 30° = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2}$$

$$\tan 30° = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\csc 30° = \frac{2}{1} = 2$$

$$\sec 30° = \frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\cot 30° = \frac{\sqrt{3}}{1} = \sqrt{3}$$

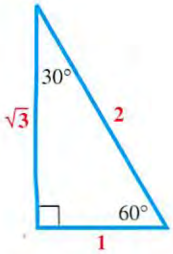


Figure 6 (repeated)

▶ EXAMPLE 5 FINDING TRIGONOMETRIC FUNCTION VALUES FOR 60°

Find the six trigonometric function values for a 60° angle.

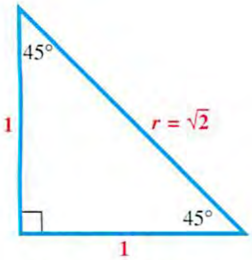
Solution Refer to Figure 6 to find the following ratios.

$$\sin 60° = \frac{\sqrt{3}}{2} \qquad \cos 60° = \frac{1}{2} \qquad \tan 60° = \frac{\sqrt{3}}{1} = \sqrt{3}$$

$$\csc 60° = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \qquad \sec 60° = \frac{2}{1} = 2 \qquad \cot 60° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

NOW TRY EXERCISES 45, 47, AND 49. ◀

▶ **Note** The results in Example 5 can also be found using the fact that cofunctions of complementary angles are equal.



45°–45° right triangle

Figure 7

We find the values of the trigonometric functions for 45° by starting with a 45°–45° right triangle, as shown in Figure 7. This triangle is isosceles; we choose the lengths of the equal sides to be 1 unit. (As before, the results are independent of the length of the equal sides.) Since the shorter sides each have length 1, if r represents the length of the hypotenuse, then

$$1^2 + 1^2 = r^2 \quad \text{Pythagorean theorem}$$

$$2 = r^2$$

$$\sqrt{2} = r. \quad \text{Choose the positive square root.}$$

Now we use the measures indicated on the 45°–45° right triangle in Figure 7.

$$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \tan 45° = \frac{1}{1} = 1$$

$$\csc 45° = \frac{\sqrt{2}}{1} = \sqrt{2} \qquad \sec 45° = \frac{\sqrt{2}}{1} = \sqrt{2} \qquad \cot 45° = \frac{1}{1} = 1$$

Function values for 30°, 45°, and 60° are summarized in the table that follows.

Function Values of Special Angles

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$

NOW TRY EXERCISES 5, 7, AND 9. ◀

TI-83

TI-84

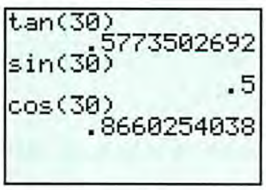


Figure 8

▶ **Note** You will be able to reproduce this table quickly if you learn the values of sin 30°, sin 45°, and sin 60°. Then you can complete the rest of the table using the reciprocal, cofunction, and quotient identities.

Since a calculator finds trigonometric function values at the touch of a key, you may wonder why we spend so much time finding values for special angles. We do this because a calculator gives only *approximate* values in most cases, while we often need *exact* values. For example, tan 30° can be found on a scientific calculator by first setting it in *degree mode,* then entering 30 and pressing the [tan] key to get

$$\tan 30^\circ \approx .57735027. \quad \approx \text{ means "is approximately equal to."}$$

Earlier, however, we found the exact value:

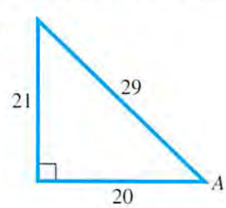
$$\tan 30^\circ = \frac{\sqrt{3}}{3}.$$

To use a graphing calculator to approximate sine, cosine, or tangent function values, press the appropriate function key *first,* and then enter the angle measure. (The calculator must be in degree mode to enter the angle measure in degrees.) See Figure 8. ■

2.1 Exercises

Find exact values or expressions for sin *A*, cos *A*, *and* tan *A*. *See Example 1.*

1.



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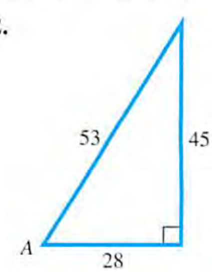

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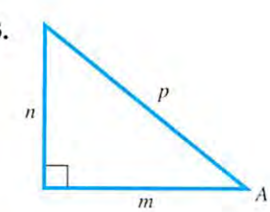

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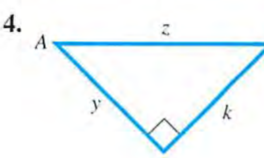


Concept Check *For each trigonometric function in Column I, choose its value from Column II.*

I		II		
5. sin 30°	**6.** cos 45°	**A.** $\sqrt{3}$	**B.** 1	**C.** $\frac{1}{2}$
7. tan 45°	**8.** sec 60°	**D.** $\frac{\sqrt{3}}{2}$	**E.** $\frac{2\sqrt{3}}{3}$	**F.** $\frac{\sqrt{3}}{3}$
9. csc 60°	**10.** cot 30°	**G.** 2	**H.** $\frac{\sqrt{2}}{2}$	**I.** $\sqrt{2}$

*Suppose ABC is a right triangle with sides of lengths a, b, and c and right angle at C. (See Figure 3.) Find the unknown side length using the Pythagorean theorem (**Appendix B**), and then find the values of the six trigonometric functions for angle B. Rationalize denominators when applicable.*

11. $a = 5, b = 12$
12. $a = 3, b = 5$
13. $a = 6, c = 7$
14. $b = 7, c = 12$
15. $a = 3, c = 5$
16. $b = 8, c = 11$

17. ***Concept Check*** Give a summary of the six cofunction relationships.

Write each function in terms of its cofunction. Assume that all angles in which an unknown appears are acute angles. See Example 2.

18. $\cot 73°$
19. $\sec 39°$
20. $\sin 27°$
21. $\sec(\theta + 15°)$
22. $\cos(\alpha + 20°)$
23. $\cot(\theta - 10°)$
24. $\tan 25.4°$
25. $\sin 38.7°$

26. With a calculator, evaluate $\sin(90° - A)$ and $\cos A$ for various values of A. (Include values greater than 90° and less than 0°.) What do you find?

Find one solution for each equation. Assume that all angles in which an unknown appears are acute angles. See Example 3.

27. $\tan \alpha = \cot(\alpha + 10°)$
28. $\cos \theta = \sin 2\theta$
29. $\sin(2\theta + 10°) = \cos(3\theta - 20°)$
30. $\sec(\beta + 10°) = \csc(2\beta + 20°)$
31. $\tan(3B + 4°) = \cot(5B - 10°)$
32. $\cot(5\theta + 2°) = \tan(2\theta + 4°)$
33. $\sin(\theta - 20°) = \cos(2\theta + 5°)$
34. $\cos(2\theta + 50°) = \sin(2\theta - 20°)$
35. $\sec(3\beta + 10°) = \csc(\beta + 8°)$
36. $\csc(\beta + 40°) = \sec(\beta - 20°)$

Determine whether each statement is true *or* false. *See Example 4.*

37. $\sin 50° > \sin 40°$
38. $\tan 28° \leq \tan 40°$
39. $\sin 46° < \cos 46°$
(*Hint:* $\cos 46° = \sin 44°$)
40. $\cos 28° < \sin 28°$
(*Hint:* $\sin 28° = \cos 62°$)
41. $\tan 41° < \cot 41°$
42. $\cot 30° < \tan 40°$
43. $\sec 60° > \sec 30°$
44. $\csc 20° < \csc 30°$

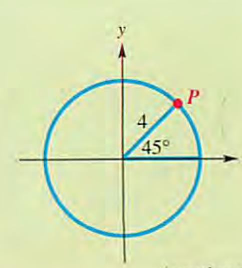
For each expression, give the exact value. See Example 5.

45. $\tan 30°$
46. $\cot 30°$
47. $\sin 30°$
48. $\cos 30°$
49. $\sec 30°$
50. $\csc 30°$
51. $\csc 45°$
52. $\sec 45°$
53. $\cos 45°$
54. $\cot 45°$
55. $\tan 45°$
56. $\sin 45°$
57. $\sin 60°$
58. $\cos 60°$
59. $\tan 60°$
60. $\csc 60°$

RELATING CONCEPTS

For individual or collaborative investigation (Exercises 61–64)

*The figure shows a 45° central angle in a circle with radius 4 units. To find the coordinates of point P on the circle, **work Exercises 61–64 in order.***



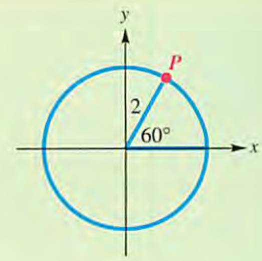
61. Sketch a line segment from P perpendicular to the x-axis.

(continued)

62. Use the trigonometric ratios for a 45° angle to label the sides of the right triangle you sketched in Exercise 61.

63. Which sides of the right triangle give the coordinates of point P? What are the coordinates of P?

64. The figure at the right shows a 60° central angle in a circle of radius 2 units. Follow the same procedure as in Exercises 61–63 to find the coordinates of P in the figure.



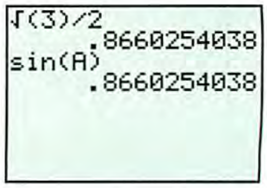
65. *Concept Check* Refer to the table. What trigonometric functions are y_1 and y_2?

$x°$	y_1	y_2
0	0	0
15	.25882	.26795
30	.5	.57735
45	.70711	1
60	.86603	1.7321
75	.96593	3.7321
90	1	undefined

66. *Concept Check* Refer to the table. What trigonometric functions are y_1 and y_2?

$x°$	y_1	y_2
0	1	undefined
15	.96593	3.8637
30	.86603	2
45	.70711	1.4142
60	.5	1.1547
75	.25882	1.0353
90	0	1

67. *Concept Check* What value of A between 0° and 90° will produce the output shown on the graphing calculator screen?



68. A student was asked to give the exact value of sin 45°. Using a calculator, he gave the answer .7071067812. The teacher did not give him credit. What was the teacher's reason for this?

69. With a graphing calculator, find the coordinates of the point of intersection of $y = x$ and $y = \sqrt{1 - x^2}$. These coordinates are the cosine and sine of what angle between 0° and 90°?

Concept Check *Work each problem.*

70. Find the equation of the line passing through the origin and making a 60° angle with the x-axis.

71. Find the equation of the line passing through the origin and making a 30° angle with the x-axis.

72. What angle does the line $y = \frac{\sqrt{3}}{3}x$ make with the positive x-axis?

73. What angle does the line $y = \sqrt{3}x$ make with the positive x-axis?

74. Construct a square with each side of length k.

(a) Draw a diagonal of the square. What is the measure of each angle formed by a side of the square and this diagonal?

(b) What is the length of the diagonal?

(c) From the results of parts (a) and (b), complete the following statement: In a $45°–45°$ right triangle, the hypotenuse has a length that is ________ times as long as either leg.

75. Construct an equilateral triangle with each side having length $2k$.

(a) What is the measure of each angle?

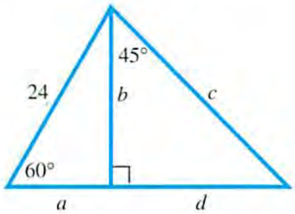
(b) Label one angle A. Drop a perpendicular from A to the side opposite A. Two $30°$ angles are formed at A, and two right triangles are formed. What is the length of the sides opposite the $30°$ angles?

(c) What is the length of the perpendicular constructed in part (b)?

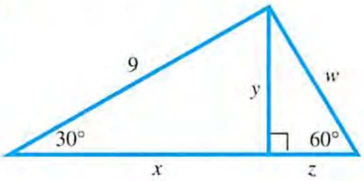
(d) From the results of parts (a)–(c), complete the following statement: In a $30°–60°$ right triangle, the hypotenuse is always ________ times as long as the shorter leg, and the longer leg has a length that is ________ times as long as that of the shorter leg. Also, the shorter leg is opposite the ________ angle, and the longer leg is opposite the ________ angle.

Find the exact value of each part labeled with a variable in each figure.

76.



77.



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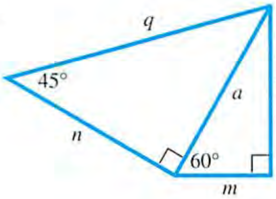

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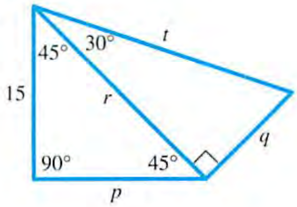
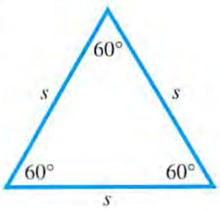
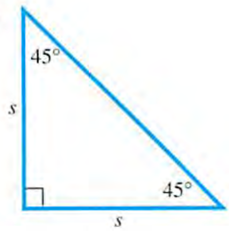


Find a formula for the area of each figure in terms of s.

80.



81.



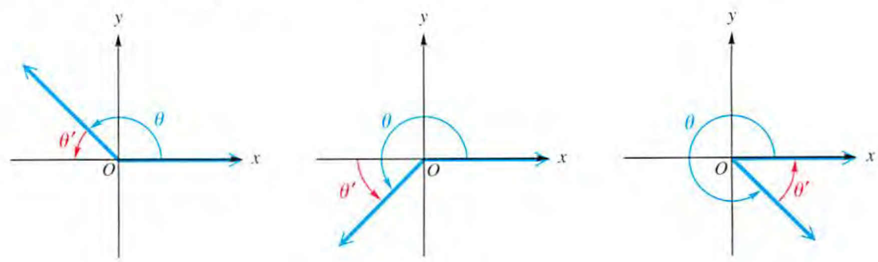
82. Suppose you know the length of one side and one acute angle of a right triangle. Can you determine the measures of all the sides and angles of the triangle?

83. Why is it important to be able to find trigonometric values for the special angles without using a calculator?

2.2 Trigonometric Functions of Non-Acute Angles

Reference Angles ▪ Special Angles as Reference Angles ▪ Finding Angle Measures with Special Angles

Reference Angles Associated with every nonquadrantal angle in standard position is a positive acute angle called its *reference angle.* A **reference angle** for an angle θ, written θ', is the positive acute angle made by the terminal side of angle θ and the x-axis. Figure 9 shows several angles θ (each less than one complete counterclockwise revolution) in quadrants II, III, and IV, respectively, with the reference angle θ' also shown. In quadrant I, θ and θ' are the same. If an angle θ is negative or has measure greater than 360°, its reference angle is found by first finding its coterminal angle that is between 0° and 360°, and then using the diagrams in Figure 9.



θ in quadrant II θ in quadrant III θ in quadrant IV

Figure 9

▶ **Caution** A common error is to find the reference angle by using the terminal side of θ and the y-axis. ***The reference angle is always found with reference to the x-axis.***

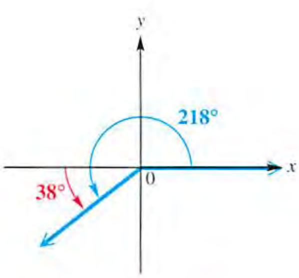
▶ EXAMPLE 1 FINDING REFERENCE ANGLES

Find the reference angle for each angle.

(a) 218° **(b)** 1387°

Solution

(a) As shown in Figure 10, the positive acute angle made by the terminal side of this angle and the x-axis is $218° - 180° = 38°$. For $\theta = 218°$, the reference angle $\theta' = 38°$.



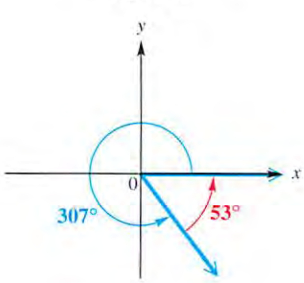
218° − 180° = 38°

Figure 10

(b) First find a coterminal angle between 0° and 360°. Divide 1387° by 360° to get a quotient of about 3.9. Begin by subtracting 360° three times (because of the whole number 3 in 3.9):

$$1387° - 3 \cdot 360° = 1387° - 1080° = 307°. \quad \text{(Section 1.1)}$$

The reference angle for 307° (and thus for 1387°) is $360° - 307° = 53°$. See Figure 11.



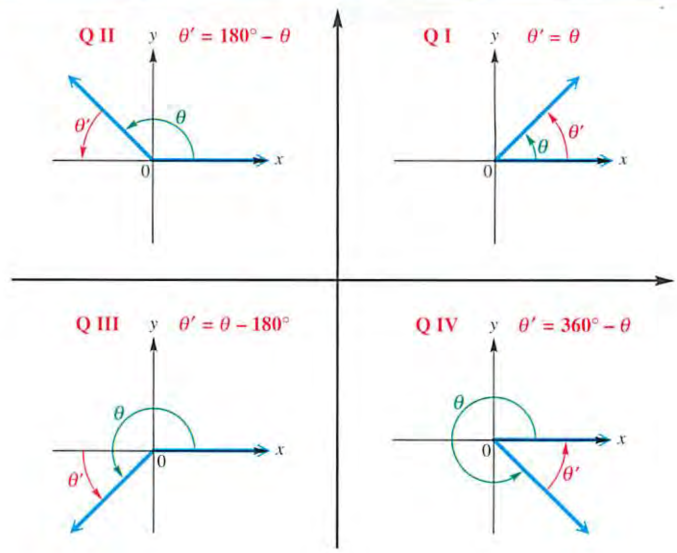
360° − 307° = 53°

Figure 11

NOW TRY EXERCISES 1 AND 5. ◀

The preceding example suggests the following table for finding the reference angle θ' for any angle θ between 0° and 360°.

Reference Angle θ' for θ, where $0° < \theta < 360°$*



Special Angles as Reference Angles

We can now find exact trigonometric function values of angles with reference angles of 30°, 45°, or 60°.

▶ EXAMPLE 2 FINDING TRIGONOMETRIC FUNCTION VALUES OF A QUADRANT III ANGLE

Find the values of the six trigonometric functions for 210°.

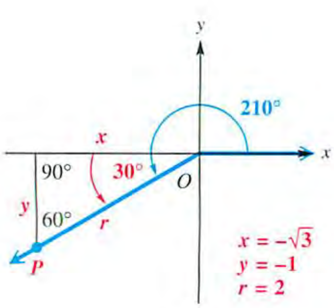


Figure 12

Solution An angle of 210° is shown in Figure 12. The reference angle is $210° - 180° = 30°$. To find the trigonometric function values of 210°, choose point P on the terminal side of the angle so that the distance from the origin O to P is 2. By the results from 30°–60° right triangles, the coordinates of point P become $\left(-\sqrt{3}, -1\right)$, with $x = -\sqrt{3}$, $y = -1$, and $r = 2$. Then, by the definitions of the trigonometric functions in **Section 2.1**,

$$\sin 210° = \frac{-1}{2} = -\frac{1}{2} \qquad \csc 210° = \frac{2}{-1} = -2$$

$$\cos 210° = \frac{-\sqrt{3}}{2} = -\frac{\sqrt{3}}{2} \qquad \sec 210° = \frac{2}{-\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

$$\tan 210° = \frac{-1}{-\sqrt{3}} = \frac{\sqrt{3}}{3} \qquad \cot 210° = \frac{-\sqrt{3}}{-1} = \sqrt{3}.$$

NOW TRY EXERCISE 19. ◀

*The authors would like to thank Bethany Vaughn and Theresa Matick, of Vincennes Lincoln High School, for their suggestions concerning this table.

Notice in Example 2 that the trigonometric function values of 210° correspond in absolute value to those of its reference angle 30°. The signs are different for the sine, cosine, secant, and cosecant functions because 210° is a quadrant III angle. These results suggest a shortcut for finding the trigonometric function values of a non-acute angle, using the reference angle. In Example 2, the reference angle for 210° is 30°. Using the trigonometric function values of 30°, and choosing the correct signs for a quadrant III angle, we obtain the same results.

We determine the values of the trigonometric functions for any nonquadrantal angle θ as follows.

FINDING TRIGONOMETRIC FUNCTION VALUES FOR ANY NONQUADRANTAL ANGLE θ

Step 1 If $\theta > 360°$, or if $\theta < 0°$, then find a coterminal angle by adding or subtracting 360° as many times as needed to get an angle greater than 0° but less than 360°.

Step 2 Find the reference angle θ'.

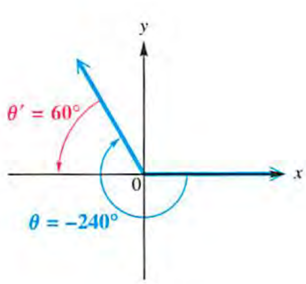
Step 3 Find the trigonometric function values for reference angle θ'.

Step 4 Determine the correct signs for the values found in Step 3. (Use the table of signs in **Section 1.4,** if necessary.) This gives the values of the trigonometric functions for angle θ.

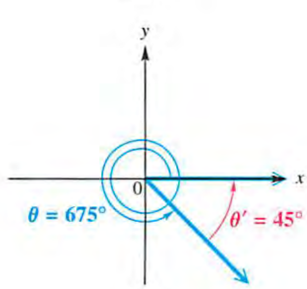
▶ EXAMPLE 3 FINDING TRIGONOMETRIC FUNCTION VALUES USING REFERENCE ANGLES

Find the exact value of each expression.

(a) $\cos(-240°)$ **(b)** $\tan 675°$



(a)



(b)

Figure 13

Solution

(a) Since an angle of $-240°$ is coterminal with an angle of

$$-240° + 360° = 120°, \quad \text{(Section 1.1)}$$

the reference angle is $180° - 120° = 60°$, as shown in Figure 13(a). Since the cosine is negative in quadrant II,

$$\cos(-240°) = \underset{\substack{\uparrow \\ \text{Coterminal} \\ \text{angle}}}{\cos 120°} = -\underset{\substack{\uparrow \\ \text{Reference} \\ \text{angle}}}{\cos 60°} = -\frac{1}{2}.$$

(b) Begin by subtracting 360° to get a coterminal angle between 0° and 360°.

$$675° - 360° = 315°$$

As shown in Figure 13(b), the reference angle is $360° - 315° = 45°$. An angle of 315° is in quadrant IV, so the tangent will be negative, and

$$\tan 675° = \tan 315° = -\tan 45° = -1.$$

NOW TRY EXERCISES 37 AND 39. ◀

▶ EXAMPLE 4 **EVALUATING AN EXPRESSION WITH FUNCTION VALUES OF SPECIAL ANGLES**

Evaluate $\cos 120° + 2\sin^2 60° - \tan^2 30°$.

Solution Since $\cos 120° = -\frac{1}{2}$, $\sin 60° = \frac{\sqrt{3}}{2}$, and $\tan 30° = \frac{\sqrt{3}}{3}$,

$$\begin{aligned}\cos 120° + 2\sin^2 60° - \tan^2 30° &= -\frac{1}{2} + 2\left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{\sqrt{3}}{3}\right)^2 \\ &= -\frac{1}{2} + 2\left(\frac{3}{4}\right) - \frac{3}{9} \\ &= \frac{2}{3}.\end{aligned}$$

NOW TRY EXERCISE 45. ◀

Recall that trigonometric function values of coterminal angles are the same.

▶ EXAMPLE 5 **USING COTERMINAL ANGLES TO FIND FUNCTION VALUES**

Evaluate each function by first expressing the function in terms of an angle between 0° and 360°.

(a) $\cos 780°$ **(b)** $\tan(-405°)$

Solution

(a) Add or subtract 360° as many times as necessary to get an angle between 0° and 360°. Subtracting 720°, which is $2 \cdot 360°$, gives

$$\begin{aligned}\cos 780° &= \cos(780° - 2 \cdot 360°) \\ &= \cos 60° \\ &= \frac{1}{2}.\end{aligned}$$

Multiply first, and then subtract.

(b) Add 360° twice to get $-405° + 2(360°) = 315°$. This angle is located in quadrant IV and its reference angle is 45°. The tangent function is negative in quadrant IV; thus

$$\tan(-405°) = \tan 315° = -\tan 45° = -1.$$

NOW TRY EXERCISES 27 AND 31. ◀

Finding Angle Measures with Special Angles The ideas discussed in this section can also be used to find the measures of certain angles, given a trigonometric function value and an interval in which the angle must lie. We are most often interested in the interval $[0°, 360°)$.

▶ EXAMPLE 6 **FINDING ANGLE MEASURES GIVEN AN INTERVAL AND A FUNCTION VALUE**

Find all values of θ, if θ is in the interval $[0°, 360°)$ and $\cos\theta = -\frac{\sqrt{2}}{2}$.

Solution Since $\cos\theta$ is negative, θ must lie in quadrant II or III. Since the absolute value of $\cos\theta$ is $\frac{\sqrt{2}}{2}$, the reference angle θ' must be 45°. The two possible angles θ are sketched in Figure 14. The quadrant II angle θ equals

$$180° - 45° = 135°,$$

and the quadrant III angle θ equals

$$180° + 45° = 225°.$$

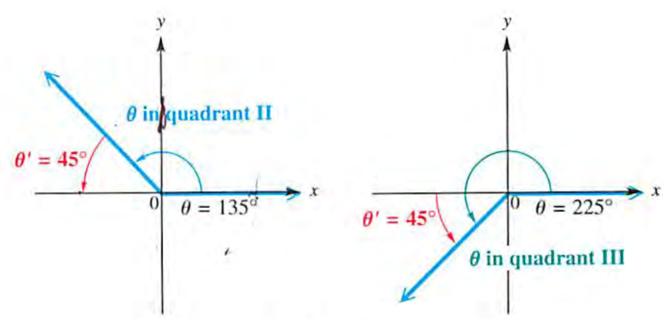


Figure 14

NOW TRY EXERCISE 71. ◀

2.2 Exercises

Match each angle in Column I with its reference angle in Column II. Choices may be used once, more than once, or not at all. See Example 1.

I		II	
1. 98°	**2.** 212°	**A.** 45°	**B.** 60°
3. −135°	**4.** −60°	**C.** 82°	**D.** 30°
5. 750°	**6.** 480°	**E.** 38°	**F.** 32°

Give a short explanation in Exercises 7–9.

7. In Example 2, why was 2 a good choice for r? Could any other positive number have been used?

8. Explain how the reference angle is used to find values of the trigonometric functions for an angle in quadrant III.

9. Explain why two coterminal angles have the same values for their trigonometric functions.

Complete the table with exact trigonometric function values. Do not use a calculator. See Examples 2 and 3.

	θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
10.	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$			$\frac{2\sqrt{3}}{3}$	2
11.	45°			1	1		
12.	60°		$\frac{1}{2}$	$\sqrt{3}$		2	
13.	120°	$\frac{\sqrt{3}}{2}$		$-\sqrt{3}$			$\frac{2\sqrt{3}}{3}$
14.	135°	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$			$-\sqrt{2}$	$\sqrt{2}$
15.	150°		$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$			2
16.	210°	$-\frac{1}{2}$		$\frac{\sqrt{3}}{3}$	$\sqrt{3}$		-2
17.	240°	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$			-2	$-\frac{2\sqrt{3}}{3}$

Find exact values of the six trigonometric functions for each angle. Rationalize denominators when applicable. See Examples 2, 3, and 5.

18. 300° **19.** 315° **20.** 405° **21.** −300° **22.** 420° **23.** 480°
24. 495° **25.** 570° **26.** 750° **27.** 1305° **28.** 1500° **29.** 2670°
30. −390° **31.** −510° **32.** −1020° **33.** −1290° **34.** −855° **35.** −1860°

Find the exact value of each expression. See Example 3.

36. sin 1305° **37.** cos(−510°) **38.** tan(−1020°) **39.** sin 1500°
40. sec(−495°) **41.** csc(−855°) **42.** cot 2280° **43.** tan 3015°

Determine whether each statement is true *or* false. *If false, tell why. See Example 4.*

44. $\sin 30° + \sin 60° = \sin(30° + 60°)$

45. $\sin(30° + 60°) = \sin 30° \cdot \cos 60° + \sin 60° \cdot \cos 30°$

46. $\cos 60° = 2\cos^2 30° - 1$

47. $\cos 60° = 2\cos 30°$

48. $\sin 120° = \sin 150° - \sin 30°$

49. $\sin 120° = \sin 180° \cdot \cos 60° - \sin 60° \cdot \cos 180°$

50. $\sin(2 \cdot 30°) = 2\sin 30° \cdot \cos 30°$

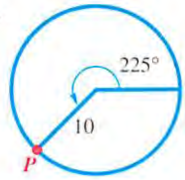
51. $\sin^2 45° + \cos^2 45° = 1$

52. $\tan^2 60° + 1 = \sec^2 60°$

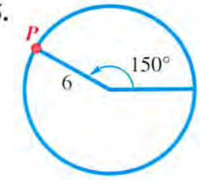
53. $\cos(30° + 60°) = \cos 30° + \cos 60°$

Concept Check *Find the coordinates of the point P on the circumference of each circle. (Hint: Add x- and y-axes, assuming that the angle is in standard position.)*

54.



55.



56. ***Concept Check*** Does there exist an angle θ with the function values $\cos\theta = .6$ and $\sin\theta = -.8$?

57. ***Concept Check*** Does there exist an angle θ with the function values $\cos\theta = \frac{2}{3}$ and $\sin\theta = \frac{3}{4}$?

Suppose θ is in the interval $(90°, 180°)$. Find the sign of each of the following.

58. $\sin\dfrac{\theta}{2}$ **59.** $\cos\dfrac{\theta}{2}$ **60.** $\cot(\theta + 180°)$

61. $\sec(\theta + 180°)$ **62.** $\cos(-\theta)$ **63.** $\sin(-\theta)$

64. Explain why $\sin\theta = \sin(\theta + n \cdot 360°)$ for any angle θ and any integer n.

65. Explain why $\cos\theta = \cos(\theta + n \cdot 360°)$ for any angle θ and any integer n.

Concept Check *Work Exercises 66–69.*

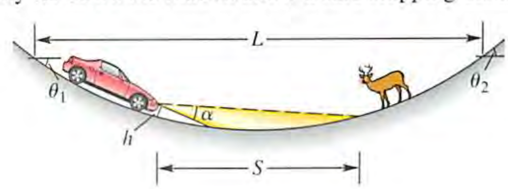
66. Without using a calculator, determine which of the following numbers is closest to cos 115°: .4, .6, 0, −.4, or −.6.

67. Without using a calculator, determine which of the following numbers is closest to sin 115°: .9, .1, 0, −.9, or −.1.

68. For what angles θ between 0° and 360° does $\cos\theta = -\sin\theta$?

69. For what angles θ between 0° and 360° does $\cos\theta = \sin\theta$?

70. ***(Modeling) Length of a Sag Curve*** When a highway goes downhill and then uphill, it is said to have a **sag curve.** Sag curves are designed so that at night, headlights shine sufficiently far down the road to allow a safe stopping distance. See the figure.



The minimum length L of a sag curve is determined by the height h of the car's headlights above the pavement, the downhill grade $\theta_1 < 0°$, the uphill grade $\theta_2 > 0°$, and the safe stopping distance S for a given speed limit. In addition, L is dependent on the vertical alignment of the headlights. Headlights are usually pointed upward at a slight angle α above the horizontal of the car. Using these quantities, for a 55 mph speed limit, L can be modeled by the formula

$$L = \frac{(\theta_2 - \theta_1)S^2}{200(h + S\tan\alpha)},$$

where $S < L$. (*Source:* Mannering, F. and W. Kilareski, *Principles of Highway Engineering and Traffic Analysis,* Second Edition, John Wiley and Sons, 1998.)

(a) Compute L if $h = 1.9$ ft, $\alpha = .9°$, $\theta_1 = -3°$, $\theta_2 = 4°$, and $S = 336$ ft.

(b) Repeat part (a) with $\alpha = 1.5°$.

(c) How does the alignment of the headlights affect the value of L?

Find all values of θ, if θ is in the interval $[0°, 360°)$ and has the given function value. See Example 6.

71. $\sin \theta = \frac{1}{2}$ **72.** $\cos \theta = \frac{\sqrt{3}}{2}$ **73.** $\tan \theta = -\sqrt{3}$

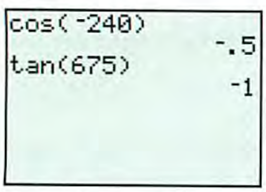
74. $\sec \theta = -\sqrt{2}$ **75.** $\cos \theta = \frac{\sqrt{2}}{2}$ **76.** $\cot \theta = -\frac{\sqrt{3}}{3}$

77. $\csc \theta = -2$ **78.** $\sin \theta = -\frac{\sqrt{3}}{2}$ **79.** $\tan \theta = \frac{\sqrt{3}}{3}$

80. $\cos \theta = -\frac{1}{2}$ **81.** $\csc \theta = -\sqrt{2}$ **82.** $\cot \theta = -1$

2.3 Finding Trigonometric Function Values Using a Calculator

Finding Function Values Using a Calculator ▪ Finding Angle Measures Using a Calculator



Degree mode

Figure 15

Finding Function Values Using a Calculator

Calculators are capable of finding trigonometric function values. For example, the values of $\cos(-240°)$ and $\tan 675°$ in Example 3 of **Section 2.2** are found with a calculator as shown in Figure 15.

▶ **Caution** ***When evaluating trigonometric functions of angles given in degrees, remember that the calculator must be set in degree mode.*** Get in the habit of always starting work by entering sin 90. If the displayed answer is 1, then the calculator is set for degree measure. Remember that most calculator values of trigonometric functions are *approximations.*

▶ EXAMPLE 1 FINDING FUNCTION VALUES WITH A CALCULATOR

Approximate the value of each expression.

(a) $\sin 49° 12'$ **(b)** $\sec 97.977°$ **(c)** $\frac{1}{\cot 51.4283°}$ **(d)** $\sin(-246°)$

Solution

(a) $49° 12' = 49\frac{12}{60}° = 49.2°$ Convert 49° 12′ to decimal degrees. (Section 1.1)

$\sin 49° 12' = \sin 49.2° \approx .75699506$ To eight decimal places

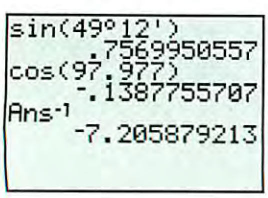
(b) Calculators do not have secant keys. However, $\sec \theta = \frac{1}{\cos \theta}$ for all angles θ where $\cos \theta \neq 0$. First find $\cos 97.977°$, and then take the reciprocal to get

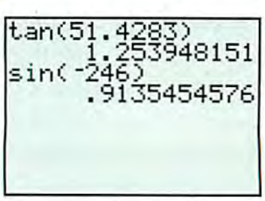
$$\sec 97.977° \approx -7.20587921.$$

(c) Use the reciprocal identity $\tan \theta = \frac{1}{\cot \theta}$ from **Section 1.4** to get

$$\frac{1}{\cot 51.4283°} = \tan 51.4283° \approx 1.25394815.$$

(d) $\sin(-246°) \approx .91354546$

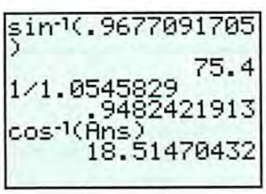




These screens support the results of Example 7. We entered the angle measure in degrees and minutes for part (a). In the fifth line of the first screen, Ans^{-1} tells the calculator to find the reciprocal of the answer given in the previous line.

NOW TRY EXERCISES 5, 7, 13, AND 17. ◀

Finding Angle Measures Using a Calculator Sometimes we need to find the measure of an angle having a certain trigonometric function value. Graphing calculators have three *inverse functions* (denoted $\mathbf{sin^{-1}}$, $\mathbf{cos^{-1}}$, and $\mathbf{tan^{-1}}$) that do just that. If x is an appropriate number, then $\sin^{-1}x$, $\cos^{-1}x$, or $\tan^{-1}x$ gives the measure of an angle whose sine, cosine, or tangent is x. For the applications in this chapter, these functions will return values of x in quadrant I.



Degree mode

Figure 16

▶ EXAMPLE 2 USING INVERSE TRIGONOMETRIC FUNCTIONS TO FIND ANGLES

Use a calculator to find an angle θ in the interval $[0°, 90°]$ that satisfies each condition.

(a) $\sin\theta \approx .9677091705$ **(b)** $\sec\theta \approx 1.0545829$

Solution

(a) Using degree mode and the inverse sine function, we find that an angle θ having sine value .9677091705 is 75.4°. (While there are infinitely many such angles, the calculator gives only this one.) We write this result as

$$\theta \approx \sin^{-1} .9677091705 \approx 75.4°.$$

See Figure 16.

(b) Use the identity $\cos\theta = \frac{1}{\sec\theta}$. Find the reciprocal of 1.0545829 to get $\cos\theta \approx .9482421913$. Now find θ as shown in part (a), using the inverse cosine function. The result is

$$\theta \approx \cos^{-1}(.9482421913) \approx 18.514704°.$$

See Figure 16.

NOW TRY EXERCISES 25 AND 29. ◀

▶ **Caution** Compare Examples 1(b) and 2(b). Note that the reciprocal is used *before* the inverse cosine key when finding the angle, but *after* the cosine key when finding the trigonometric function value.

▶ EXAMPLE 3 FINDING GRADE RESISTANCE

When an automobile travels uphill or downhill on a highway, it experiences a force due to gravity. This force F in pounds is called **grade resistance** and is modeled by the equation $F = W\sin\theta$, where θ is the grade and W is the weight of the automobile. If the automobile is moving uphill, then $\theta > 0°$; if downhill, then $\theta < 0°$. See Figure 17. (*Source:* Mannering, F. and W. Kilareski, *Principles of Highway Engineering and Traffic Analysis,* Second Edition, John Wiley and Sons, 1998.)

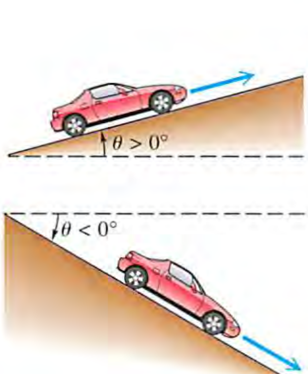


Figure 17

(a) Calculate F to the nearest 10 lb for a 2500-lb car traveling an uphill grade with $\theta = 2.5°$.

(b) Calculate F to the nearest 10 lb for a 5000-lb truck traveling a downhill grade with $\theta = -6.1°$.

(c) Calculate F for $\theta = 0°$ and $\theta = 90°$. Do these answers agree with your intuition?

Solution

(a) $F = W \sin \theta = 2500 \sin 2.5° \approx 110$ lb

(b) $F = W \sin \theta = 5000 \sin(-6.1°) \approx -530$ lb

F is negative because the truck is moving downhill.

(c) $F = W \sin \theta = W \sin 0° = W(0) = 0$ lb

$F = W \sin \theta = W \sin 90° = W(1) = W$ lb

This agrees with intuition because if $\theta = 0°$, then there is level ground and gravity does not cause the vehicle to roll. If $\theta = 90°$, the road would be vertical and the full weight of the vehicle would be pulled downward by gravity, so $F = W$.

NOW TRY EXERCISES 57 AND 59.

2.3 Exercises

Concept Check *Fill in the blanks to complete each statement.*

1. The CAUTION at the beginning of this section suggests verifying that a calculator is in degree mode by finding ______ (sin/cos/tan) 90°. If the calculator is in degree mode, the display should be ______.

2. When a scientific or graphing calculator is used to find a trigonometric function value, in most cases the result is an ______ (exact/approximate) value.

3. To find values of the cotangent, secant, and cosecant functions with a calculator, it is necessary to find the ______ of the ______ function value.

4. The reciprocal is used ______ (before/after) the inverse function key when finding the angle, but ______ (before/after) the function key when finding the trigonometric function value.

Use a calculator to find a decimal approximation for each value. Give as many digits as your calculator displays. In Exercises 16–22, simplify the expression before using the calculator. See Example 1.

5. $\sin 38° 42'$	**6.** $\cot 41° 24'$	**7.** $\sec 13° 15'$
8. $\csc 145° 45'$	**9.** $\cot 183° 48'$	**10.** $\cos 421° 30'$
11. $\sec 312° 12'$	**12.** $\tan(-80° 6')$	**13.** $\sin(-317° 36')$
14. $\cot(-512° 20')$	**15.** $\cos(-15')$	**16.** $\dfrac{1}{\sec 14.8°}$
17. $\dfrac{1}{\cot 23.4°}$	**18.** $\dfrac{\sin 33°}{\cos 33°}$	**19.** $\dfrac{\cos 77°}{\sin 77°}$
20. $\cos(90° - 3.69°)$	**21.** $\cot(90° - 4.72°)$	**22.** $\dfrac{1}{\tan(90° - 22°)}$

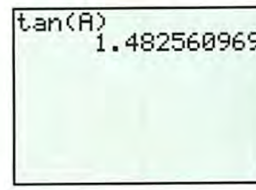
Find a value of θ in the interval $[0°, 90°]$ that satisfies each statement. Leave answers in decimal degrees. See Example 2.

23. $\tan \theta = 1.4739716$	**24.** $\tan \theta = 6.4358841$	**25.** $\sin \theta = .27843196$
26. $\sec \theta = 1.1606249$	**27.** $\cot \theta = 1.2575516$	**28.** $\csc \theta = 1.3861147$
29. $\sec \theta = 2.7496222$	**30.** $\sin \theta = .84802194$	**31.** $\cos \theta = .70058013$

32. A student, wishing to use a calculator to verify the value of sin 30°, enters the information correctly but gets a display of $-.98803162$. He knows that the display should be .5, and he also knows that his calculator is in good working order. What do you think is the problem?

33. At one time, a certain make of calculator did not allow the input of angles outside of a particular interval when finding trigonometric function values. For example, trying to find cos 2000° using the methods of this section gave an error message, despite the fact that cos 2000° can be evaluated. Explain how you would find cos 2000° using this calculator.

34. What value of A between 0° and 90° will produce the output in the graphing calculator screen?



35. What value of A will produce the output (in degrees) in the graphing calculator screen?

Use a calculator to evaluate each expression.

36. $\sin 35° \cos 55° + \cos 35° \sin 55°$

37. $\cos 100° \cos 80° - \sin 100° \sin 80°$

38. $\cos 75° 29' \cos 14° 31' - \sin 75° 29' \sin 14° 31'$

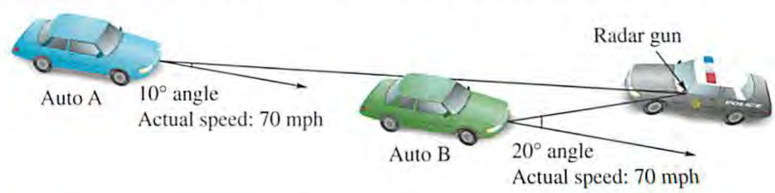
39. $\sin 28° 14' \cos 61° 46' + \cos 28° 14' \sin 61° 46'$

40. $\sin^2 36° + \cos^2 36°$

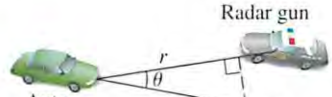
41. $2 \sin 25° 13' \cos 25° 13' - \sin 50° 26'$

Work each problem.

42. ***Measuring Speed by Radar*** Any offset between a stationary radar gun and a moving target creates a "cosine effect" that reduces the radar mileage reading by the cosine of the angle between the gun and the vehicle. That is, the radar speed reading is the product of the actual reading and the cosine of the angle. Find the radar readings to the nearest hundredth for Auto A and Auto B shown in the figure. (*Source:* Fischetti, M., "Working Knowledge," *Scientific American,* March 2001.)



43. ***Measuring Speed by Radar*** In Exercise 42, we saw that the mileage reported by a radar gun is reduced by the cosine of angle θ, shown in the figure. In the figure, r represents reduced speed and a represents the actual speed. Use the figure to show why this "cosine effect" occurs.



Use a calculator to decide whether each statement is true *or* false. *It may be that a true statement will lead to results that differ in the last decimal place due to rounding error.*

44. $\cos 40° = 2 \cos 20°$

45. $\sin 10° + \sin 10° = \sin 20°$

46. $\cos 70° = 2 \cos^2 35° - 1$

47. $\sin 50° = 2 \sin 25° \cos 25°$

48. $2 \cos 38° 22' = \cos 76° 44'$

49. $\cos 40° = 1 - 2 \sin^2 80°$

50. $\frac{1}{2} \sin 40° = \sin \frac{1}{2}(40°)$

51. $\sin 39° 48' + \cos 39° 48' = 1$

52. $\cos(30° + 20°) = \cos 30° + \cos 20°$

53. $\cos(30° + 20°) = \cos 30° \cos 20° - \sin 30° \sin 20°$

54. $\tan^2 72° 25' + 1 = \sec^2 72° 25'$

55. $1 + \cot^2 42.5° = \csc^2 42.5°$

(Modeling) Grade Resistance *See Example 3 to work Exercises 56–61.*

56. What is the grade resistance of a 2400-lb car traveling on a $-2.4°$ downhill grade?

57. What is the grade resistance of a 2100-lb car traveling on a $1.8°$ uphill grade?

58. A car traveling on a $-3°$ downhill grade has a grade resistance of -145 lb. What is the weight of the car?

59. A 2600-lb car traveling downhill has a grade resistance of -130 lb. What is the angle of the grade?

60. Which has the greater grade resistance: a 2200-lb car on a $2°$ uphill grade or a 2000-lb car on a $2.2°$ uphill grade?

61. A 3000-lb car traveling uphill has a grade resistance of 150 lb. What is the angle of the grade?

62. ***Highway Grades*** Complete the table for the values of $\sin \theta$, $\tan \theta$, and $\frac{\pi\theta}{180}$ to four decimal places.

θ	$\sin \theta$	$\tan \theta$	$\frac{\pi\theta}{180}$
0°			
.5°			
1°			
1.5°			
2°			
2.5°			
3°			
3.5°			
4°			

(a) How do $\sin \theta$, $\tan \theta$, and $\frac{\pi\theta}{180}$ compare for small grades θ?

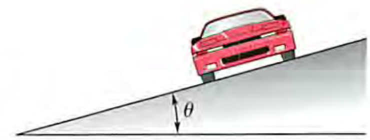
(b) Highway grades are usually small. Give two approximations of the grade resistance $F = W \sin \theta$ that do not use the sine function.

(c) A stretch of highway has a 4-ft vertical rise for every 100 ft of horizontal run. Use an approximation from part (a) to estimate the grade resistance for a 2000-lb car on this stretch of highway.

(d) Without evaluating a trigonometric function, estimate the grade resistance for an 1800-lb car on a stretch of highway that has a 3.75° grade.

(Modeling) *Solve each problem.*

63. ***Design of Highway Curves*** When highway curves are designed, the outside of the curve is often slightly elevated or inclined above the inside of the curve. See the figure. This inclination is called **superelevation.** For safety reasons, it is important that both the curve's radius and superelevation are correct for a given speed limit. If an automobile is traveling at velocity V (in feet per second), the safe radius R for a curve with superelevation θ is modeled by the formula



$$R = \frac{V^2}{g(f + \tan \theta)},$$

where f and g are constants. (*Source:* Mannering, F. and W. Kilareski, *Principles of Highway Engineering and Traffic Analysis,* Second Edition, John Wiley and Sons, 1998.)

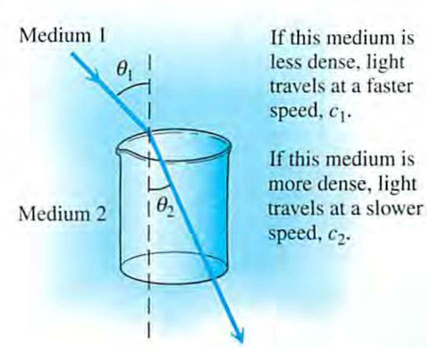
(a) A roadway is being designed for automobiles traveling at 45 mph. If $\theta = 3°$, $g = 32.2$, and $f = .14$, calculate R to the nearest foot. (*Hint:* 45 mph = 66 ft per sec)

(b) Determine the radius of the curve, to the nearest foot, if the speed in part (a) is increased to 70 mph.

(c) How would increasing the angle θ affect the results? Verify your answer by repeating parts (a) and (b) with $\theta = 4°$.

64. ***Speed Limit on a Curve*** Refer to Exercise 63 and use the same values for f and g. A highway curve has radius $R = 1150$ ft and a superelevation of $\theta = 2.1°$. What should the speed limit (in miles per hour) be for this curve?

(Modeling) Speed of Light *When a light ray travels from one medium, such as air, to another medium, such as water or glass, the speed of the light changes, and the direction in which the ray is traveling changes. (This is why a fish under water is in a different position than it appears to be.) These changes are given by* ***Snell's law***



$$\frac{c_1}{c_2} = \frac{\sin \theta_1}{\sin \theta_2},$$

where c_1 is the speed of light in the first medium, c_2 is the speed of light in the second medium, and θ_1 and θ_2 are the angles shown in the figure. (Source: The Physics Classroom, www.glenbrook.k12.il.us) *In Exercises 65 and 66, assume that $c_1 = 3 \times 10^8$ m per sec.*

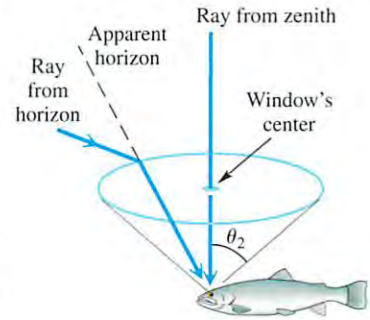
65. Find the speed of light in the second medium for each of the following.

(a) $\theta_1 = 46°$, $\theta_2 = 31°$ **(b)** $\theta_1 = 39°$, $\theta_2 = 28°$

66. Find θ_2 for each of the following values of θ_1 and c_2. Round to the nearest degree.

(a) $\theta_1 = 40°$, $c_2 = 1.5 \times 10^8$ m per sec **(b)** $\theta_1 = 62°$, $c_2 = 2.6 \times 10^8$ m per sec

(Modeling) Fish's View of the World *The figure shows a fish's view of the world above the surface of the water. (Source:* Walker, J., "The Amateur Scientist," *Scientific American,* March 1984.*) Suppose that a light ray comes from the horizon, enters the water, and strikes the fish's eye.*



67. Assume that this ray gives a value of 90° for angle θ_1 in the formula for Snell's law. (In a practical situation, this angle would probably be a little less than 90°.) The speed of light in water is about 2.254×10^8 m per sec. Find angle θ_2 to the nearest tenth.

68. Refer to Exercise 67. Suppose an object is located at a true angle of 29.6° above the horizon. Find the apparent angle above the horizon to a fish.

69. *(Modeling) Braking Distance* If aerodynamic resistance is ignored, the braking distance D (in feet) for an automobile to change its velocity from V_1 to V_2 (feet per second) can be modeled using the equation

$$D = \frac{1.05(V_1^2 - V_2^2)}{64.4(K_1 + K_2 + \sin \theta)}.$$

K_1 is a constant determined by the efficiency of the brakes and tires, K_2 is a constant determined by the rolling resistance of the automobile, and θ is the grade of the highway. (*Source:* Mannering, F. and W. Kilareski, *Principles of Highway Engineering and Traffic Analysis,* Second Edition, John Wiley and Sons, 1998.)

(a) Compute the number of feet required to slow a car from 55 mph to 30 mph while traveling uphill with a grade of $\theta = 3.5°$. Let $K_1 = .4$ and $K_2 = .02$. (*Hint:* Change miles per hour to feet per second.)

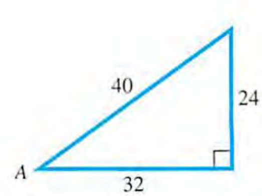
(b) Repeat part (a) with $\theta = -2°$.

(c) How is braking distance affected by grade θ? Does this agree with your driving experience?

70. *(Modeling) Car's Speed at Collision* Refer to Exercise 69. An automobile is traveling at 90 mph on a highway with a downhill grade of $\theta = -3.5°$. The driver sees a stalled truck in the road 200 ft away and immediately applies the brakes. Assuming that a collision cannot be avoided, how fast (in miles per hour) is the car traveling when it hits the truck? (Use the same values for K_1 and K_2 as in Exercise 69.)

CHAPTER 2 ▶ Quiz (Sections 2.1–2.3)

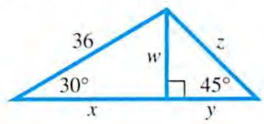
1. Find the exact values for sin A, cos A, and tan A in the figure.



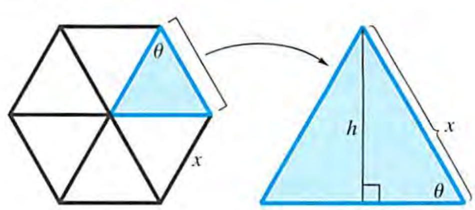
2. Find exact values of the trigonometric functions to complete the table.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
30°						
45°						
60°						

3. Find the exact value of each variable in the figure.



4. ***Area of a Solar Cell*** A solar cell converts the energy of sunlight directly into electrical energy. The amount of energy a cell produces depends on its area. Suppose a solar cell is hexagonal, as shown in the figure on the left. Express its area in terms of $\sin\theta$ and any side x. (*Hint:* Consider one of the six equilateral triangles from the hexagon. See the figure on the right.) (*Source:* Kastner, B., *Space Mathematics*, NASA, 1985.)



Find exact values of the six trigonometric functions for each angle. Rationalize denominators when applicable.

5. 135° **6.** −150° **7.** 1020°

Find all values of θ *in the interval* [0°, 360°) *that have the given function value.*

8. $\sin\theta = \frac{\sqrt{3}}{2}$ **9.** $\sec\theta = -\sqrt{2}$

Use a calculator to approximate each value.

10. $\sin 42^\circ 18'$ **11.** $\sec(-212^\circ 12')$

Use a calculator to find the value of θ *in the interval* [0°, 90°] that satisfies each statement.

12. $\tan\theta = 2.6743210$ **13.** $\csc\theta = 2.3861147$

Determine whether each statement is true *or* false.

14. $\sin(60^\circ + 30^\circ) = \sin 60^\circ + \sin 30^\circ$

15. $\tan(90^\circ - 35^\circ) = \cot 35^\circ$

2.4 Solving Right Triangles

Significant Digits ▪ Solving Triangles ▪ Angles of Elevation or Depression

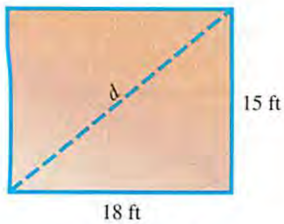


Figure 18

Significant Digits

A number that represents the result of counting, or a number that results from theoretical work and is not the result of measurement, is an **exact number.** There are 50 states in the United States, so in that statement, 50 is an exact number.

Most values obtained for trigonometric applications are measured values that are *not* exact. Suppose we quickly measure a room as 15 ft by 18 ft. See Figure 18. To calculate the length of a diagonal of the room, we can use the Pythagorean theorem.

$$d^2 = 15^2 + 18^2 \quad \text{(Appendix B)}$$
$$d^2 = 549$$
$$d = \sqrt{549} \approx 23.430749$$

Should this answer be given as the length of the diagonal of the room? Of course not. The number 23.430749 contains six decimal places, while the original data of 15 ft and 18 ft are only accurate to the nearest foot. In practice, the results of a calculation can be no more accurate than the least accurate number in the calculation. Thus, we should indicate that the diagonal of the 15-by-18-ft room is approximately 23 ft.

If a wall measured to the nearest foot is 18 ft long, this actually means that the wall has length between 17.5 ft and 18.5 ft. If the wall is measured more accurately as 18.3 ft long, then its length is really between 18.25 ft and 18.35 ft. The results of physical measurement are only approximately accurate and depend on the precision of the measuring instrument as well as the aptness of the observer. The digits obtained by actual measurement are called **significant digits.** The measurement 18 ft is said to have two significant digits; 18.3 ft has three significant digits.

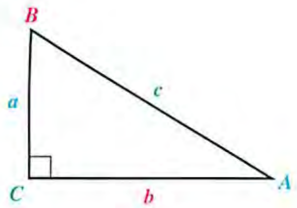
The following numbers have their significant digits identified in color.

408 21.5 18.00 6.700 .0025 .09810 7300

Notice that 18.00 has four significant digits. The zeros in this number represent measured digits and are significant. The number .0025 has only two significant digits, 2 and 5, because the zeros here are used only to locate the decimal point. The number 7300 causes some confusion because it is impossible to determine whether the zeros are measured values. The number 7300 may have two, three, or four significant digits. When presented with this situation, we assume that the zeros are not significant, unless the context of the problem indicates otherwise.

To determine the number of significant digits for answers in applications of angle measure, use the following table.

Angle Measure to Nearest	Examples	Answer to Number of Significant Digits
Degree	62°, 36°	2
Ten minutes, or nearest tenth of a degree	52° 30′, 60.4°	3
Minute, or nearest hundredth of a degree	81° 48′, 71.25°	4
Ten seconds, or nearest thousandth of a degree	10° 52′ 20″, 21.264°	5



When solving triangles, a labeled sketch is an important aid.

Figure 19

To perform calculations with measured numbers, start by identifying the number with the fewest significant digits. Round your final answer to the same number of significant digits as this number. ***Remember that your answer is no more accurate than the least accurate number in your calculation.***

Solving Triangles To *solve a triangle* means to find the measures of all the angles and sides of the triangle. As shown in Figure 19, we use a to represent the length of the side opposite angle A, b for the length of the side opposite angle B, and so on. In a right triangle, the letter c is reserved for the hypotenuse.

▶ EXAMPLE 1 SOLVING A RIGHT TRIANGLE GIVEN AN ANGLE AND A SIDE

Solve right triangle ABC, if $A = 34° 30'$ and $c = 12.7$ in. See Figure 20.

Solution To solve the triangle, find the measures of the remaining sides and angles. To find the value of a, use a trigonometric function involving the known values of angle A and side c. Since the sine of angle A is given by the quotient of the side opposite A and the hypotenuse, use $\sin A$.

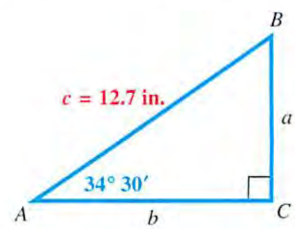


Figure 20

$$\sin A = \frac{a}{c} \qquad \sin A = \frac{\text{side opposite}}{\text{hypotenuse}} \text{ (Section 2.1)}$$

$$\sin 34° 30' = \frac{a}{12.7} \qquad A = 34° 30',\ c = 12.7$$

$$a = 12.7 \sin 34° 30' \qquad \text{Multiply by 12.7; rewrite.}$$

$$a = 12.7 \sin 34.5° \qquad \text{Convert to decimal degrees. (Section 1.1)}$$

$$a = 12.7(.56640624) \qquad \text{Use a calculator.}$$

$$a \approx 7.19 \text{ in.} \qquad \text{Three significant digits}$$

Assuming that 34° 30′ is given to the nearest ten minutes, we rounded the answer to three significant digits.

To find the value of b, we could substitute the value of a just calculated and the given value of c in the Pythagorean theorem. It is better, however, to use the information given in the problem rather than a result just calculated. If a mistake is made in finding a, then b also would be incorrect. Also, rounding more than once may cause the result to be less accurate. Use $\cos A$.

$$\cos A = \frac{b}{c} \qquad \cos A = \frac{\text{side adjacent}}{\text{hypotenuse}} \text{ (Section 2.1)}$$

$$\cos 34° 30' = \frac{b}{12.7}$$

$$b = 12.7 \cos 34° 30'$$

$$b \approx 10.5 \text{ in.}$$

Once b is found, the Pythagorean theorem can be used as a check.

▼ LOOKING AHEAD TO CALCULUS

The derivatives of the **parametric equations** $x = f(t)$ and $y = g(t)$ often represent the rate of change of physical quantities, such as velocities. When x and y are related by an equation, the derivatives are called **related rates** because a change in one causes a related change in the other. Determining these rates in calculus often requires solving a right triangle.

All that remains to solve triangle ABC is to find the measure of angle B. Since $A + B = 90°$,

$$B = 90° - A \qquad \text{(Section 1.1)}$$
$$B = 89° 60' - 34° 30' \qquad A = 34° 30'$$
$$B = 55° 30'.$$

NOW TRY EXERCISE 21.

▶ **Note** In Example 1, we could have found the measure of angle B first, and then used the trigonometric function values of B to find the unknown sides. The process of solving a right triangle can usually be done in several ways, each producing the correct answer. ***To maintain accuracy, always use given information as much as possible, and avoid rounding off in intermediate steps.***

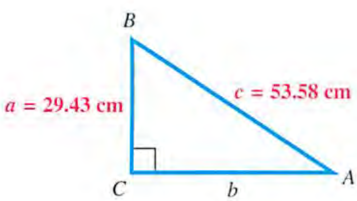


Figure 21

▶ EXAMPLE 2 SOLVING A RIGHT TRIANGLE GIVEN TWO SIDES

Solve right triangle ABC, if $a = 29.43$ cm and $c = 53.58$ cm.

Solution We draw a sketch showing the given information, as in Figure 21. One way to begin is to find angle A by using the sine function.

$$\sin A = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{29.43}{53.58} \approx .5492721165 \qquad \text{(Section 2.1)}$$
$$A = \sin^{-1} .5492721165 \approx 33.32° \qquad \text{(Section 2.3)}$$

The measure of B is approximately

$$90° - 33.32° = 56.68°.$$

We now find b from the Pythagorean theorem.

$$b^2 = c^2 - a^2 \qquad \text{Pythagorean theorem solved for } b^2 \text{ (Appendix B)}$$
$$b^2 = 53.58^2 - 29.43^2 \qquad c = 53.58,\ a = 29.43$$
$$b = \sqrt{2004.6915}$$
$$b \approx 44.77 \text{ cm}$$

Keep only the positive square root.

NOW TRY EXERCISE 31.

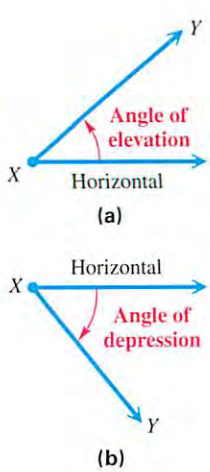


Figure 22

Angles of Elevation or Depression

Many applications of right triangles involve angles of elevation or depression. The **angle of elevation** from point X to point Y (above X) is the acute angle formed by ray XY and a horizontal ray with endpoint at X. See Figure 22(a). The **angle of depression** from point X to point Y (below X) is the acute angle formed by ray XY and a horizontal ray with endpoint X. See Figure 22(b).

▶ **Caution** Be careful when interpreting the angle of depression. ***Both the angle of elevation and the angle of depression are measured between the line of sight and a horizontal line.***

To solve applied trigonometry problems, follow the same procedure as solving a triangle.

SOLVING AN APPLIED TRIGONOMETRY PROBLEM

Step 1 Draw a sketch, and label it with the given information. Label the quantity to be found with a variable.

Step 2 Use the sketch to write an equation relating the given quantities to the variable.

Step 3 Solve the equation, and check that your answer makes sense.

Drawing a triangle and labeling it correctly in Step 1 is crucial.

▶ EXAMPLE 3 FINDING A LENGTH WHEN THE ANGLE OF ELEVATION IS KNOWN

Shelly McCarthy knows that when she stands 123 ft from the base of a flagpole, the angle of elevation to the top of the flagpole is 26° 40′. If her eyes are 5.30 ft above the ground, find the height of the flagpole.

Solution

Step 1 The length of the side adjacent to Shelly is known, and the length of the side opposite her must be found. See Figure 23.

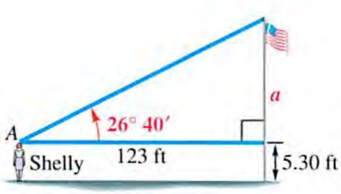


Figure 23

Step 2 The tangent ratio involves the given values. Write an equation.

$$\tan A = \frac{\text{side opposite}}{\text{side adjacent}} \qquad \text{Tangent ratio (Section 2.1)}$$

$$\tan 26^\circ\, 40' = \frac{a}{123} \qquad A = 26^\circ\, 40';\ \text{side adjacent} = 123$$

Step 3

$$a = 123 \tan 26^\circ\, 40' \qquad \text{Multiply by 123; rewrite.}$$

$$a = 123 \tan 26.66666667^\circ \qquad \text{Convert to decimal degrees.}$$

$$a = 123(.50221888) \qquad \text{Use a calculator.}$$

$$a \approx 61.8 \text{ ft} \qquad \text{Three significant digits}$$

Since Shelly's eyes are 5.30 ft above the ground, the height of the flagpole is

$$61.8 + 5.30 = 67.1 \text{ ft.}$$

NOW TRY EXERCISE 51. ◀

▶ EXAMPLE 4 FINDING THE ANGLE OF ELEVATION WHEN LENGTHS ARE KNOWN

The length of the shadow of a building 34.09 m tall is 37.62 m. Find the angle of elevation of the sun.

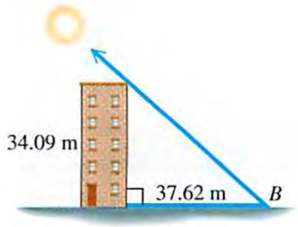


Figure 24

Solution As shown in Figure 24, the angle of elevation of the sun is angle B. Since the side opposite B and the side adjacent to B are known, use the tangent ratio to find B.

$$\tan B = \frac{34.09}{37.62}, \quad \text{so} \quad B = \tan^{-1}\frac{34.09}{37.62} \approx 42.18^\circ. \quad \text{(Section 2.3)}$$

The angle of elevation of the sun is 42.18°.

NOW TRY EXERCISE 47. ◀

George Polya (1887–1985)

CONNECTIONS George Polya proposed an excellent general outline for solving applied problems in his classic book *How to Solve It*.

1. Understand the problem.
2. Devise a plan.
3. Carry out the plan.
4. Look back.

Polya, a native of Budapest, Hungary, wrote more than 250 papers in many languages, as well as a number of books. He was a brilliant lecturer and teacher, and numerous mathematical properties and theorems bear his name. He once was asked why so many good mathematicians came out of Hungary at the turn of the century. He theorized that it was because mathematics was the cheapest science, requiring no expensive equipment, only pencil and paper.

For Discussion or Writing

Look back at the problem-solving steps given in this section, and compare them to Polya's four steps.

2.4 Exercises

Concept Check Refer to the discussion of accuracy and significant digits in this section to work Exercises 1–8.

1. ***Leading NFL Receiver*** At the end of the 2004 National Football League season, Jerry Rice was the leading career receiver with 22,895 yd. State the range represented by this number. (*Source:* www.nfl.com)

2. ***Height of Mt. Everest*** When Mt. Everest was first surveyed, the surveyors obtained a height of 29,000 ft to the nearest foot. State the range represented by this number. (The surveyors thought no one would believe a measurement of 29,000 ft, so they reported it as 29,002.) (*Source:* Dunham, W., *The Mathematical Universe,* John Wiley and Sons, 1994.)

3. ***Longest Vehicular Tunnel*** The E. Johnson Memorial Tunnel in Colorado, which measures 8959 ft, is one of the longest land vehicular tunnels in the United States. What is the range of this number? (*Source: The World Almanac and Book of Facts,* 2003.)

4. ***Top WNBA Scorer*** Women's National Basketball Association player Lauren Jackson of the Seattle Storm received the 2007 award for most points scored, 604. Is it appropriate to consider this number as between 603.5 and 604.5? Why or why not? (*Source:* www.wnba.com)

5. ***Circumference of a Circle*** The formula for the circumference of a circle is $C = 2\pi r$. Suppose you use the $\boxed{\pi}$ key on your calculator to find the circumference of a circle with radius 54.98 cm, getting 345.44953. Since 2 has only one significant digit, the answer should be given as 3×10^2, or 300 cm. Is this conclusion correct? If not, explain how the answer should be given.

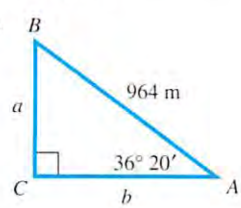
6. Explain the difference between a measurement of 23.0 ft and a measurement of 23.00 ft.

7. If h is the actual height of a building and the height is measured as 58.6 ft, then $|h - 58.6| \leq$ _____.

8. If w is the actual weight of a car and the weight is measured as 1542 lb, then $|w - 1542| \leq$ _____.

Solve each right triangle. When two sides are given, give angles in degrees and minutes. See Example 1.

9.



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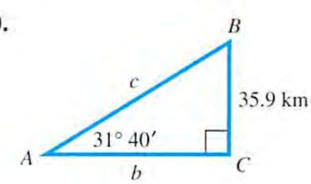

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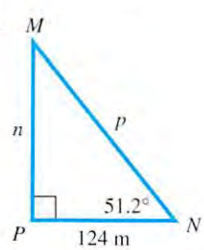

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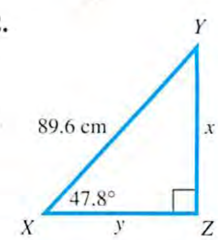

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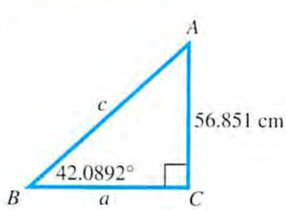

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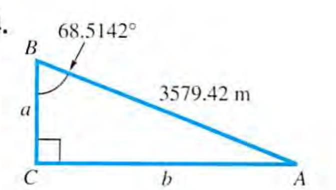

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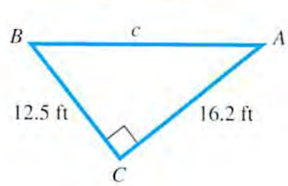

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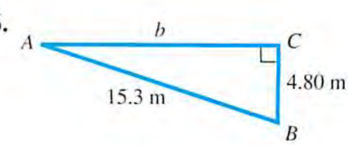


17. Can a right triangle be solved if we are given measures of its two acute angles and no side lengths? Explain.

18. ***Concept Check*** If we are given an acute angle and a side in a right triangle, what unknown part of the triangle requires the least work to find?

19. Explain why you can always solve a right triangle if you know the measures of one side and one acute angle.

20. Explain why you can always solve a right triangle if you know the lengths of two sides.

Solve each right triangle. In each case, $C = 90°$. If angle information is given in degrees and minutes, give answers in the same way. If given in decimal degrees, do likewise in answers. When two sides are given, give angles in degrees and minutes. See Examples 1 and 2.

21. $A = 28.0°$, $c = 17.4$ ft

22. $B = 46.0°$, $c = 29.7$ m

23. $B = 73.0°$, $b = 128$ in.

24. $A = 61.0°$, $b = 39.2$ cm

25. $A = 62.5°$, $a = 12.7$ m

26. $B = 51.7°$, $a = 28.1$ ft

27. $a = 13$ m, $c = 22$ m

28. $b = 32$ ft, $c = 51$ ft

29. $a = 76.4$ yd, $b = 39.3$ yd

30. $a = 958$ m, $b = 489$ m

31. $a = 18.9$ cm, $c = 46.3$ cm

32. $b = 219$ m, $c = 647$ m

33. $A = 53° 24'$, $c = 387.1$ ft

34. $A = 13° 47'$, $c = 1285$ m

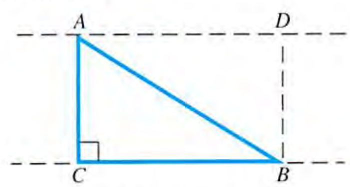
35. $B = 39° 9'$, $c = .6231$ m

36. $B = 82° 51'$, $c = 4.825$ cm

37. Explain the meaning of *angle of elevation.*

38. ***Concept Check*** Can an angle of elevation be more than 90°?

39. Explain why the angle of depression *DAB* has the same measure as the angle of elevation *ABC* in the figure.

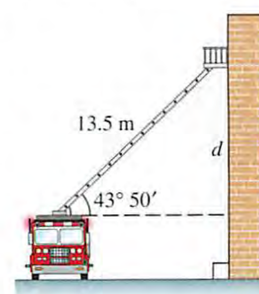


AD is parallel to *BC*.

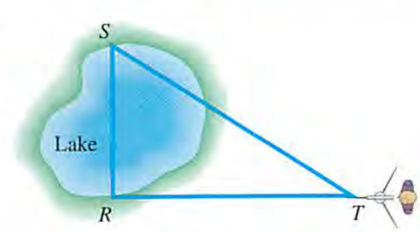
40. Why is angle *CAB* *not* an angle of depression in the figure for Exercise 39?

Solve each problem involving triangles. See Examples 1–4.

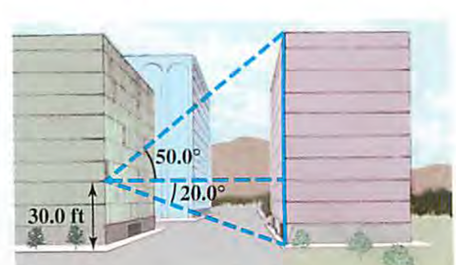
41. ***Height of a Ladder on a Wall*** A 13.5-m fire truck ladder is leaning against a wall. Find the distance *d* the ladder goes up the wall (above the top of the fire truck) if the ladder makes an angle of 43° 50′ with the horizontal.



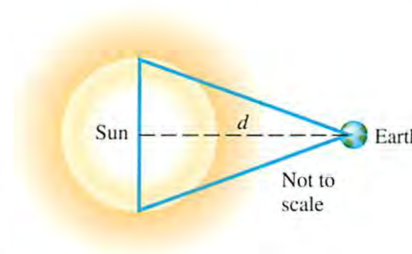
42. ***Distance Across a Lake*** To find the distance *RS* across a lake, a surveyor lays off length $RT = 53.1$ m, so that angle $T = 32° 10'$ and angle $S = 57° 50'$. Find length *RS*.



43. ***Height of a Building*** From a window 30.0 ft above the street, the angle of elevation to the top of the building across the street is 50.0° and the angle of depression to the base of this building is 20.0°. Find the height of the building across the street.



44. ***Diameter of the Sun*** To determine the diameter of the sun, an astronomer might sight with a **transit** (a device used by surveyors for measuring angles) first to one edge of the sun and then to the other, estimating that the included angle equals 32′. Assuming that the distance d from Earth to the sun is 92,919,800 mi, approximate the diameter of the sun.

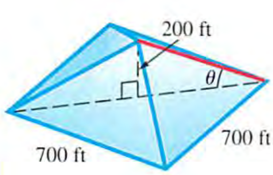


45. ***Side Lengths of a Triangle*** The length of the base of an isosceles triangle is 42.36 in. Each base angle is 38.12°. Find the length of each of the two equal sides of the triangle. (*Hint:* Divide the triangle into two right triangles.)

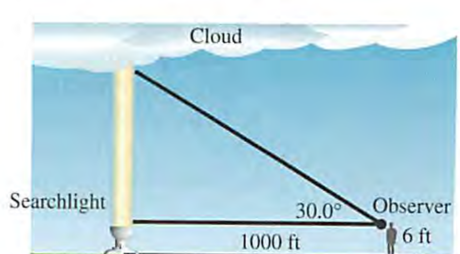
46. ***Altitude of a Triangle*** Find the altitude of an isosceles triangle having base 184.2 cm if the angle opposite the base is 68° 44′.

Solve each problem involving an angle of elevation or depression. See Examples 3 and 4.

47. ***Angle of Elevation of Pyramid of the Sun*** The Pyramid of the Sun in the ancient Mexican city of Teotihuacan was the largest and most important structure in the city. The base is a square with sides 700 ft long, and the height of the pyramid is 200 ft. Find the angle of elevation of the edge indicated in the figure to two significant digits. (*Hint:* The base of the triangle in the figure is half the diagonal of the square base of the pyramid.)



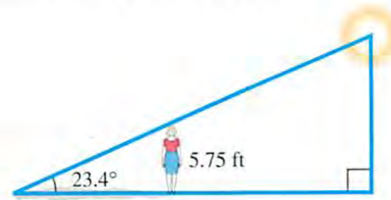
48. ***Cloud Ceiling*** The U.S. Weather Bureau defines a **cloud ceiling** as the altitude of the lowest clouds that cover more than half the sky. To determine a cloud ceiling, a powerful searchlight projects a circle of light vertically on the bottom of the cloud. An observer sights the circle of light in the crosshairs of a tube called a **clinometer.** A pendant hanging vertically from the tube and resting on a protractor gives the angle of elevation. Find the cloud ceiling if the searchlight is located 1000 ft from the observer and the angle of elevation is 30.0° as measured with a clinometer at eye-height 6 ft. (Assume three significant digits.)



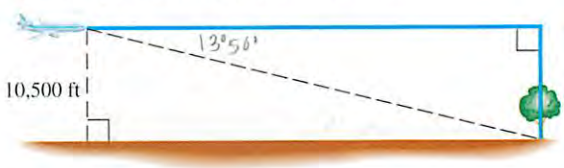
49. ***Height of a Tower*** The shadow of a vertical tower is 40.6 m long when the angle of elevation of the sun is 34.6°. Find the height of the tower.

50. ***Distance from the Ground to the Top of a Building*** The angle of depression from the top of a building to a point on the ground is 32° 30′. How far is the point on the ground from the top of the building if the building is 252 m high?

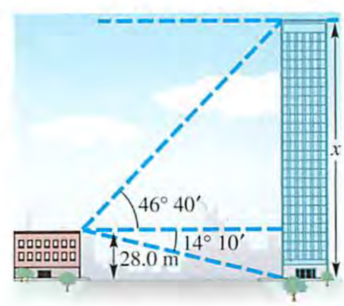
51. ***Length of a Shadow*** Suppose that the angle of elevation of the sun is 23.4°. Find the length of the shadow cast by Diane Carr, who is 5.75 ft tall.



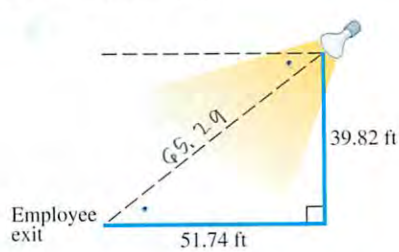
52. ***Airplane Distance*** An airplane is flying 10,500 ft above the level ground. The angle of depression from the plane to the base of a tree is 13° 50′. How far horizontally must the plane fly to be directly over the tree?



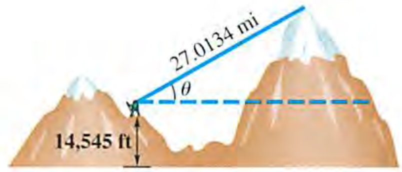
53. ***Height of a Building*** The angle of elevation from the top of a small building to the top of a nearby taller building is 46° 40′, while the angle of depression to the bottom is 14° 10′. If the shorter building is 28.0 m high, find the height of the taller building.



54. ***Angle of Depression of a Light*** A company safety committee has recommended that a floodlight be mounted in a parking lot so as to illuminate the employee exit. Find the angle of depression of the light.



55. ***Height of Mt. Everest*** The highest mountain peak in the world is Mt. Everest, located in the Himalayas. The height of this enormous mountain was determined in 1856 by surveyors using trigonometry long before it was first climbed in 1953. This difficult measurement had to be done from a great distance. At an altitude of 14,545 ft on a different mountain, the straight line distance to the peak of Mt. Everest is 27.0134 mi and its angle of elevation is $\theta = 5.82°$. (*Source:* Dunham, W., *The Mathematical Universe*, John Wiley and Sons, 1994.)



(a) Approximate the height (in feet) of Mt. Everest.

(b) In the actual measurement, Mt. Everest was over 100 mi away and the curvature of Earth had to be taken into account. Would the curvature of Earth make the peak appear taller or shorter than it actually is?

56. ***Error in Measurement*** A degree may seem like a very small unit, but an error of one degree in measuring an angle may be very significant. For example, suppose a laser beam directed toward the visible center of the moon misses its assigned target by 30 sec. How far is it (in miles) from its assigned target? Take the distance from the surface of Earth to that of the moon to be 234,000 mi. (*Source: A Sourcebook of Applications of School Mathematics* by Donald Bushaw et al. Copyright © 1980 by The Mathematical Association of America.)

2.5 Further Applications of Right Triangles

Bearing ▪ Further Applications

Bearing Other applications of right triangles involve **bearing,** an important concept in navigation. There are two methods for expressing bearing. When a single angle is given, such as 164°, it is understood that the bearing is measured in a clockwise direction from due north. Several sample bearings using this first method are shown in Figure 25.

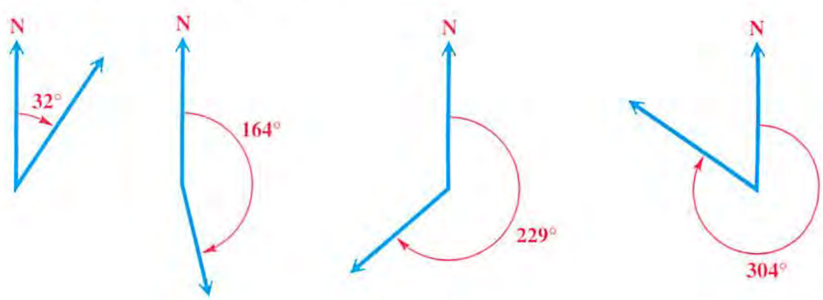


Figure 25

▶ EXAMPLE 1 SOLVING A PROBLEM INVOLVING BEARING (FIRST METHOD)

Radar stations A and B are on an east-west line, 3.7 km apart. Station A detects a plane at C, on a bearing of 61°. Station B simultaneously detects the same plane, on a bearing of 331°. Find the distance from A to C.

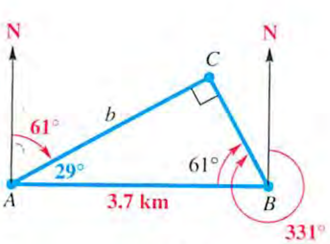


Figure 26

Solution Draw a sketch showing the given information, as in Figure 26. Since a line drawn due north is perpendicular to an east-west line, right angles are formed at A and B, so angles CAB and CBA can be found as shown in Figure 26. Angle C is a right angle because angles CAB and CBA are complementary. Find distance b by using the cosine function for angle A.

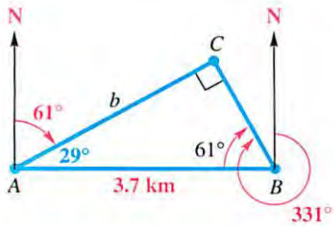


Figure 26 (repeated)

$$\cos 29^\circ = \frac{b}{3.7} \quad \text{(Section 2.1)}$$

$$3.7 \cos 29^\circ = b \quad \text{Multiply by 3.7.}$$

$$b \approx 3.2 \text{ km} \quad \text{Use a calculator; round to the nearest tenth.}$$

NOW TRY EXERCISE 15.

> ▶ **Caution** ***A correctly labeled sketch is crucial*** when solving applications like that in Example 1. Some of the necessary information is often not directly stated in the problem and can be determined only from the sketch.

The second method for expressing bearing starts with a north-south line and uses an acute angle to show the direction, either east or west, from this line. Figure 27 shows several sample bearings using this system. Either N or S always comes first, followed by an acute angle, and then E or W.

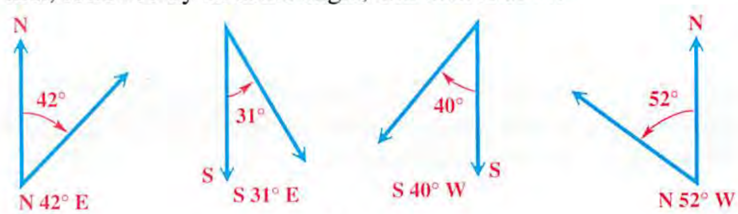


Figure 27

▶ EXAMPLE 2 SOLVING A PROBLEM INVOLVING BEARING (SECOND METHOD)

The bearing from A to C is S 52° E. The bearing from A to B is N 84° E. The bearing from B to C is S 38° W. A plane flying at 250 mph takes 2.4 hr to go from A to B. Find the distance from A to C.

Solution Make a sketch. First draw the two bearings from point A. Choose a point B on the bearing N 84° E from A, and draw the bearing to C. Point C will be located where the bearing lines from A and B intersect, as shown in Figure 28.

Since the bearing from A to B is N 84° E, angle ABD is $180^\circ - 84^\circ = 96^\circ$. Thus, angle ABC is $180^\circ - (96^\circ + 38^\circ) = 46^\circ$. Also, angle BAC is $180^\circ - (84^\circ + 52^\circ) = 44^\circ$. Angle C is $180^\circ - (44^\circ + 46^\circ) = 90^\circ$. Since a plane flying at 250 mph takes 2.4 hr to go from A to B, the distance from A to B is

$$c = \text{rate} \times \text{time} = 250(2.4) = 600 \text{ mi}.$$

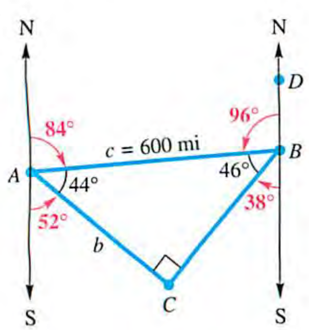


Figure 28

To find b, the distance from A to C, use the sine. (The cosine could also be used.)

$$\sin 46^\circ = \frac{b}{c} \quad \text{(Section 2.1)}$$

$$\sin 46^\circ = \frac{b}{600} \quad \text{Let } c = 600.$$

$$600 \sin 46^\circ = b \quad \text{Multiply by 600.}$$

$$b \approx 430 \text{ mi}$$

NOW TRY EXERCISE 19.

Further Applications

EXAMPLE 3 USING TRIGONOMETRY TO MEASURE A DISTANCE

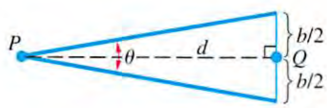


Figure 29

A method that surveyors use to determine a small distance d between two points P and Q is called the **subtense bar method.** The subtense bar with length b is centered at Q and situated perpendicular to the line of sight between P and Q. See Figure 29. Angle θ is measured, and then the distance d can be determined.

(a) Find d when $\theta = 1° 23' 12''$ and $b = 2.0000$ cm.

(b) Angle θ usually cannot be measured more accurately than to the nearest $1''$. How much change would there be in the value of d if θ were measured $1''$ larger?

Solution

(a) From Figure 29, we see that

$$\cot \frac{\theta}{2} = \frac{d}{\frac{b}{2}}$$

$$d = \frac{b}{2} \cot \frac{\theta}{2}. \quad \text{Multiply; rewrite.}$$

Let $b = 2$. To evaluate $\frac{\theta}{2}$, we change θ to decimal degrees.

$$1° 23' 12'' \approx 1.386667° \quad \text{(Section 1.1)}$$

Then $$d = \frac{2}{2} \cot \frac{1.386667°}{2} \approx 82.634110 \text{ cm}.$$

(b) Since θ is $1''$ larger, use $\theta = 1° 23' 13'' \approx 1.386944°$.

$$d = \frac{2}{2} \cot \frac{1.386944°}{2} \approx 82.617558 \text{ cm}$$

The difference is $82.634110 - 82.617558 = .016552$ cm.

NOW TRY EXERCISE 33.

EXAMPLE 4 SOLVING A PROBLEM INVOLVING ANGLES OF ELEVATION

Francisco needs to know the height of a tree. From a given point on the ground, he finds that the angle of elevation to the top of the tree is 36.7°. He then moves back 50 ft. From the second point, the angle of elevation to the top of the tree is 22.2°. See Figure 30. Find the height of the tree to the nearest foot.

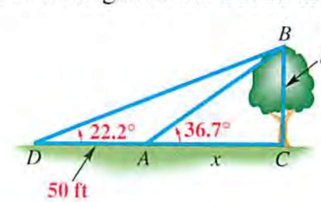


Figure 30

Algebraic Solution

Figure 30 shows two unknowns: x, the distance from the center of the trunk of the tree to the point where the first observation was made, and h, the height of the tree. See Figure 31 in the Graphing Calculator Solution. Since nothing is given about the length of the hypotenuse of either triangle ABC or triangle BCD, use a ratio that does not involve the hypotenuse—namely, the tangent.

In triangle ABC, $\tan 36.7° = \frac{h}{x}$ or $h = x \tan 36.7°$.

In triangle BCD, $\tan 22.2° = \frac{h}{50 + x}$ or $h = (50 + x) \tan 22.2°$.

Each expression equals h, so the expressions must be equal.

$$x \tan 36.7° = (50 + x) \tan 22.2°$$
Solve for x.

$$x \tan 36.7° = 50 \tan 22.2° + x \tan 22.2°$$
Distributive property

$$x \tan 36.7° - x \tan 22.2° = 50 \tan 22.2°$$
Get x-terms on one side.

$$x(\tan 36.7° - \tan 22.2°) = 50 \tan 22.2°$$
Factor out x.

$$x = \frac{50 \tan 22.2°}{\tan 36.7° - \tan 22.2°}$$
Divide by the coefficient of x.

We saw above that $h = x \tan 36.7°$. Substituting for x,

$$h = \left(\frac{50 \tan 22.2°}{\tan 36.7° - \tan 22.2°}\right) \tan 36.7°.$$

Using a calculator,

$$\tan 36.7° = .74537703 \quad \text{and} \quad \tan 22.2° = .40809244,$$

so $\tan 36.7° - \tan 22.2° = .74537703 - .40809244 = .33728459$

and $$h = \left(\frac{50(.40809244)}{.33728459}\right).74537703 \approx 45.$$

The height of the tree is approximately 45 ft.

Graphing Calculator Solution*

In Figure 31, we superimposed Figure 30 on coordinate axes with the origin at D. By definition, the tangent of the angle between the x-axis and the graph of a line with equation $y = mx + b$ is the slope of the line, m. For line DB, $m = \tan 22.2°$. Since b equals 0, the equation of line DB is $y_1 = (\tan 22.2°)x$. The equation of line AB is $y_2 = (\tan 36.7°)x + b$. Since $b \neq 0$ here, we use the point $A(50, 0)$ and the point-slope form to find the equation.

$$y_2 - y_1 = m(x - x_1)$$
$$y_2 - 0 = m(x - 50) \quad x_1 = 50, y_1 = 0$$
$$y_2 = \tan 36.7°(x - 50)$$

Lines y_1 and y_2 are graphed in Figure 32. The y-coordinate of the point of intersection of the graphs gives the length of BC, or h. Thus, $h \approx 45$.

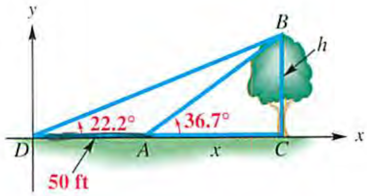


Figure 31

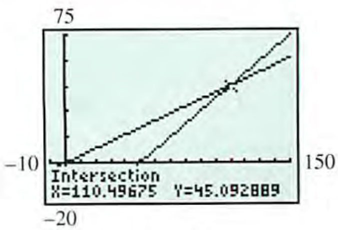


Figure 32

NOW TRY EXERCISE 27.

▶ **Note** In practice, we usually do not write down intermediate calculator approximation steps. We did in Example 4 so you could follow the steps more easily.

*Source: Adapted with permission from "Letter to the Editor," by Robert Ruzich (*Mathematics Teacher*, Volume 88, Number 1). Copyright © 1995 by the National Council of Teachers of Mathematics.

2.5 Exercises

Concept Check *Give a short written answer to each question.*

1. When bearing is given as a single angle measure, how is the angle represented in a sketch?
2. When bearing is given as N (or S), then the angle measure, and then E (or W), how is the angle represented in a sketch?
3. Why is it important to draw a sketch before solving trigonometric problems like those in the last two sections of this chapter?
4. How should the angle of elevation (or depression) from a point X to a point Y be represented?

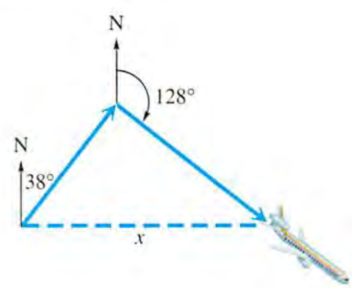
Concept Check *An observer for a radar station is located at the origin of a coordinate system. For each of the points in Exercises 5–12, find the bearing of an airplane located at that point. Express the bearing using both methods.*

5. $(-4, 0)$	6. $(-3, -3)$	7. $(-5, 5)$	8. $(0, -2)$
9. $(0, 4)$	10. $(2, 2)$	11. $(2, -2)$	12. $(5, 0)$

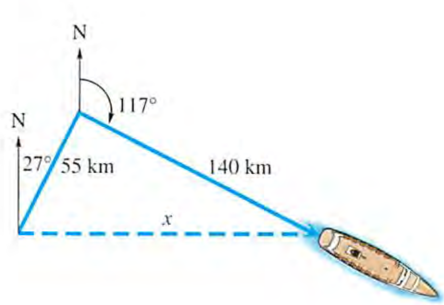
13. The ray $y = x$, $x \geq 0$, contains the origin and all points in the coordinate system whose bearing is 45°. Determine the equation of a ray consisting of the origin and all points whose bearing is 240°.
14. Repeat Exercise 13 for a bearing of 150°.

Work each problem. In these exercises, assume the course of a plane or ship is on the indicated bearing. See Examples 1 and 2.

15. *Distance Flown by a Plane* A plane flies 1.3 hr at 110 mph on a bearing of 38°. It then turns and flies 1.5 hr at the same speed on a bearing of 128°. How far is the plane from its starting point?



16. *Distance Traveled by a Ship* A ship travels 55 km on a bearing of 27°, then travels on a bearing of 117° for 140 km. Find the distance traveled from the starting point to the ending point.



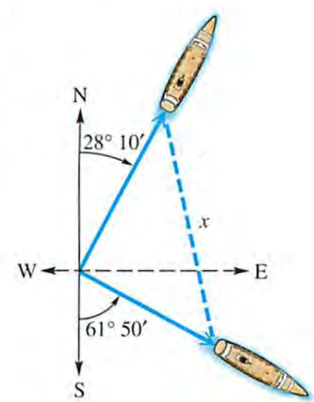
17. *Distance Between Two Ships* Two ships leave a port at the same time. The first ship sails on a bearing of 40° at 18 knots (nautical miles per hour) and the second at a bearing of 130° at 26 knots. How far apart are they after 1.5 hr?

18. *Distance Between Two Lighthouses* Two lighthouses are located on a north-south line. From lighthouse A, the bearing of a ship 3742 m away is 129° 43′. From lighthouse B, the bearing of the ship is 39° 43′. Find the distance between the lighthouses.

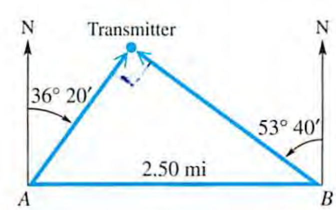
19. *Distance Between Two Cities* The bearing from Winston-Salem, North Carolina, to Danville, Virginia, is N 42° E. The bearing from Danville to Goldsboro, North Carolina, is S 48° E. A car driven by Mark Ferrari, traveling at 65 mph, takes 1.1 hr to go from Winston-Salem to Danville and 1.8 hr to go from Danville to Goldsboro. Find the distance from Winston-Salem to Goldsboro.

20. *Distance Between Two Cities* The bearing from Atlanta to Macon is S 27° E, and the bearing from Macon to Augusta is N 63° E. An automobile traveling at 62 mph needs $1\frac{1}{4}$ hr to go from Atlanta to Macon and $1\frac{3}{4}$ hr to go from Macon to Augusta. Find the distance from Atlanta to Augusta.

21. *Distance Between Two Ships* A ship leaves its home port and sails on a bearing of S 61° 50′ E. Another ship leaves the same port at the same time and sails on a bearing of N 28° 10′ E. If the first ship sails at 24.0 mph and the second sails at 28.0 mph, find the distance between the two ships after 4 hr.



22. *Distance Between Transmitters* Radio direction finders are set up at two points A and B, which are 2.50 mi apart on an east-west line. From A, it is found that the bearing of a signal from a radio transmitter is N 36° 20′ E, while from B the bearing of the same signal is N 53° 40′ W. Find the distance of the transmitter from B.



23. Solve the equation $ax = b + cx$ for x in terms of a, b, and c. (*Note:* This is in essence the calculation carried out in Example 4.)

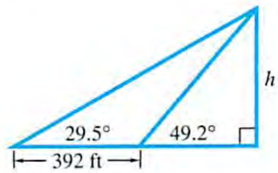
24. Explain why the line $y = (\tan \theta)(x - a)$ passes through the point $(a, 0)$ and makes an angle θ with the x-axis.

25. Find the equation of the line passing through the point $(25, 0)$ that makes an angle of 35° with the x-axis.

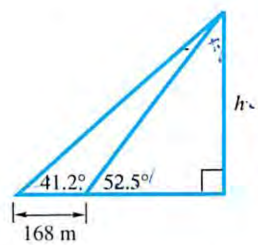
26. Find the equation of the line passing through the point $(5, 0)$ that makes an angle of 15° with the x-axis.

In Exercises 27–32, use the method of Example 4.

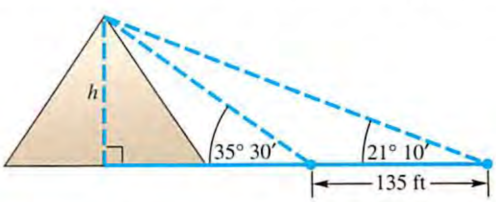
27. Find h as indicated in the figure.



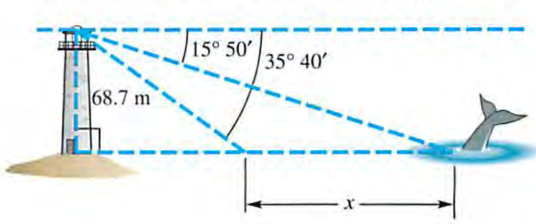
28. Find h as indicated in the figure.



29. *Height of a Pyramid* The angle of elevation from a point on the ground to the top of a pyramid is 35° 30′. The angle of elevation from a point 135 ft farther back to the top of the pyramid is 21° 10′. Find the height of the pyramid.



30. *Distance Between a Whale and a Lighthouse* Debbie Glockner-Ferrari, a whale researcher, is watching a whale approach directly toward a lighthouse as she observes from the top of this lighthouse. When she first begins watching the whale, the angle of depression to the whale is 15° 50′. Just as the whale turns away from the lighthouse, the angle of depression is 35° 40′. If the height of the lighthouse is 68.7 m, find the distance traveled by the whale as it approaches the lighthouse.

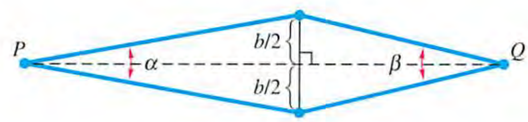


31. *Height of an Antenna* A scanner antenna is on top of the center of a house. The angle of elevation from a point 28.0 m from the center of the house to the top of the antenna is 27° 10′, and the angle of elevation to the bottom of the antenna is 18° 10′. Find the height of the antenna.

32. *Height of Mt. Whitney* The angle of elevation from Lone Pine to the top of Mt. Whitney is 10° 50′. Van Dong Le, traveling 7.00 km from Lone Pine along a straight, level road toward Mt. Whitney, finds the angle of elevation to be 22° 40′. Find the height of the top of Mt. Whitney above the level of the road.

Solve each problem.

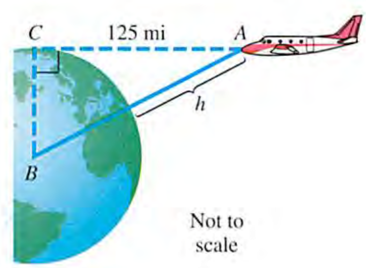
33. ***(Modeling) Distance Between Two Points*** Refer to Example 3. A variation of the subtense bar method that surveyors use to determine larger distances d between two points P and Q is shown in the figure. In this case the subtense bar with length b is placed between the points P and Q so that the bar is centered on and perpendicular to the line of sight connecting P and Q. The angles α and β are measured from points P and Q, respectively. (*Source:* Mueller, I. and K. Ramsayer, *Introduction to Surveying,* Frederick Ungar Publishing Co., 1979.)



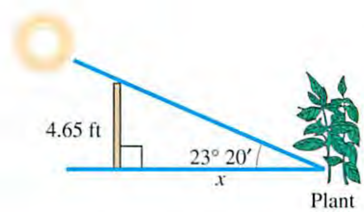
(a) Find a formula for d involving α, β, and b.

(b) Use your formula to determine d if $\alpha = 37' 48''$, $\beta = 42' 3''$, and $b = 2.000$ cm.

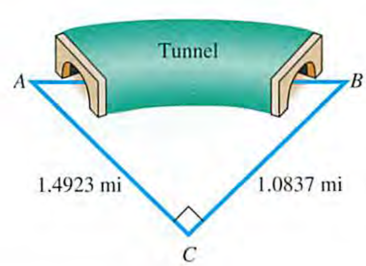
34. ***Height of a Plane Above Earth*** Find the minimum height h above the surface of Earth so that a pilot at point A in the figure can see an object on the horizon at C, 125 mi away. Assume that the radius of Earth is 4.00×10^3 mi.



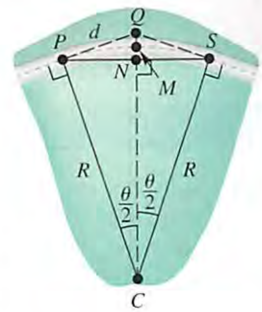
35. ***Distance of a Plant from a Fence*** In one area, the lowest angle of elevation of the sun in winter is 23° 20′. Find the minimum distance x that a plant needing full sun can be placed from a fence 4.65 ft high.



36. ***Distance Through a Tunnel*** A tunnel is to be dug from A to B. Both A and B are visible from C. If AC is 1.4923 mi and BC is 1.0837 mi, and if C is 90°, find the measures of angles A and B.



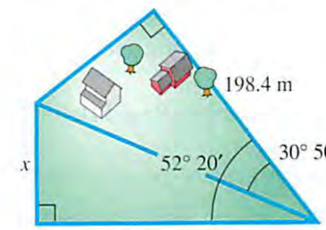
37. ***(Modeling) Highway Curves*** A basic highway curve connecting two straight sections of road is often circular. In the figure, the points P and S mark the beginning and end of the curve. Let Q be the point of intersection where the two straight sections of highway leading into the curve would meet if extended. The radius of the curve is R, and the central angle θ denotes how many degrees the curve turns. (*Source:* Mannering, F. and W. Kilareski, *Principles of Highway Engineering and Traffic Analysis,* Second Edition, John Wiley and Sons, 1998.)



(a) If $R = 965$ ft and $\theta = 37°$, find the distance d between P and Q.

(b) Find an expression in terms of R and θ for the distance between points M and N.

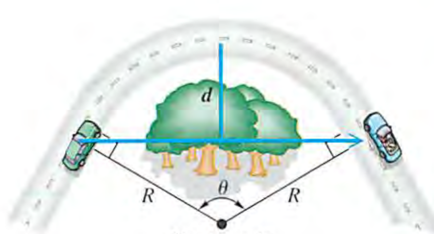
38. ***Length of a Side of a Piece of Land*** A piece of land has the shape shown in the figure. Find x.



39. ***(Modeling) Stopping Distance on a Curve*** Refer to Exercise 37. When an automobile travels along a circular curve, objects like trees and buildings situated on the inside of the curve can obstruct the driver's vision. These obstructions prevent the driver from seeing sufficiently far down the highway to ensure a safe stopping distance. In the figure, the *minimum* distance d that should be cleared on the inside of the highway is modeled by the equation

$$d = R\left(1 - \cos\frac{\theta}{2}\right).$$

(*Source:* Mannering, F. and W. Kilareski, *Principles of Highway Engineering and Traffic Analysis,* Second Edition, John Wiley and Sons, 1998.)



(a) It can be shown that if θ is measured in degrees, then $\theta \approx \frac{57.3S}{R}$, where S is the safe stopping distance for the given speed limit. Compute d to the nearest foot for a 55 mph speed limit if $S = 336$ ft and $R = 600$ ft.

(b) Compute d to the nearest foot for a 65 mph speed limit if $S = 485$ ft and $R = 600$ ft.

(c) How does the speed limit affect the amount of land that should be cleared on the inside of the curve?

Chapter 2 Summary

KEY TERMS

2.1 side opposite, side adjacent, cofunctions

2.2 reference angle

2.4 exact number, significant digits, angle of elevation, angle of depression

2.5 bearing

QUICK REVIEW

CONCEPTS	EXAMPLES

2.1 Trigonometric Functions of Acute Angles

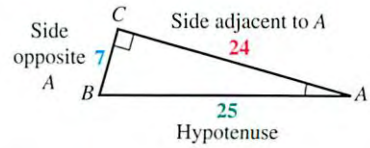
Right-Triangle-Based Definitions of the Trigonometric Functions

For any acute angle A in standard position,

$$\sin A = \frac{y}{r} = \frac{\text{side opposite}}{\text{hypotenuse}} \qquad \csc A = \frac{r}{y} = \frac{\text{hypotenuse}}{\text{side opposite}}$$

$$\cos A = \frac{x}{r} = \frac{\text{side adjacent}}{\text{hypotenuse}} \qquad \sec A = \frac{r}{x} = \frac{\text{hypotenuse}}{\text{side adjacent}}$$

$$\tan A = \frac{y}{x} = \frac{\text{side opposite}}{\text{side adjacent}} \qquad \cot A = \frac{x}{y} = \frac{\text{side adjacent}}{\text{side opposite}}.$$



$$\sin A = \frac{7}{25} \qquad \cos A = \frac{24}{25} \qquad \tan A = \frac{7}{24}$$

$$\csc A = \frac{25}{7} \qquad \sec A = \frac{25}{24} \qquad \cot A = \frac{24}{7}$$

Cofunction Identities

For any acute angle A,

$$\sin A = \cos(90^\circ - A) \qquad \cos A = \sin(90^\circ - A)$$

$$\sec A = \csc(90^\circ - A) \qquad \csc A = \sec(90^\circ - A)$$

$$\tan A = \cot(90^\circ - A) \qquad \cot A = \tan(90^\circ - A).$$

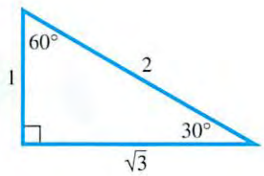
$$\sin 55^\circ = \cos(90^\circ - 55^\circ) = \cos 35^\circ$$

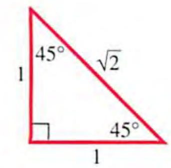
$$\sec 48^\circ = \csc(90^\circ - 48^\circ) = \csc 42^\circ$$

$$\tan 72^\circ = \cot(90^\circ - 72^\circ) = \cot 18^\circ$$

Function Values of Special Angles

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$





CONCEPTS	EXAMPLES

2.2 Trigonometric Functions of Non-Acute Angles

Reference Angle θ' for θ in (0°, 360°)

θ in Quadrant	I	II	III	IV
θ' is	θ	$180° - \theta$	$\theta - 180°$	$360° - \theta$

See the figure on page 60 for illustrations of reference angles.

Quadrant I: For $\theta = 25°$, $\theta' = 25°$
Quadrant II: For $\theta = 152°$, $\theta' = 28°$
Quadrant III: For $\theta = 200°$, $\theta' = 20°$
Quadrant IV: For $\theta = 320°$, $\theta' = 40°$

Finding Trigonometric Function Values for Any Nonquadrantal Angle θ

Step 1 Add or subtract 360° as many times as needed to get an angle greater than 0° but less than 360°.
Step 2 Find the reference angle θ'.
Step 3 Find the trigonometric function values for θ'.
Step 4 Determine the correct signs for the values found in Step 3.

Find sin 1050°.

$$1050° - 2(360°) = 330°$$

Thus, $\theta' = 30°$.

$$\sin 1050° = -\sin 30° = -\frac{1}{2}$$

2.3 Finding Trigonometric Function Values Using a Calculator

To approximate a trigonometric function value of an angle in degrees, make sure your calculator is in degree mode.

Approximate each value.

$$\cos 50°\ 15' = \cos 50.25° \approx .63943900$$

$$\csc 32.5° = \frac{1}{\sin 32.5°} \approx 1.86115900$$

To find the corresponding angle measure given a trigonometric function value, use an appropriate inverse function.

Find an angle θ in the interval [0°, 90°] that satisfies each condition.

$$\cos \theta \approx .73677482$$
$$\theta \approx \cos^{-1}(.73677482)$$
$$\theta \approx 42.542600°$$

$$\csc \theta \approx 1.04766792$$
$$\sin \theta \approx \frac{1}{1.04766792} \qquad \sin \theta = \frac{1}{\csc \theta}$$
$$\theta \approx \sin^{-1}\left(\frac{1}{1.04766792}\right)$$
$$\theta \approx 72.65°$$

(continued)

CONCEPTS	EXAMPLES

2.4 Solving Right Triangles

Solving an Applied Trigonometry Problem

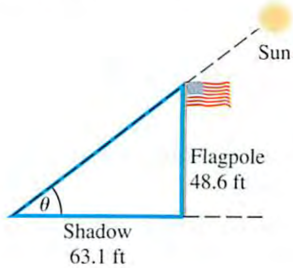
Step 1 Draw a sketch, and label it with the given information. Label the quantity to be found with a variable.

Step 2 Use the sketch to write an equation relating the given quantities to the variable.

Step 3 Solve the equation, and check that your answer makes sense.

Find the angle of elevation of the sun if a 48.6-ft flagpole casts a shadow 63.1 ft long.

Step 1 See the sketch. We must find θ.



Step 2 $$\tan\theta = \frac{48.6}{63.1} \approx .770206$$

Step 3 $$\theta = \tan^{-1} .770206 \approx 37.6°$$

The angle of elevation rounded to three significant digits is 37.6°, or 37° 40′.

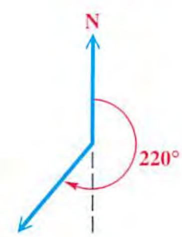
2.5 Further Applications of Right Triangles

Expressing Bearing

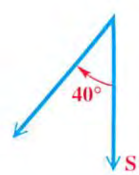
Method 1 When a single angle is given, such as 220°, this bearing is measured in a clockwise direction from north.

Method 2 Start with a north-south line and use an acute angle to show direction, either east or west, from this line.

220°



S 40° W

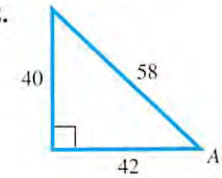


CHAPTER 2 ▶ Review Exercises

Find the values of the six trigonometric functions for each angle A.

1.

2.



Find one solution for each equation. Assume that all angles are acute angles.

3. $\sin 4\beta = \cos 5\beta$

4. $\sec(2\theta + 10°) = \csc(4\theta + 20°)$

5. $\tan(5x + 11°) = \cot(6x + 2°)$

6. $\cos\left(\frac{3\theta}{5} + 11°\right) = \sin\left(\frac{7\theta}{10} + 40°\right)$

Tell whether each statement is true *or* false. *If false, tell why.*

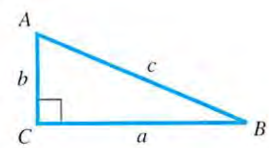
7. $\sin 46° < \sin 58°$

8. $\cos 47° < \cos 58°$

9. $\sec 48° \geq \cos 42°$

10. $\sin 22° \geq \csc 68°$

11. Explain why, in the figure, the cosine of angle A is equal to the sine of angle B.



Find exact values of the six trigonometric functions for each angle. Do not use a calculator. Rationalize denominators when applicable.

12. $120°$ **13.** $1020°$ **14.** $-225°$ **15.** $-1470°$

Find all values of θ, if θ is in the interval [0°, 360°) and θ has the given function value.

16. $\sin\theta = -\frac{1}{2}$

17. $\cos\theta = -\frac{1}{2}$

18. $\cot\theta = -1$

19. $\sec\theta = -\frac{2\sqrt{3}}{3}$

Evaluate each expression. Give exact values.

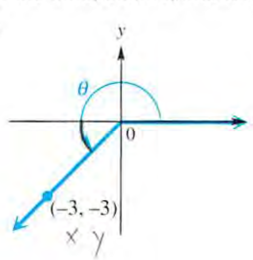
20. $\cos 60° + 2\sin^2 30°$

21. $\tan^2 120° - 2\cot 240°$

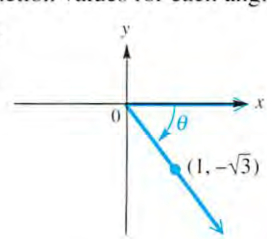
22. $\sec^2 300° - 2\cos^2 150° + \tan 45°$

23. Find the sine, cosine, and tangent function values for each angle.

(a)



(b)



Use a calculator to find each value.

24. $\sin 72° 30'$ **25.** $\sec 222° 30'$ **26.** $\cot 305.6°$

27. $\csc 78° 21'$ **28.** $\sec 58.9041°$ **29.** $\tan 11.7689°$

30. ***Concept Check*** Which one of the following cannot be *exactly* determined using the methods of this chapter?

A. $\cos 135°$ **B.** $\cot(-45°)$ **C.** $\sin 300°$ **D.** $\tan 140°$

Use a calculator to find each value of θ, where θ is in the interval $[0°, 90°)$. Give answers in decimal degrees.

31. $\sin \theta = .82584121$ **32.** $\cot \theta = 1.1249386$ **33.** $\cos \theta = .97540415$

34. $\sec \theta = 1.2637891$ **35.** $\tan \theta = 1.9633124$ **36.** $\csc \theta = 9.5670466$

Find two angles in the interval $[0°, 360°)$ that satisfy each of the following. Leave answers in decimal degrees rounded to the nearest tenth.

37. $\sin \theta = .73254290$ **38.** $\tan \theta = 1.3865342$

Tell whether each statement is true *or* false. *If false, tell why. Use a calculator for Exercises 39 and 42.*

39. $\sin 50° + \sin 40° = \sin 90°$

40. $\cos 210° = \cos 180° \cdot \cos 30° - \sin 180° \cdot \sin 30°$

41. $\sin 240° = 2 \sin 120° \cdot \cos 120°$

42. $\sin 42° + \sin 42° = \sin 84°$

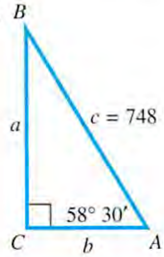
43. A student wants to use a calculator to find the value of $\cot 25°$. However, instead of entering $\frac{1}{\tan 25}$, he enters $\tan^{-1} 25$. Assuming the calculator is in degree mode, will this produce the correct answer? Explain.

For each angle θ, use a calculator to find $\cos \theta$ *and* $\sin \theta$. *Use your results to decide in which quadrant the angle lies.*

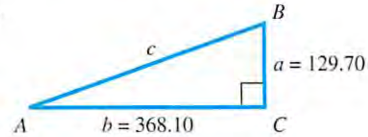
44. $\theta = 2976°$ **45.** $\theta = 1997°$ **46.** $\theta = 4000°$

Solve each right triangle. In Exercise 48, give angles to the nearest minute. In Exercises 49 and 50, label the triangle ABC as in Exercises 47 and 48.

47.



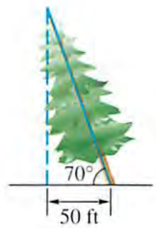
48.



49. $A = 39.72°$, $b = 38.97$ m **50.** $B = 47° 53'$, $b = 298.6$ m

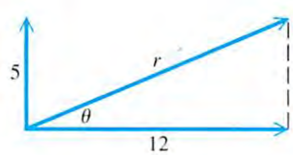
Solve each problem. (*Source for Exercises 51 and 52:* Parker, M., Editor, *She Does Math,* Mathematical Association of America, 1995.)

51. ***Height of a Tree*** A civil engineer must determine the height of the tree shown in the figure. The given angle was measured with a **clinometer.** Find the height of the tree to the nearest whole number.

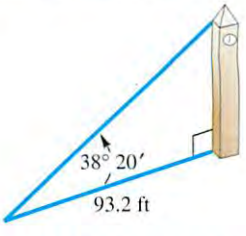


This is a picture of one type of clinometer, called an Abney hand level and clinometer. (Courtesy of Keuffel & Esser Co.)

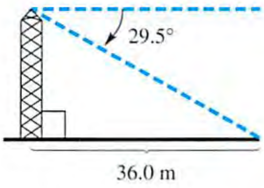
52. ***(Modeling) Double Vision*** To correct mild double vision, a small amount of prism is added to a patient's eyeglasses. The amount of light shift this causes is measured in **prism diopters.** A patient needs 12 prism diopters horizontally and 5 prism diopters vertically. A prism that corrects for both requirements should have length r and be set at angle θ. Find the values of r and θ in the figure.



53. ***Height of a Tower*** The angle of elevation from a point 93.2 ft from the base of a tower to the top of the tower is 38° 20′. Find the height of the tower.



54. ***Height of a Tower*** The angle of depression of a television tower to a point on the ground 36.0 m from the bottom of the tower is 29.5°. Find the height of the tower.



55. ***Length of a Diagonal*** One side of a rectangle measures 15.24 cm. The angle between the diagonal and that side is 35.65°. Find the length of the diagonal.

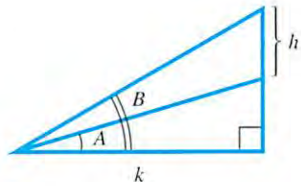
56. ***Length of Sides of an Isosceles Triangle*** An isosceles triangle has a base of length 49.28 m. The angle opposite the base is 58.746°. Find the length of each of the two equal sides.

57. ***Distance Between Two Points*** The bearing of point B from point C is 254°. The bearing of point A from point C is 344°. The bearing of point A from point B is 32°. If the distance from A to C is 780 m, find the distance from A to B.

58. ***Distance a Ship Sails*** The bearing from point A to point B is S 55° E and from point B to point C is N 35° E. If a ship sails from A to B, a distance of 81 km, and then from B to C, a distance of 74 km, how far is it from A to C?

59. ***Distance Between Two Points*** Two cars leave an intersection at the same time. One heads due south at 55 mph. The other travels due west. After 2 hr, the bearing of the car headed west from the car headed south is 324°. How far apart are they at that time?

60. Find a formula for h in terms of k, A, and B. Assume $A < B$.



61. Make up a right triangle problem whose solution is 3 tan 25°.

62. Make up a right triangle problem whose solution is found from $\sin\theta = \frac{3}{4}$.

63. ***(Modeling) Height of a Satellite*** Artificial satellites that orbit Earth often use VHF signals to communicate with the ground. VHF signals travel in straight lines. The height h of the satellite above Earth and the time T that the satellite can communicate with a fixed location on the ground are related by the model

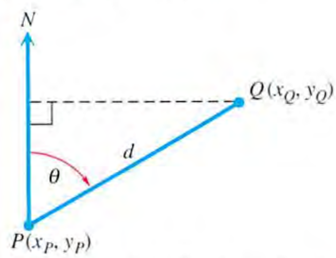
$$h = R\left(\frac{1}{\cos\frac{180T}{P}} - 1\right),$$

where $R = 3955$ mi is the radius of Earth and P is the period for the satellite to orbit Earth. (*Source:* Schlosser, W., T. Schmidt-Kaler, and E. Milone, *Challenges of Astronomy*, Springer-Verlag, 1991.)

(a) Find h to the nearest mile when $T = 25$ min and $P = 140$ min. (Evaluate the cosine function in degree mode.)

(b) What is the value of h to the nearest mile if T is increased to 30 min?

64. ***(Modeling) Fundamental Surveying Problem*** The first fundamental problem of surveying is to determine the coordinates of a point Q given the coordinates of a point P, the distance between P and Q, and the bearing θ from P to Q. See the figure. (*Source:* Mueller, I. and K. Ramsayer, *Introduction to Surveying*, Frederick Ungar Publishing Co., 1979.)

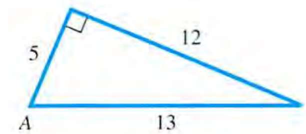


(a) Find a formula for the coordinates (x_Q, y_Q) of the point Q given θ, the coordinates (x_P, y_P) of P, and the distance d between P and Q.

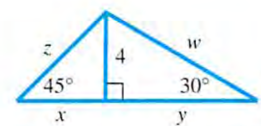
(b) Use your formula to determine (x_Q, y_Q) if $(x_P, y_P) = (123.62, 337.95)$, $\theta = 17° 19' 22''$, and $d = 193.86$ ft.

CHAPTER 2 ▶ Test

1. Give the six trigonometric function values of angle A.



2. Find the exact values of each part labeled with a letter.



3. Find a solution for $\sin(B + 15°) = \cos(2B + 30°)$.

4. Determine whether each statement is *true* or *false*. If false, tell why.

(a) $\sin 24° < \sin 48°$ (b) $\cos 24° < \cos 48°$

(c) $\cos(60° + 30°) = \cos 60° \cdot \cos 30° - \sin 60° \cdot \sin 30°$

Find the exact values of the six trigonometric functions for each angle. Rationalize denominators when applicable.

5. 240° **6.** −135° **7.** 990°

Find all values of θ *in the interval* $[0°, 360°)$ *that have the given function value.*

8. $\cos\theta = -\dfrac{\sqrt{2}}{2}$ **9.** $\csc\theta = -\dfrac{2\sqrt{3}}{3}$ **10.** $\tan\theta = 1$

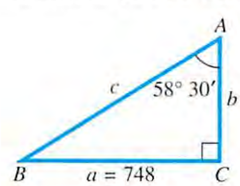
11. How would you find $\cot\theta$ using a calculator, if $\tan\theta = 1.6778490$? Give $\cot\theta$.

12. Use a calculator to approximate each value.

(a) $\sin 78° 21'$ (b) $\tan 117.689°$ (c) $\sec 58.9041°$

13. Find a value of θ in the interval $[0°, 90°)$ in decimal degrees, if $\sin\theta = .27843196$.

14. Solve the triangle.



15. ***Antenna Mast Guy Wire*** A guy wire 77.4 m long is attached to the top of an antenna mast that is 71.3 m high. Find the angle that the wire makes with the ground.

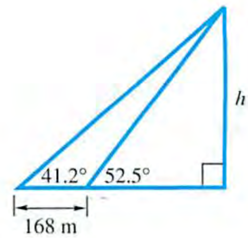
16. ***Height of a Flagpole*** To measure the height of a flagpole, Amado Carillo found that the angle of elevation from a point 24.7 ft from the base to the top is 32° 10′. What is the height of the flagpole?

17. ***Altitude of a Mountain*** The highest point in Texas is Guadalupe Peak. The angle of depression from the top of this peak to a small miner's cabin at an approximate elevation of 2000 ft is 26°. The cabin is located 14,000 ft horizontally from a point directly under the top of the mountain. Find the altitude of the top of the mountain to the nearest hundred feet.

18. ***Distance Between Two Points*** Two ships leave a port at the same time. The first ship sails on a bearing of 32° at 16 knots (nautical miles per hour) and the second on a bearing of 122° at 24 knots. How far apart are they after 2.5 hr?

19. ***Distance of a Ship from a Pier*** A ship leaves a pier on a bearing of S 62° E and travels for 75 km. It then turns and continues on a bearing of N 28° E for 53 km. How far is the ship from the pier?

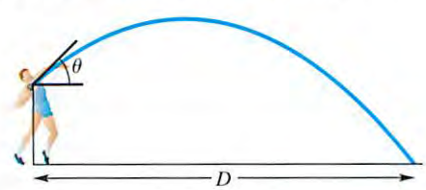
20. Find h as indicated in the figure.



CHAPTER 2 ▶ Quantitative Reasoning

Can trigonometry be used to win an Olympic medal?

A shot-putter trying to improve performance may wonder: Is there an optimal angle to aim for, or is the velocity (speed) at which the ball is thrown more important? The figure shows the path of a steel ball thrown by a shot-putter. The distance D depends on initial velocity v, height h, and angle θ when the ball is released.



One model developed for this situation gives D as

$$D = \frac{v^2 \sin\theta\cos\theta + v\cos\theta\sqrt{(v\sin\theta)^2 + 64h}}{32}.$$

Typical ranges for the variables are v: 33–46 ft per sec; h: 6–8 ft; and θ: 40°–45°. (*Source:* Kreighbaum, E. and K. Barthels, *Biomechanics*, Allyn & Bacon, 1996.)

1. To see how angle θ affects distance D, let $v = 44$ ft per sec and $h = 7$ ft. Calculate D for $\theta = 40°, 42°$, and $45°$. How does distance D change as θ increases?

2. To see how velocity v affects distance D, let $h = 7$ and $\theta = 42°$. Calculate D for $v = 43, 44$, and 45 ft per sec. How does distance D change as v increases?

3. Which affects distance D more, v or θ? What should the shot-putter do to improve performance?