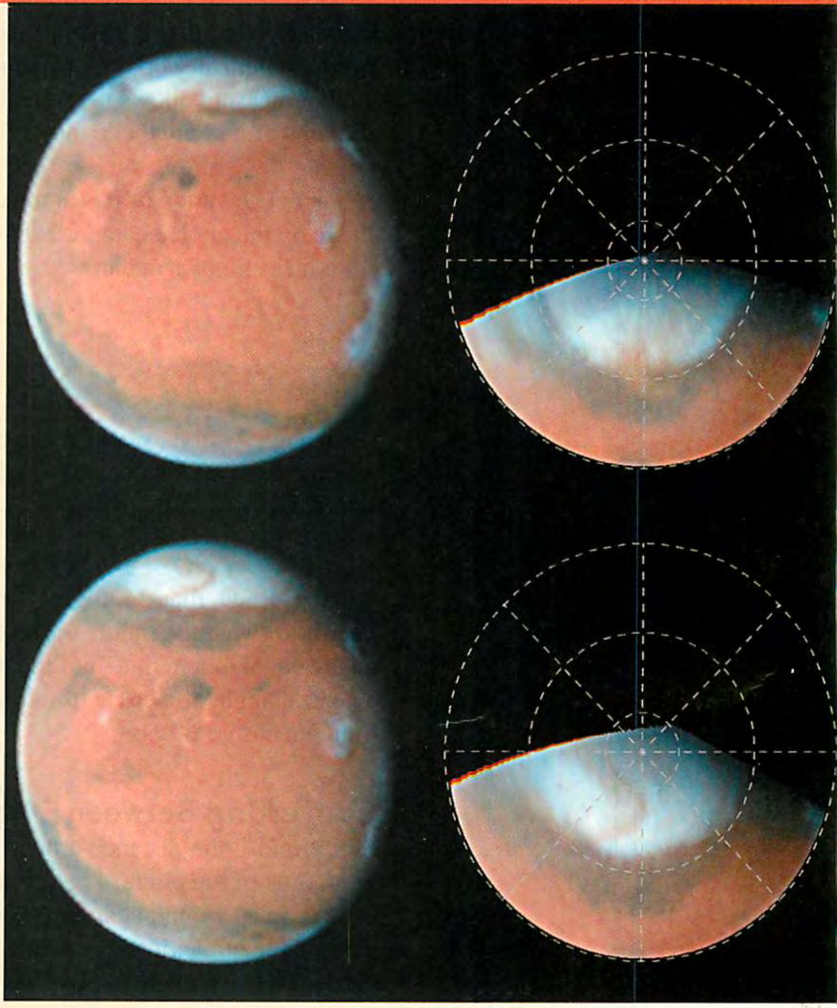
3 Radian Measure and Circular Functions



In August 2003, the planet Mars passed closer to Earth than it had in almost 60,000 years. Like Earth, Mars rotates on its axis and thus has days (also called *sols*) and nights. The photo shows a dust cloud/streak in the north polar cap of Mars, taken by the Hubble Space Telescope in 1996. In early 2004, the rovers *Spirit* and *Opportunity* landed on Mars and have provided scientists a wealth of information about the "Red Planet." (*Source:* www.hubblesite.org)

In Exercise 36 of Section 3.4, we examine the length of a Martian *sol* using *radian measure,* an alternative to measuring with degrees.

3.1 Radian Measure

Radian Measure ▪ Converting Between Degrees and Radians ▪ Finding Function Values for Angles in Radians

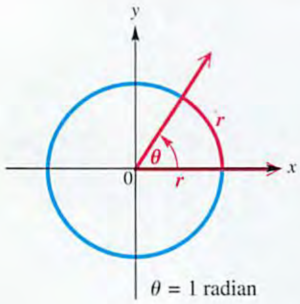


Figure 1

Radian Measure We have seen that angles can be measured in degrees. In more theoretical work in mathematics, *radian measure* of angles is preferred. Radian measure allows us to treat the trigonometric functions as functions with domains of *real numbers,* rather than angles.

Figure 1 shows an angle θ in standard position along with a circle of radius r. The vertex of θ is at the center of the circle. Because angle θ intercepts an arc on the circle equal in length to the radius of the circle, we say that angle θ has a measure of 1 radian.

RADIAN

An angle with its vertex at the center of a circle that intercepts an arc on the circle equal in length to the radius of the circle has a measure of **1 radian.**

It follows that an angle of measure 2 radians intercepts an arc equal in length to twice the radius of the circle, an angle of measure $\frac{1}{2}$ radian intercepts an arc equal in length to half the radius of the circle, and so on. In general, if θ is a central angle of a circle of radius r and θ intercepts an arc of length s, then the radian measure of θ is $\frac{s}{r}$.

Converting Between Degrees and Radians The **circumference** of a circle—the distance around the circle—is given by $C = 2\pi r$, where r is the radius of the circle. The formula $C = 2\pi r$ shows that the radius can be laid off 2π times around a circle. Therefore, an angle of 360°, which corresponds to a complete circle, intercepts an arc equal in length to 2π times the radius of the circle. Thus, an angle of 360° has a measure of 2π radians:

$$360° = 2\pi \text{ radians.}$$

An angle of 180° is half the size of an angle of 360°, so an angle of 180° has half the radian measure of an angle of 360°.

$$180° = \frac{1}{2}(2\pi) \text{ radians} = \pi \text{ radians} \quad \text{Degree/radian relationship}$$

We can use the relationship $180° = \pi$ radians to develop a method for converting between degrees and radians as follows.

$$180° = \pi \text{ radians}$$

$$1° = \frac{\pi}{180} \text{ radian} \quad \text{Divide by 180.} \qquad \text{or} \qquad 1 \text{ radian} = \frac{180°}{\pi} \quad \text{Divide by } \pi.$$

CONVERTING BETWEEN DEGREES AND RADIANS

1. Multiply a degree measure by $\frac{\pi}{180}$ radian and simplify to convert to radians.
2. Multiply a radian measure by $\frac{180°}{\pi}$ and simplify to convert to degrees.

▶ EXAMPLE 1 CONVERTING DEGREES TO RADIANS

Convert each degree measure to radians.

(a) $45°$ **(b)** $-270°$ **(c)** $249.8°$

Solution

(a) $45° = 45\left(\frac{\pi}{180}\text{ radian}\right) = \frac{\pi}{4}\text{ radian}$ Multiply by $\frac{\pi}{180}$ radian.

(b) $-270° = -270\left(\frac{\pi}{180}\text{ radian}\right) = -\frac{270\pi}{180}\text{ radians}$

$= -\frac{3\pi}{2}\text{ radians}$ Lowest terms

(c) $249.8° = 249.8\left(\frac{\pi}{180}\text{ radian}\right) \approx 4.360\text{ radians}$ Nearest thousandth

NOW TRY EXERCISES 7, 13, AND 43. ◀

▶ EXAMPLE 2 CONVERTING RADIANS TO DEGREES

Convert each radian measure to degrees.

(a) $\frac{9\pi}{4}$ **(b)** $-\frac{5\pi}{6}$ **(c)** 4.25

Solution

(a) $\frac{9\pi}{4}\text{ radians} = \frac{9\pi}{4}\left(\frac{180°}{\pi}\right) = 405°$ Multiply by $\frac{180°}{\pi}$.

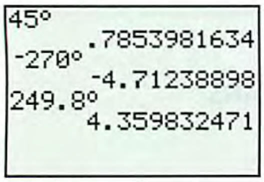
(b) $-\frac{5\pi}{6}\text{ radians} = -\frac{5\pi}{6}\left(\frac{180°}{\pi}\right) = -150°$

(c) $4.25\text{ radians} = 4.25\left(\frac{180°}{\pi}\right) \approx 243.5° = 243°\ 30'$ Use a calculator.

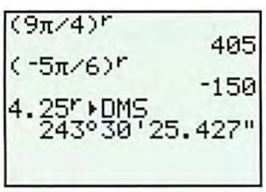
NOW TRY EXERCISES 27, 31, AND 55. ◀

▶ **Note** Another way to convert a radian measure that is a rational multiple of π, such as $\frac{9\pi}{4}$, to degrees is to just substitute 180° for π. In Example 2(a), this would be $\frac{9(180°)}{4} = 405°$.

One of the most important facts to remember when working with angles and their measures is summarized in the following statement.



A TI-83/84 Plus calculator can convert directly between degrees and radians. This radian mode screen shows the conversions for Example 1. Verify that the first two results are *approximations* for the *exact* values of $\frac{\pi}{4}$ and $-\frac{3\pi}{2}$.

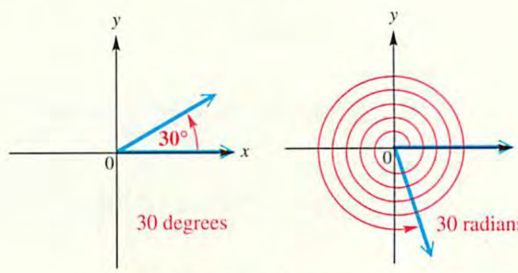


This degree mode screen shows how a TI-83/84 Plus calculator converts the radian measures in Example 2 to degree measures.

AGREEMENT ON ANGLE MEASUREMENT UNITS

If no unit of angle measure is specified, then the angle is understood to be measured in radians.

For example, Figure 2(a) shows an angle of 30°, while Figure 2(b) shows an angle of 30 (which means 30 radians).



Note the difference between an angle of 30 *degrees* and an angle of 30 *radians*.

(a) (b)

Figure 2

The following table and Figure 3 on the next page give some equivalent angle measures in degrees and radians. Keep in mind that **180° = π radians.**

Degrees	Radians		Degrees	Radians	
	Exact	Approximate		Exact	Approximate
0°	0	0	90°	$\frac{\pi}{2}$	1.57
30°	$\frac{\pi}{6}$	.52	180°	π	3.14
45°	$\frac{\pi}{4}$	.79	270°	$\frac{3\pi}{2}$	4.71
60°	$\frac{\pi}{3}$	1.05	360°	2π	6.28

These exact values are *rational multiples of π.*

▼ LOOKING AHEAD TO CALCULUS

In calculus, radian measure is much easier to work with than degree measure. If x is measured in radians, then the derivative of $f(x) = \sin x$ is

$$f'(x) = \cos x.$$

However, if x is measured in degrees, then the derivative of $f(x) = \sin x$ is

$$f'(x) = \frac{\pi}{180} \cos x.$$

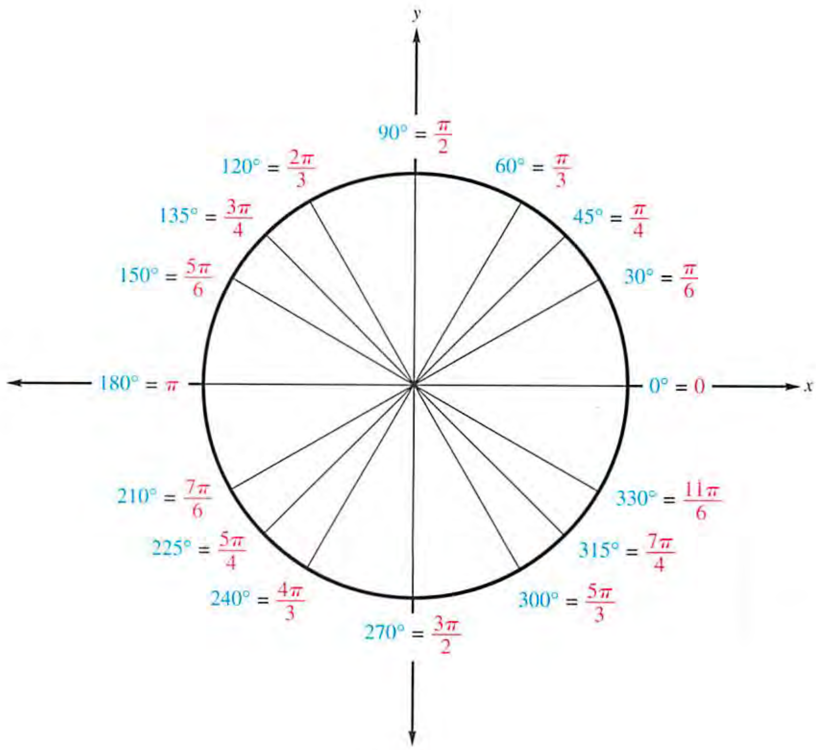


Figure 3

The angles marked in Figure 3 are extremely important in the study of trigonometry. ***You should learn these equivalences, as they will appear often in the chapters to follow.***

Finding Function Values for Angles in Radians Trigonometric function values for angles measured in radians can be found by first converting radian measure to degrees. *(You should try to skip this intermediate step as soon as possible, and find the function values directly from radian measure.)*

▶ EXAMPLE 3 FINDING FUNCTION VALUES OF ANGLES IN RADIAN MEASURE

Find each function value.

(a) $\tan \frac{2\pi}{3}$ **(b)** $\sin \frac{3\pi}{2}$ **(c)** $\cos\left(-\frac{4\pi}{3}\right)$

Solution

(a) First convert $\frac{2\pi}{3}$ radians to degrees.

$$\tan \frac{2\pi}{3} = \tan\left(\frac{2\pi}{3} \cdot \frac{180°}{\pi}\right) \quad \text{Multiply by } \tfrac{180°}{\pi}.$$

$$= \tan 120°$$

$$= -\sqrt{3} \quad \text{(Section 2.2)}$$

(b) From the table on page 104 and Figure 3 on the preceding page, $\frac{3\pi}{2}$ radians = 270°, so

$$\sin \frac{3\pi}{2} = \sin 270° = -1.$$

(c)
$$\cos\left(-\frac{4\pi}{3}\right) = \cos\left(-\frac{4\pi}{3} \cdot \frac{180°}{\pi}\right)$$
$$= -\cos 60° \qquad \text{(Section 2.2)}$$
$$= -\frac{1}{2}$$

NOW TRY EXERCISES 65, 75, AND 79.

3.1 Exercises

Concept Check *In Exercises 1–6, each angle θ is an integer when measured in radians. Give the radian measure of the angle.*

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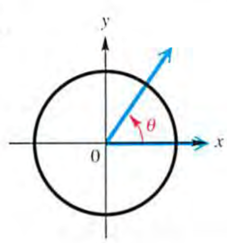

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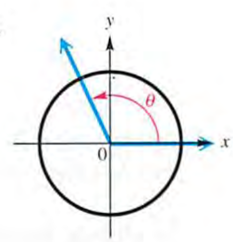

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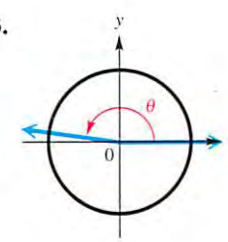

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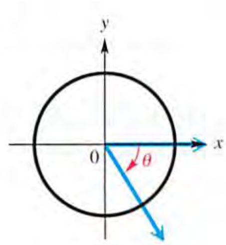

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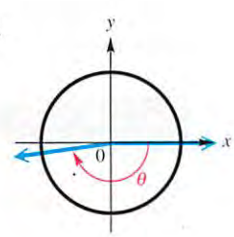

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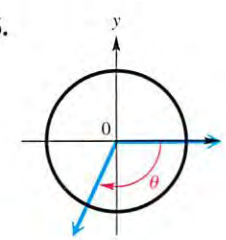


Convert each degree measure to radians. Leave answers as multiples of π. See Examples 1(a) and 1(b).

7. 60°	**8.** 30°	**9.** 90°	**10.** 120°
11. 150°	**12.** 270°	**13.** −300°	**14.** −315°
15. 450°	**16.** 480°	**17.** 1800°	**18.** −3600°

Give a short explanation in Exercises 19–24.

19. In your own words, explain how to convert degree measure to radian measure.

20. In your own words, explain how to convert radian measure to degree measure.

21. In your own words, explain the meaning of radian measure.
22. Explain the difference between degree measure and radian measure.
23. Use an example to show that you can convert from radian measure to degree measure by multiplying by $\frac{180°}{\pi}$.
24. Explain why an angle of radian measure t in standard position intercepts an arc of length t on a circle of radius 1.

Convert each radian measure to degrees. See Examples 2(a) and 2(b).

25. $\frac{\pi}{3}$	**26.** $\frac{8\pi}{3}$	**27.** $\frac{7\pi}{4}$	**28.** $\frac{2\pi}{3}$
29. $\frac{11\pi}{6}$	**30.** $\frac{15\pi}{4}$	**31.** $-\frac{\pi}{6}$	**32.** $-\frac{8\pi}{5}$
33. $\frac{7\pi}{10}$	**34.** $\frac{11\pi}{15}$	**35.** $-\frac{4\pi}{15}$	**36.** $-\frac{7\pi}{20}$
37. $\frac{17\pi}{20}$	**38.** $\frac{11\pi}{30}$	**39.** -5π	**40.** 15π

Convert each degree measure to radians. See Example 1(c).

41. 39°	**42.** 74°	**43.** 42.5°	**44.** 264.9°
45. 139° 10′	**46.** 174° 50′	**47.** 64.29°	**48.** 85.04°
49. 56° 25′	**50.** 122° 37′	**51.** 47.6925°	**52.** 23.0143°

Convert each radian measure to degrees. Write answers to the nearest minute. See Example 2(c).

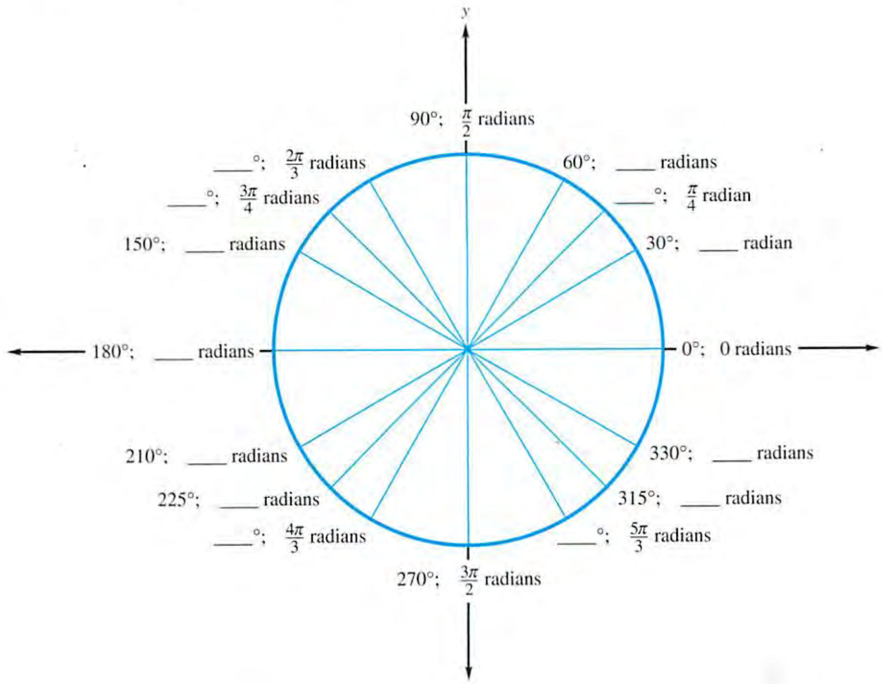
53. 2	**54.** 5	**55.** 1.74	**56.** 3.06
57. .3417	**58.** 9.84763	**59.** −5.01095	**60.** −3.47189

61. **Concept Check** The value of sin 30 is not $\frac{1}{2}$. Why is this true?
62. Explain in your own words what is meant by an angle of one radian.

Find the exact value of each expression without using a calculator. See Example 3.

63. $\sin\frac{\pi}{3}$	**64.** $\cos\frac{\pi}{6}$	**65.** $\tan\frac{\pi}{4}$	**66.** $\cot\frac{\pi}{3}$
67. $\sec\frac{\pi}{6}$	**68.** $\csc\frac{\pi}{4}$	**69.** $\sin\frac{\pi}{2}$	**70.** $\csc\frac{\pi}{2}$
71. $\tan\frac{5\pi}{3}$	**72.** $\cot\frac{2\pi}{3}$	**73.** $\sin\frac{5\pi}{6}$	**74.** $\tan\frac{5\pi}{6}$
75. $\cos 3\pi$	**76.** $\sec \pi$	**77.** $\sin\left(-\frac{8\pi}{3}\right)$	**78.** $\cot\left(-\frac{2\pi}{3}\right)$
79. $\sin\left(-\frac{7\pi}{6}\right)$	**80.** $\cos\left(-\frac{\pi}{6}\right)$	**81.** $\tan\left(-\frac{14\pi}{3}\right)$	**82.** $\csc\left(-\frac{13\pi}{3}\right)$

83. *Concept Check* The figure shows the same angles measured in both degrees and radians. Complete the missing measures.



Solve each problem.

84. *Railroad Engineering* The term **grade** has several different meanings in construction work. Some engineers use the term **grade** to represent $\frac{1}{100}$ of a right angle and express grade as a percent. For instance, an angle of $.9°$ would be referred to as a 1% grade. (*Source:* Hay, W., *Railroad Engineering,* John Wiley and Sons, 1982.)

(a) By what number should you multiply a grade (disregarding the % symbol) to convert it to radians?

(b) In a rapid-transit rail system, the maximum grade allowed between two stations is 3.5%. Express this angle in degrees and radians.

85. *Rotating Hour Hand on a Clock* Through how many radians will the hour hand on a clock rotate in **(a)** 24 hr and **(b)** 4 hr?

86. *Rotating Pulley* A circular pulley is rotating about its center. Through how many radians would it turn in **(a)** 8 rotations and **(b)** 30 rotations?

87. *Orbits of a Space Vehicle* A space vehicle is orbiting Earth in a circular orbit. What radian measure corresponds to **(a)** 2.5 orbits and **(b)** $\frac{4}{3}$ orbit?

88. *Revolutions of a Carousel* A stationary horse on a carousel makes 12 complete revolutions. Through what radian measure angle does the horse revolve?

3.2 Applications of Radian Measure

Arc Length on a Circle ▪ Area of a Sector of a Circle

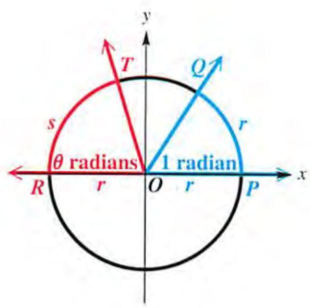


Figure 4

Arc Length on a Circle We use radian measure in the formula to find the length of an arc of a circle. This formula is derived from the fact (proved in geometry) that the length of an arc is proportional to the measure of its central angle. In Figure 4, angle *QOP* has measure 1 radian and intercepts an arc of length r on the circle. Angle *ROT* has measure θ radians and intercepts an arc of length s on the circle. Since the lengths of the arcs are proportional to the measures of their central angles,

$$\frac{s}{r} = \frac{\theta}{1}.$$

Multiplying both sides by r gives the following result.

ARC LENGTH

The length s of the arc intercepted on a circle of radius r by a central angle of measure θ radians is given by the product of the radius and the radian measure of the angle, or

$$s = r\theta, \qquad \theta \text{ in radians.}$$

▶ **Caution** ***Avoid the common error of applying this formula with θ in degree mode. When applying the formula $s = r\theta$, the value of θ* MUST *be expressed in radians.***

▶ EXAMPLE 1 FINDING ARC LENGTH USING $s = r\theta$

A circle has radius 18.20 cm. Find the length of the arc intercepted by a central angle having each of the following measures.

(a) $\frac{3\pi}{8}$ radians **(b)** 144°

Solution

(a) As shown in Figure 5, $r = 18.20$ cm and $\theta = \frac{3\pi}{8}$.

$s = r\theta$ Arc length formula

$s = 18.20\left(\frac{3\pi}{8}\right)$ cm Substitute for r and θ.

$s \approx 21.44$ cm

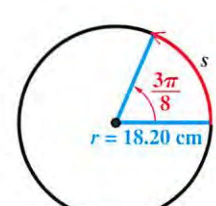


Figure 5

(b) The formula $s = r\theta$ requires that θ be measured in radians. First, convert θ to radians by multiplying 144° by $\frac{\pi}{180}$ radian.

$$144^\circ = 144\left(\frac{\pi}{180}\right) = \frac{4\pi}{5} \text{ radians}$$ Convert from degrees to radians. (Section 3.1)

The length s is given by

$$s = r\theta = 18.20\left(\frac{4\pi}{5}\right) \approx 45.74 \text{ cm.}$$

Be sure to use radians for θ in $s = r\theta$.

NOW TRY EXERCISES 11 AND 15.

▶ EXAMPLE 2 USING LATITUDES TO FIND THE DISTANCE BETWEEN TWO CITIES

Latitude gives the measure of a central angle with vertex at Earth's center whose initial side goes through the equator and whose terminal side goes through the given location. Reno, Nevada is approximately due north of Los Angeles. The latitude of Reno is 40° N, while that of Los Angeles is 34° N. (The N in 34° N means *north* of the equator.) The radius of Earth is 6400 km. Find the north-south distance between the two cities.

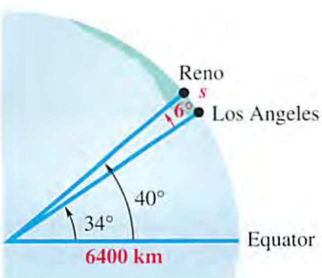


Figure 6

Solution As shown in Figure 6, the central angle between Reno and Los Angeles is $40^\circ - 34^\circ = 6^\circ$. The distance between the two cities can be found by the formula $s = r\theta$, after 6° is first converted to radians.

$$6^\circ = 6\left(\frac{\pi}{180}\right) = \frac{\pi}{30} \text{ radian}$$

The distance between the two cities is

$$s = r\theta = 6400\left(\frac{\pi}{30}\right) \approx 670 \text{ km.}$$ Let $r = 6400$ and $\theta = \frac{\pi}{30}$.

NOW TRY EXERCISE 21.

▶ EXAMPLE 3 FINDING A LENGTH USING $s = r\theta$

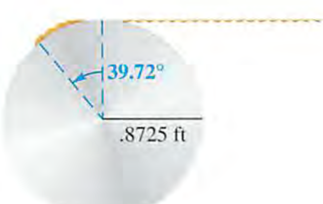


Figure 7

A rope is being wound around a drum with radius .8725 ft. (See Figure 7.) How much rope will be wound around the drum if the drum is rotated through an angle of 39.72°?

Solution The length of rope wound around the drum is the arc length for a circle of radius .8725 ft and a central angle of 39.72°. Use the formula $s = r\theta$, with the angle converted to radian measure. The length of the rope wound around the drum is approximately

$$s = r\theta = .8725\left[39.72\left(\frac{\pi}{180}\right)\right] \approx .6049 \text{ ft.}$$

NOW TRY EXERCISE 33(a).

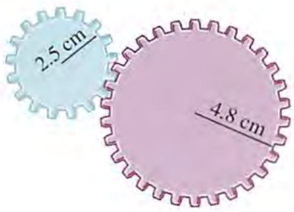


Figure 8

▶ EXAMPLE 4 FINDING AN ANGLE MEASURE USING $s = r\theta$

Two gears are adjusted so that the smaller gear drives the larger one, as shown in Figure 8. If the smaller gear rotates through an angle of 225°, through how many degrees will the larger gear rotate?

Solution First find the radian measure of the angle, and then find the arc length on the smaller gear that determines the motion of the larger gear. Since $225° = \frac{5\pi}{4}$ radians, for the smaller gear,

$$s = r\theta = 2.5\left(\frac{5\pi}{4}\right) = \frac{12.5\pi}{4} = \frac{25\pi}{8} \text{ cm.}$$

An arc with this length on the larger gear corresponds to an angle measure θ, in radians, where

$$s = r\theta$$

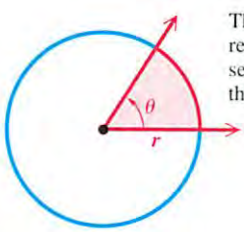
$$\frac{25\pi}{8} = 4.8\theta \quad \text{Substitute } \tfrac{25\pi}{8} \text{ for } s \text{ and } 4.8 \text{ for } r.$$

$$\frac{125\pi}{192} = \theta. \quad 4.8 = \tfrac{48}{10} = \tfrac{24}{5}\text{; multiply by } \tfrac{5}{24} \text{ to solve for } \theta.$$

Converting θ back to degrees shows that the larger gear rotates through

$$\frac{125\pi}{192}\left(\frac{180°}{\pi}\right) \approx 117°. \quad \text{Convert } \theta = \tfrac{125\pi}{192} \text{ to degrees.}$$

NOW TRY EXERCISE 27. ◀



The shaded region is a sector of the circle.

Figure 9

Area of a Sector of a Circle

A **sector of a circle** is the portion of the interior of a circle intercepted by a central angle. Think of it as a "piece of pie." See Figure 9. A complete circle can be thought of as an angle with measure 2π radians. If a central angle for a sector has measure θ radians, then the sector makes up the fraction $\frac{\theta}{2\pi}$ of a complete circle. The area of a complete circle with radius r is $A = \pi r^2$. Therefore,

$$\text{area of the sector} = \frac{\theta}{2\pi}(\pi r^2) = \frac{1}{2}r^2\theta, \qquad \theta \text{ in radians.}$$

This discussion is summarized as follows.

AREA OF A SECTOR

The area A of a sector of a circle of radius r and central angle θ is given by

$$A = \frac{1}{2}r^2\theta, \qquad \theta \text{ in radians.}$$

▶ **Caution** As in the formula for arc length, ***the value of θ must be in radians when using this formula for the area of a sector.***

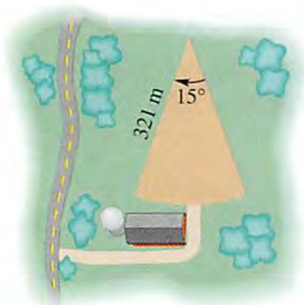


Figure 10

▶ EXAMPLE 5 FINDING THE AREA OF A SECTOR-SHAPED FIELD

Find the area of the sector-shaped field shown in Figure 10.

Solution First, convert 15° to radians.

$$15^\circ = 15\left(\frac{\pi}{180}\right) = \frac{\pi}{12} \text{ radian}$$

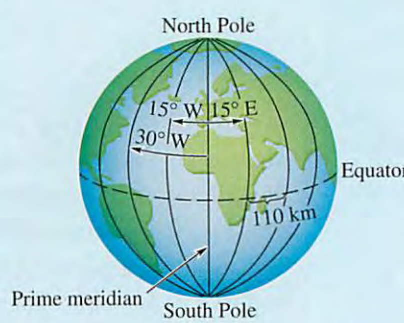
Now use the formula to find the area of a sector of a circle with radius $r = 321$.

$$A = \frac{1}{2}r^2\theta = \frac{1}{2}(321)^2\left(\frac{\pi}{12}\right) \approx 13{,}500 \text{ m}^2$$

NOW TRY EXERCISE 51. ◀

CONNECTIONS

Longitude is the angular distance (expressed in degrees) East or West of the prime meridian, which goes from the North Pole to the South Pole through Greenwich, England. Arcs of longitude are 110 km apart at the equator. As the figure shows, these sections are similar to those of an orange.



Because Earth revolves 15° per hr, longitude is found by taking the difference between time zones multiplied by 15°. For example, if it is 12 noon where you are (in the United States) and 5 P.M. in Greenwich, you are located at longitude 5(15°) = longitude 75° W. Thus, determining longitude requires only an accurate measure of time. Before 1772, sailors were unable to determine their position at sea because there were no clocks capable of precise measure of time at sea. In 1772, a clock invented by John Harrison, a carpenter's son with no formal education, solved the problem. (*Source:* Ola, P. and E. D'Aulaire, "Taking the Measure of Time," *Smithsonian*, December 1999.)

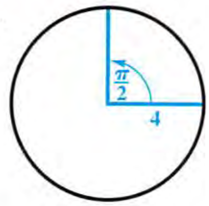
FOR DISCUSSION OR WRITING

Use time zones to determine the longitude where you live. What would the longitude be at Greenwich, England? Visit the Internet to learn more about longitude.

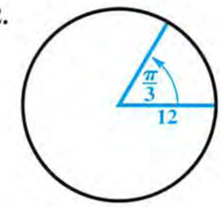
3.2 Exercises

Concept Check *Find the exact length of each arc intercepted by the given central angle.*

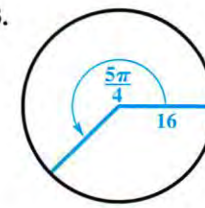
1.



2.

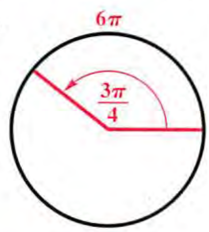


3.



Concept Check *Find the radius of each circle.*

4.



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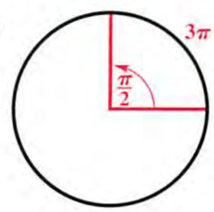

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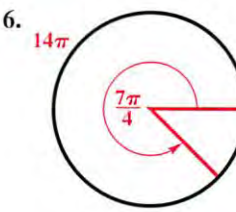
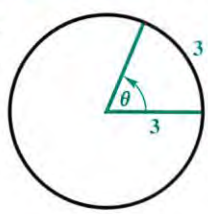


Concept Check *Find the measure of each central angle (in radians).*

7.



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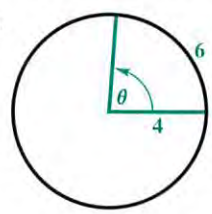
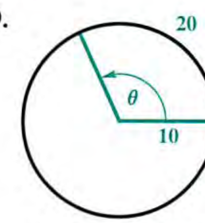


9.



10. Explain in your own words how to find the *degree* measure of a central angle in a circle if both the radius and the length of the intercepted arc are known.

Unless otherwise directed, give calculator approximations in your answers in the rest of this exercise set.

Find the length to three significant digits of each arc intercepted by a central angle θ in a circle of radius r. See Example 1.

11. $r = 12.3$ cm, $\theta = \frac{2\pi}{3}$ radians

12. $r = .892$ cm, $\theta = \frac{11\pi}{10}$ radians

13. $r = 1.38$ ft, $\theta = \frac{5\pi}{6}$ radians

14. $r = 3.24$ mi, $\theta = \frac{7\pi}{6}$ radians

15. $r = 4.82$ m, $\theta = 60°$

16. $r = 71.9$ cm, $\theta = 135°$

17. $r = 15.1$ in., $\theta = 210°$

18. $r = 12.4$ ft, $\theta = 330°$

19. ***Concept Check*** If the radius of a circle is doubled, how is the length of the arc intercepted by a fixed central angle changed?

20. ***Concept Check*** Radian measure simplifies many formulas, such as the formula for arc length, $s = r\theta$. Give the corresponding formula when θ is measured in degrees instead of radians.

Distance Between Cities *Find the distance in kilometers between each pair of cities, assuming they lie on the same north-south line. See Example 2.*

21. Panama City, Panama, 9° N, and Pittsburgh, Pennsylvania, 40° N

22. Farmersville, California, 36° N, and Penticton, British Columbia, 49° N

23. New York City, New York, 41° N, and Lima, Peru, 12° S

24. Halifax, Nova Scotia, 45° N, and Buenos Aires, Argentina, 34° S

25. ***Latitude of Madison*** Madison, South Dakota, and Dallas, Texas, are 1200 km apart and lie on the same north-south line. The latitude of Dallas is 33° N. What is the latitude of Madison?

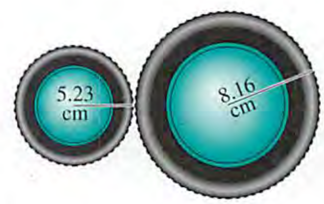
26. ***Latitude of Toronto*** Charleston, South Carolina, and Toronto, Canada, are 1100 km apart and lie on the same north-south line. The latitude of Charleston is 33° N. What is the latitude of Toronto?

Work each problem. See Examples 3 and 4.

27. ***Gear Movement*** Two gears are adjusted so that the smaller gear drives the larger one, as shown in the figure. If the smaller gear rotates through an angle of 300°, through how many degrees will the larger gear rotate?

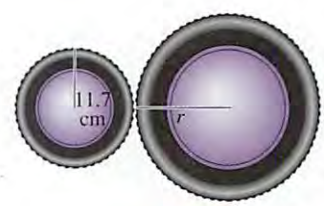
28. ***Gear Movement*** Repeat Exercise 27 for gear radii of 4.8 in. and 7.1 in., and for an angle of 315° for the smaller gear.

29. ***Rotating Wheels*** The rotation of the smaller wheel in the figure causes the larger wheel to rotate. Through how many degrees will the larger wheel rotate if the smaller one rotates through 60.0°?



30. ***Rotating Wheels*** Repeat Exercise 29 for wheel radii of 6.84 in. and 12.46 in. and an angle of 150° for the smaller wheel.

31. ***Rotating Wheels*** Find the radius of the larger wheel in the figure if the smaller wheel rotates 80.0° when the larger wheel rotates 50.0°.



32. ***Rotating Wheels*** Repeat Exercise 31 if the smaller wheel of radius 14.6 in. rotates 120° when the larger wheel rotates 60°.

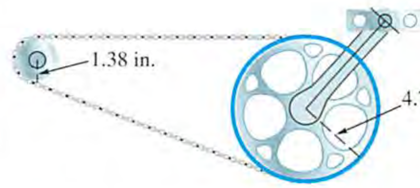
33. ***Pulley Raising a Weight***

(a) How many inches will the weight in the figure rise if the pulley is rotated through an angle of 71° 50′?

(b) Through what angle, to the nearest minute, must the pulley be rotated to raise the weight 6 in.?

34. ***Pulley Raising a Weight*** Find the radius of the pulley in the figure if a rotation of 51.6° raises the weight 11.4 cm.

35. ***Bicycle Chain Drive*** The figure shows the chain drive of a bicycle. How far will the bicycle move if the pedals are rotated through 180°? Assume the radius of the bicycle wheel is 13.6 in.



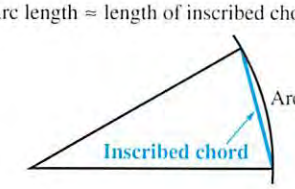
36. ***Car Speedometer*** The speedometer of Terry's Honda CR-V is designed to be accurate with tires of radius 14 in.

(a) Find the number of rotations of a tire in 1 hr if the car is driven at 55 mph.

(b) Suppose that oversize tires of radius 16 in. are placed on the car. If the car is now driven for 1 hr with the speedometer reading 55 mph, how far has the car gone? If the speed limit is 55 mph, does Terry deserve a speeding ticket?

If a central angle is very small, there is little difference in length between an arc and the inscribed chord. See the figure. Approximate each of the following lengths by finding the necessary arc length. (Note: When a central angle intercepts an arc, the arc is said to ***subtend*** *the angle.)*

Arc length ≈ length of inscribed chord

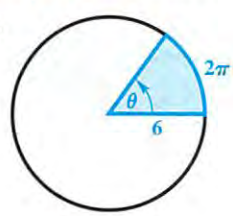


37. ***Length of a Train*** A railroad track in the desert is 3.5 km away. A train on the track subtends (horizontally) an angle of $3° \, 20'$. Find the length of the train.

38. ***Distance to a Boat*** The mast of Brent Simon's boat is 32 ft high. If it subtends an angle of $2° \, 10'$, how far away is it?

Concept Check *Find the area of each sector.*

39.



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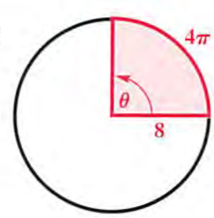

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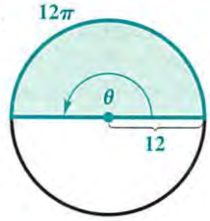

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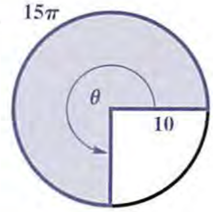
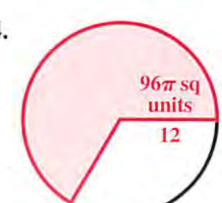


Concept Check *Find the measure (in degrees) of each central angle. The number inside the sector is the area.*

43.

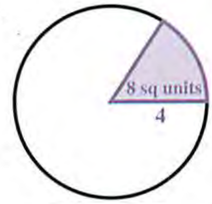
44.



Concept Check *Find the measure (in radians) of each central angle. The number inside the sector is the area.*

45.

46.



Find the area of a sector of a circle having radius r and central angle θ. Express answers to the nearest tenth. See Example 5.

47. $r = 29.2$ m, $\theta = \frac{5\pi}{6}$ radians

48. $r = 59.8$ km, $\theta = \frac{2\pi}{3}$ radians

49. $r = 30.0$ ft, $\theta = \frac{\pi}{2}$ radians

50. $r = 90.0$ yd, $\theta = \frac{5\pi}{6}$ radians

51. $r = 12.7$ cm, $\theta = 81°$

52. $r = 18.3$ m, $\theta = 125°$

53. $r = 40.0$ mi, $\theta = 135°$

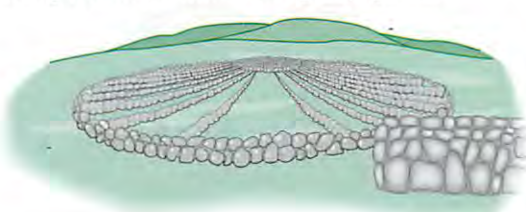
54. $r = 90.0$ km, $\theta = 270°$

Work each problem.

55. Find the measure (in radians) of a central angle of a sector of area 16 in.^{2} in a circle of radius 3.0 in.

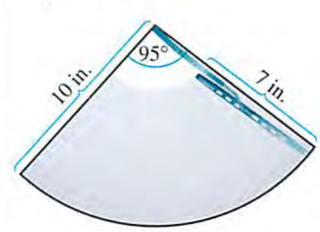
56. Find the radius of a circle in which a central angle of $\frac{\pi}{6}$ radian determines a sector of area 64 m^2.

57. ***Measures of a Structure*** The figure shows Medicine Wheel, a Native American structure in northern Wyoming. This circular structure is perhaps 2500 yr old. There are 27 aboriginal spokes in the wheel, all equally spaced.



(a) Find the measure of each central angle in degrees and in radians.
(b) If the radius of the wheel is 76.0 ft, find the circumference.
(c) Find the length of each arc intercepted by consecutive pairs of spokes.
(d) Find the area of each sector formed by consecutive spokes.

58. ***Area Cleaned by a Windshield Wiper*** The Ford Model A, built from 1928 to 1931, had a single windshield wiper on the driver's side. The total arm and blade was 10 in. long and rotated back and forth through an angle of 95°. The shaded region in the figure is the portion of the windshield cleaned by the 7-in. wiper blade. What is the area of the region cleaned?



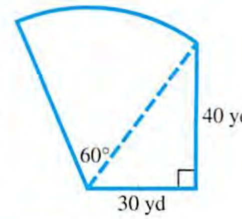
59. ***Circular Railroad Curves*** In the United States, circular railroad curves are designated by the **degree of curvature,** the central angle subtended by a chord of 100 ft. Suppose a portion of track has curvature 42.0°. (*Source:* Hay, W., *Railroad Engineering,* John Wiley and Sons, 1982.)

(a) What is the radius of the curve?
(b) What is the length of the arc determined by the 100-ft chord?

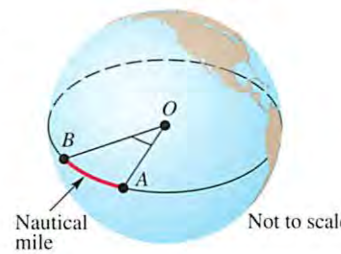
(c) What is the area of the portion of the circle bounded by the arc and the 100-ft chord?

60. *Land Required for a Solar-Power Plant* A 300-megawatt solar-power plant requires approximately 950,000 m^2 of land area in order to collect the required amount of energy from sunlight. If this land area is circular, what is its radius? If this land area is a 35° sector of a circle, what is its radius?

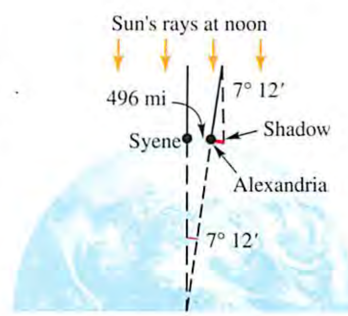
61. *Area of a Lot* A frequent problem in surveying city lots and rural lands adjacent to curves of highways and railways is that of finding the area when one or more of the boundary lines is the arc of a circle. Find the area of the lot (to two significant digits) shown in the figure. (*Source:* Anderson, J. and E. Michael, *Introduction to Surveying,* McGraw-Hill, 1985.)



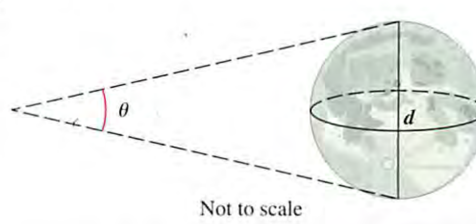
62. *Nautical Miles* **Nautical miles** are used by ships and airplanes. They are different from **statute miles,** which equal 5280 ft. A nautical mile is defined to be the arc length along the equator intercepted by a central angle AOB of 1 min, as illustrated in the figure. If the equatorial radius of Earth is 3963 mi, use the arc length formula to approximate the number of statute miles in 1 nautical mile. Round your answer to two decimal places.



63. *Circumference of Earth* The first accurate estimate of the distance around Earth was done by the Greek astronomer Eratosthenes (276–195 B.C.), who noted that the noontime position of the sun at the summer solstice differed by 7° 12′ from the city of Syene to the city of Alexandria. (See the figure.) The distance between these two cities is 496 mi. Use the arc length formula to estimate the radius of Earth. Then find the circumference of Earth. (*Source:* Zeilik, M., *Introductory Astronomy and Astrophysics,* Third Edition, Saunders College Publishers, 1992.)

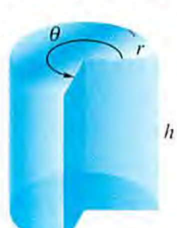


64. *Diameter of the Moon* The distance to the moon is approximately 238,900 mi. Use the arc length formula to estimate the diameter d of the moon if angle θ in the figure is measured to be .5170°.

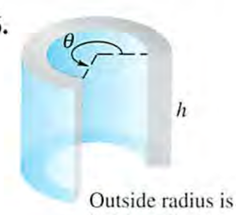


Volume of a Solid *Multiply the area of the base by the height to find a formula for the volume V of each solid.*

65.



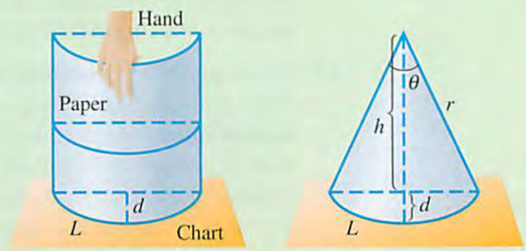
66.



RELATING CONCEPTS

For individual or collaborative investigation
(Exercises 67–70)

(Modeling) Measuring Paper Curl *Manufacturers of paper determine its quality by its* curl. *The curl of a sheet of paper is measured by holding it at the center of one edge and comparing the arc formed by the free end to arcs on a chart lying flat on a table. Each arc in the chart corresponds to a number d that gives the* depth *of the arc. See the figure. (Source:* Tabakovic, H., J. Paullet, and R. Bertram, "Measuring the Curl of Paper," *The College Mathematics Journal,* Vol. 30 No. 4, September 1999.)



To produce the chart, it is necessary to find a function that relates d to the length of arc L. **Work Exercises 67–70 in order,** *to determine that function. Refer to the figure on the right.*

67. Express L in terms of r and θ, and then solve for r.

68. Use a right triangle to relate r, h, and θ. Solve for h.

69. Express d in terms of r and h, then substitute your answer from Exercise 68 for h. Factor out r.

70. Use your answer from Exercise 67 to substitute for r in the result from Exercise 69. This result is a formula that gives d for specific values of θ.

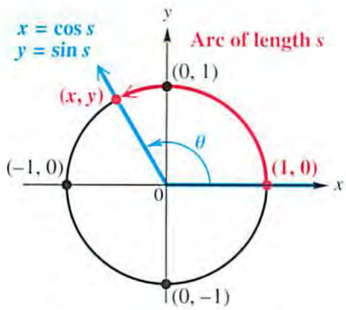
71. *Concept Check* If the radius of a circle is doubled and the central angle of a sector is unchanged, how is the area of the sector changed?

72. *Concept Check* Give the corresponding formula for the area of a sector when the angle is measured in degrees.

3.3 The Unit Circle and Circular Functions

Circular Functions ▪ Finding Values of Circular Functions ▪ Determining a Number with a Given Circular Function Value ▪ Applying Circular Functions

In **Section 1.3,** we defined the six trigonometric functions in such a way that the domain of each function was a set of *angles* in standard position. These angles can be measured in degrees or in radians. In advanced courses, such as calculus, it is necessary to modify the trigonometric functions so that their domains consist of *real numbers* rather than angles. We do this by using the relationship between an angle θ and an arc of length s on a circle.



The unit circle $x^2 + y^2 = 1$

Figure 11

Circular Functions In Figure 11, we start at the point $(1, 0)$ and measure an arc of length s along the circle. If $s > 0$, then the arc is measured in a counterclockwise direction, and if $s < 0$, then the direction is clockwise. (If $s = 0$, then no arc is measured.) Let the endpoint of this arc be at the point (x, y). The circle in Figure 11 is the **unit circle**—it has center at the origin and radius 1 unit (hence the name *unit circle*). Recall from algebra that the equation of this circle is

$$x^2 + y^2 = 1. \quad \text{(Appendix B)}$$

Recall that the radian measure of θ is related to the arc length s. For θ measured in radians, we know that $s = r\theta$. Here, $r = 1$, so s, which is measured in linear units such as inches or centimeters, is equal to θ, measured in radians. Thus, the trigonometric functions of angle θ in radians found by choosing a point (x, y) on the unit circle can be rewritten as functions of the arc length s, a real number. When interpreted this way, they are called **circular functions.**

▼ LOOKING AHEAD TO CALCULUS

If you plan to study calculus, you must become very familiar with radian measure. In calculus, the trigonometric or circular functions are always understood to have real number domains.

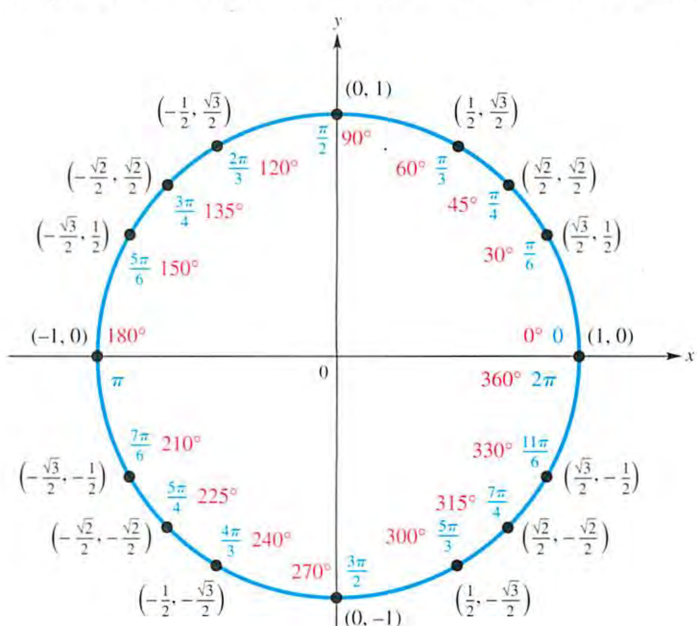
CIRCULAR FUNCTIONS

For any real number s represented by a directed arc on the unit circle,

$$\sin s = y \qquad \cos s = x \qquad \tan s = \frac{y}{x} \quad (x \neq 0)$$

$$\csc s = \frac{1}{y} \quad (y \neq 0) \qquad \sec s = \frac{1}{x} \quad (x \neq 0) \qquad \cot s = \frac{x}{y} \quad (y \neq 0).$$

Since x represents the cosine of s and y represents the sine of s, and because of the discussion in **Section 3.1** on converting between degrees and radians, we can summarize a great deal of information in a concise manner, as seen in Figure 12.*



The unit circle $x^2 + y^2 = 1$

Figure 12

*The authors thank Professor Marvel Townsend of the University of Florida for her suggestion to include this figure.

The unit circle is symmetric with respect to the x-axis, the y-axis, and the origin. (See **Appendix D.**) Thus, if a point (a, b) lies on the unit circle, so does $(a, -b)$, $(-a, b)$, and $(-a, -b)$. Furthermore, each of these points has a *reference arc* of equal magnitude. For a point on the unit circle, its **reference arc** is the shortest arc from the point itself to the nearest point on the x-axis. (This concept is analogous to the reference angle concept introduced in **Chapter 2.**) Using the concept of symmetry makes determining sines and cosines of the real numbers identified in Figure 12 a relatively simple procedure if we know the coordinates of the points labeled in quadrant I.

For example, the quadrant I real number $\frac{\pi}{3}$ is associated with the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ on the unit circle. Therefore, we can use symmetry to identify the coordinates of the points associated with

$$\pi - \frac{\pi}{3} = \underbrace{\frac{2\pi}{3}}_{\text{Quadrant II}}, \quad \pi + \frac{\pi}{3} = \underbrace{\frac{4\pi}{3}}_{\text{Quadrant III}}, \quad \text{and} \quad 2\pi - \frac{\pi}{3} = \underbrace{\frac{5\pi}{3}}_{\text{Quadrant IV}}.$$

The following chart summarizes this information.

s	Quadrant of s	Symmetry Type and Corresponding Point	$\cos s$	$\sin s$
$\frac{\pi}{3}$	I	not applicable; $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\pi - \frac{\pi}{3} = \frac{2\pi}{3}$	II	y-axis; $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\pi + \frac{\pi}{3} = \frac{4\pi}{3}$	III	origin; $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
$2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$	IV	x-axis; $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$

▶ **Note** Since $\sin s = y$ and $\cos s = x$, we can replace x and y in the equation of the unit circle

$$x^2 + y^2 = 1$$

and obtain the Pythagorean identity

$$\mathbf{\cos^2 s + \sin^2 s = 1.} \quad \text{(Section 1.4)}$$

The ordered pair (x, y) represents a point on the unit circle, and therefore

$$-1 \leq x \leq 1 \quad \text{and} \quad -1 \leq y \leq 1,$$

so $$\mathbf{-1 \leq \cos s \leq 1} \quad \text{and} \quad \mathbf{-1 \leq \sin s \leq 1.}$$

For any value of s, both $\sin s$ and $\cos s$ exist, so the domain of these functions is the set of all real numbers.

For tan s, defined as $\frac{y}{x}$, x must not equal 0. The only way x can equal 0 is when the arc length s is $\frac{\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, -\frac{3\pi}{2}$, and so on. To avoid a 0 denominator, the domain of the tangent function must be restricted to those values of s satisfying

$$s \neq (2n + 1)\frac{\pi}{2}, \qquad n \text{ any integer.}$$

The definition of secant also has x in the denominator, so the domain of secant is the same as the domain of tangent. Both cotangent and cosecant are defined with a denominator of y. To guarantee that $y \neq 0$, the domain of these functions must be the set of all values of s satisfying

$$s \neq n\pi, \qquad n \text{ any integer.}$$

DOMAINS OF THE CIRCULAR FUNCTIONS

The domains of the circular functions are as follows:

Sine and Cosine Functions: $(-\infty, \infty)$

Tangent and Secant Functions: $\left\{s \,\middle|\, s \neq (2n + 1)\frac{\pi}{2}, \text{ where } n \text{ is any integer}\right\}$

Cotangent and Cosecant Functions: $\{s \,|\, s \neq n\pi, \text{ where } n \text{ is any integer}\}$

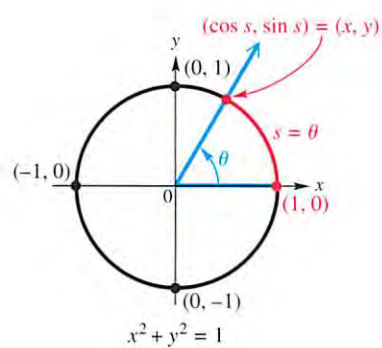


Figure 13

Finding Values of Circular Functions The circular functions of real numbers correspond to the trigonometric functions of angles measured in radians. Let us assume that angle θ is in standard position, superimposed on the unit circle. See Figure 13. Suppose that θ is the *radian* measure of this angle. Using the arc length formula $s = r\theta$ with $r = 1$, we have $s = \theta$. Thus, the length of the intercepted arc is the real number that corresponds to the radian measure of θ. Using the trigonometric function definitions from **Section 1.3,**

$$\sin\theta = \frac{y}{r} = \frac{y}{1} = y = \sin s, \qquad \text{and} \qquad \cos\theta = \frac{x}{r} = \frac{x}{1} = x = \cos s,$$

and so on. As shown here, the trigonometric functions and the circular functions lead to the same function values, provided we think of the angles as being in radian measure. This leads to the following important result.

EVALUATING A CIRCULAR FUNCTION

Circular function values of real numbers are obtained in the same manner as trigonometric function values of angles measured in radians. This applies both to methods of finding exact values (such as reference angle analysis) and to calculator approximations. ***Calculators must be in radian mode when finding circular function values.***

▶ EXAMPLE 1 FINDING EXACT CIRCULAR FUNCTION VALUES

Find the exact values of $\sin\frac{3\pi}{2}$, $\cos\frac{3\pi}{2}$, and $\tan\frac{3\pi}{2}$.

Solution Evaluating a circular function at the real number $\frac{3\pi}{2}$ is equivalent to evaluating it at $\frac{3\pi}{2}$ radians. An angle of $\frac{3\pi}{2}$ radians intersects the unit circle at the point $(0, -1)$, as shown in Figure 14. Since

$$\sin s = y, \quad \cos s = x, \quad \text{and} \quad \tan s = \frac{y}{x},$$

it follows that

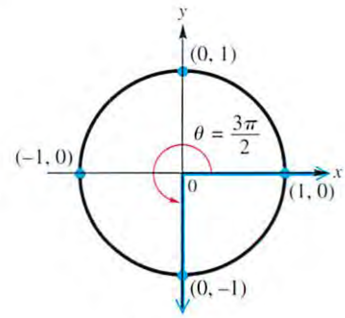


Figure 14

$$\sin\frac{3\pi}{2} = -1, \ \cos\frac{3\pi}{2} = 0, \ \text{and} \ \tan\frac{3\pi}{2} \text{ is undefined.}$$

NOW TRY EXERCISE 1. ◀

▶ EXAMPLE 2 FINDING EXACT CIRCULAR FUNCTION VALUES

(a) Use Figure 12 to find the exact values of $\cos\frac{7\pi}{4}$ and $\sin\frac{7\pi}{4}$.

(b) Use Figure 12 and the definition of tangent to find the exact value of $\tan\left(-\frac{5\pi}{3}\right)$.

(c) Use reference angles and degree/radian conversion to find the exact value of $\cos\frac{2\pi}{3}$.

Solution

(a) In Figure 12, we see that the real number $\frac{7\pi}{4}$ corresponds to the unit circle point $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$. Thus,

$$\cos\frac{7\pi}{4} = \frac{\sqrt{2}}{2} \quad \text{and} \quad \sin\frac{7\pi}{4} = -\frac{\sqrt{2}}{2}.$$

(b) Moving around the unit circle $\frac{5\pi}{3}$ units in the *negative* direction yields the same ending point as moving around $\frac{\pi}{3}$ units in the positive direction. Thus, $-\frac{5\pi}{3}$ corresponds to $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, and

tan $s = \frac{y}{x}$

$$\tan\left(-\frac{5\pi}{3}\right) = \tan\frac{\pi}{3} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{\sqrt{3}}{2} \div \frac{1}{2} = \frac{\sqrt{3}}{2} \cdot \frac{2}{1} = \sqrt{3}.$$

(c) An angle of $\frac{2\pi}{3}$ radians corresponds to an angle of 120°. In standard position, 120° lies in quadrant II with a reference angle of 60°, so

Cosine is negative in quadrant II.

$$\cos\frac{2\pi}{3} = \cos 120° = -\cos 60° = -\frac{1}{2}.$$

Reference angle (Section 2.2)

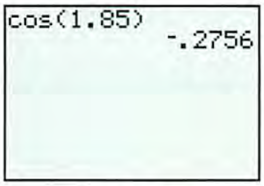
NOW TRY EXERCISES 7, 17, AND 21. ◀

▶ EXAMPLE 3 **APPROXIMATING CIRCULAR FUNCTION VALUES**

Find a calculator approximation for each circular function value.

(a) cos 1.85 **(b)** cos .5149 **(c)** cot 1.3209 **(d)** sec(−2.9234)

Solution



Radian mode

This is how the TI-83/84 Plus calculator displays the result of Example 3(a), fixed to four decimal digits.

(a) With a calculator in radian mode, we find $\cos 1.85 \approx -.2756$.

(b) $\cos .5149 \approx .8703$ Use a calculator in radian mode.

(c) As before, to find cotangent, secant, and cosecant function values, we must use the appropriate reciprocal functions. To find cot 1.3209, first find tan 1.3209 and then find the reciprocal.

$$\cot 1.3209 = \frac{1}{\tan 1.3209} \approx .2552$$

(d) $\sec(-2.9234) = \dfrac{1}{\cos(-2.9234)} \approx -1.0243$

NOW TRY EXERCISES 23, 29, AND 33. ◀

▶ **Caution** A common error in trigonometry is using a calculator in degree mode when radian mode should be used. ***Remember, if you are finding a circular function value of a real number, the calculator must be in radian mode.***

Determining a Number with a Given Circular Function Value

Recall from **Section 2.3** how we used a calculator to determine an angle measure, given a trigonometric function value of the angle. ***Remember that the keys marked*** $\mathbf{sin^{-1}}$, $\mathbf{cos^{-1}}$, ***and*** $\mathbf{tan^{-1}}$ ***do not represent reciprocal functions.***

▶ EXAMPLE 4 **FINDING A NUMBER GIVEN ITS CIRCULAR FUNCTION VALUE**

(a) Approximate the value of s in the interval $\left[0, \frac{\pi}{2}\right]$, if $\cos s = .9685$.

(b) Find the exact value of s in the interval $\left[\pi, \frac{3\pi}{2}\right]$, if $\tan s = 1$.

Solution

(a) Since we are given a cosine value and want to determine the real number in $\left[0, \frac{\pi}{2}\right]$ having this cosine value, we use the *inverse cosine* function of a calculator. With the calculator in radian mode, we find

$$\cos^{-1}(.9685) \approx .2517. \quad \text{(Section 2.3)}$$

See Figure 15. (Refer to your owner's manual to determine how to evaluate the $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ functions with your calculator.)

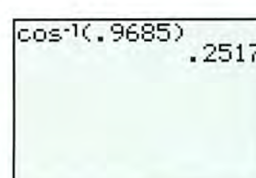


Figure 15

(b) Recall that $\tan \frac{\pi}{4} = 1$, and in quadrant III $\tan s$ is positive. Therefore,

$$\tan\left(\pi + \frac{\pi}{4}\right) = \tan \frac{5\pi}{4} = 1,$$

and $s = \frac{5\pi}{4}$. Figure 16 supports this result.

Figure 16

NOW TRY EXERCISES 55 AND 65. ◀

Applying Circular Functions

▶ EXAMPLE 5 MODELING THE ANGLE OF ELEVATION OF THE SUN

The angle of elevation θ of the sun in the sky at any latitude L is calculated with the formula

$$\sin \theta = \cos D \cos L \cos \omega + \sin D \sin L,$$

where $\theta = 0$ corresponds to sunrise and $\theta = \frac{\pi}{2}$ occurs if the sun is directly overhead. ω (the Greek letter *omega*) is the number of radians that Earth has rotated through since noon, when $\omega = 0$. D is the declination of the sun, which varies because Earth is tilted on its axis. (*Source:* Winter, C., R. Sizmann, and L. L. Vant-Hull, Editors, *Solar Power Plants,* Springer-Verlag, 1991.)

Sacramento, California, has latitude $L = 38.5°$ or .6720 radian. Find the angle of elevation θ of the sun at 3 P.M. on February 29, 2008, where at that time $D \approx -.1425$ and $\omega \approx .7854$.

Solution Use the given formula for $\sin \theta$.

$$\begin{aligned}\sin \theta &= \cos D \cos L \cos \omega + \sin D \sin L \\ &= \cos(-.1425)\cos(.6720)\cos(.7854) + \sin(-.1425)\sin(.6720) \\ &\approx .4593426188\end{aligned}$$

Thus, $\theta \approx .4773$ radian or 27.3°. Use inverse sine.

NOW TRY EXERCISE 77. ◀

CONNECTIONS The diagram shown in Figure 17 illustrates a correspondence that ties together the right triangle ratio definitions of the trigonometric functions introduced in **Chapter 2** with the unit circle interpretation presented in this section. The arc SR is the first quadrant portion of the unit circle, and the standard-position angle POQ is designated θ. By definition, the coordinates of P are $(\cos \theta, \sin \theta)$. The six trigonometric functions of θ can be interpreted as lengths of line segments found in Figure 17.

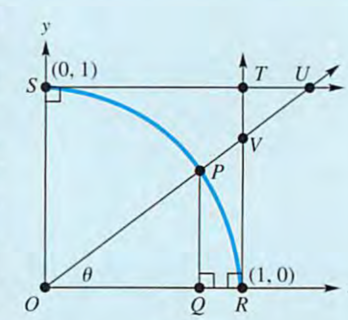


Figure 17

cos θ = OQ, sin θ = PQ

Use right triangle POQ and right triangle ratios:

$$\cos\theta = \frac{\text{side adjacent to }\theta}{\text{hypotenuse}} = \frac{OQ}{OP} = \frac{OQ}{1} = OQ,$$

$$\sin\theta = \frac{\text{side opposite }\theta}{\text{hypotenuse}} = \frac{PQ}{OP} = \frac{PQ}{1} = PQ.$$

tan θ = VR, sec θ = OV

Use right triangle VOR and right triangle ratios:

$$\tan\theta = \frac{\text{side opposite }\theta}{\text{side adjacent to }\theta} = \frac{VR}{OR} = \frac{VR}{1} = VR,$$

$$\sec\theta = \frac{\text{hypotenuse}}{\text{side adjacent to }\theta} = \frac{OV}{OR} = \frac{OV}{1} = OV.$$

cot θ = US, csc θ = OU

US and OR are parallel, and, thus, angle SUO is equal to θ as it is an alternate interior angle to angle $POQ = \theta$. Use right triangle USO and right triangle ratios:

$$\cot SUO = \cot\theta = \frac{\text{side adjacent to }\theta}{\text{side opposite }\theta} = \frac{US}{OS} = \frac{US}{1} = US,$$

$$\csc SUO = \csc\theta = \frac{\text{hypotenuse}}{\text{side opposite }\theta} = \frac{OU}{OS} = \frac{OU}{1} = OU.$$

Figure 18 uses color to illustrate the results found above.

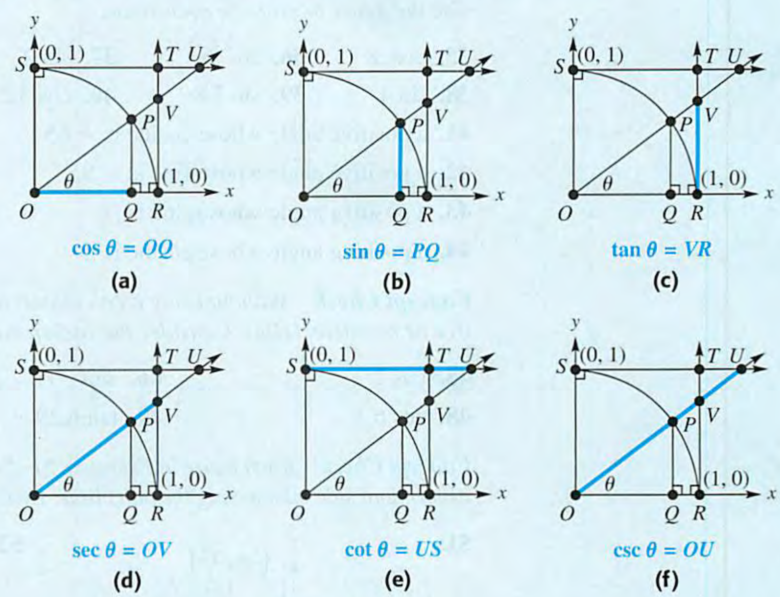


Figure 18

FOR DISCUSSION OR WRITING

1. See if you can find other ways to justify the results given above.
2. How can this interpretation be extended to angles in other quadrants?

3.3 Exercises

For each value of the real number s, find **(a)** sin *s*, **(b)** cos *s*, *and* **(c)** tan *s*. *See Example 1.*

1. $s = \frac{\pi}{2}$ **2.** $s = \pi$ **3.** $s = 2\pi$

4. $s = 3\pi$ **5.** $s = -\pi$ **6.** $s = -\frac{3\pi}{2}$

Find the exact circular function value for each of the following. See Example 2.

7. $\sin \frac{7\pi}{6}$ **8.** $\cos \frac{5\pi}{3}$ **9.** $\tan \frac{3\pi}{4}$ **10.** $\sec \frac{2\pi}{3}$

11. $\csc \frac{11\pi}{6}$ **12.** $\cot \frac{5\pi}{6}$ **13.** $\cos\left(-\frac{4\pi}{3}\right)$ **14.** $\tan \frac{17\pi}{3}$

15. $\cos \frac{7\pi}{4}$ **16.** $\sec \frac{5\pi}{4}$ **17.** $\sin\left(-\frac{4\pi}{3}\right)$ **18.** $\sin\left(-\frac{5\pi}{6}\right)$

19. $\sec \frac{23\pi}{6}$ **20.** $\csc \frac{13\pi}{3}$ **21.** $\tan \frac{5\pi}{6}$ **22.** $\cos \frac{3\pi}{4}$

Find a calculator approximation for each circular function value. See Example 3.

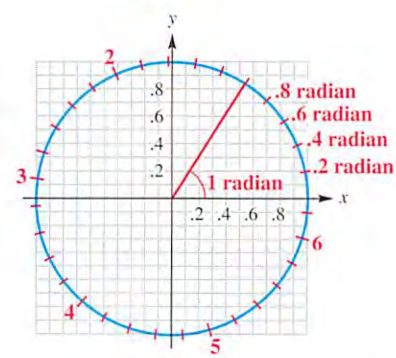
23. sin .6109 **24.** sin .8203 **25.** cos(−1.1519)

26. cos(−5.2825) **27.** tan 4.0203 **28.** tan 6.4752

29. csc(−9.4946) **30.** csc 1.3875 **31.** sec 2.8440

32. sec(−8.3429) **33.** cot 6.0301 **34.** cot 3.8426

Concept Check *The figure displays a unit circle and an angle of 1 radian. The tick marks on the circle are spaced at every two-tenths radian. Use the figure to estimate each value.*



35. cos .8 **36.** cos .6 **37.** sin 2

38. sin 4 **39.** sin 3.8 **40.** cos 3.2

41. a positive angle whose cosine is −.65

42. a positive angle whose sine is −.95

43. a positive angle whose sine is .7

44. a positive angle whose cosine is .3

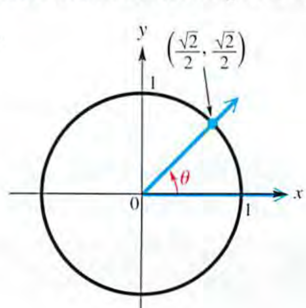
Concept Check *Without using a calculator, decide whether each function value is positive or negative. (Hint: Consider the radian measures of the quadrantal angles.)*

45. cos 2 **46.** sin(−1) **47.** sin 5

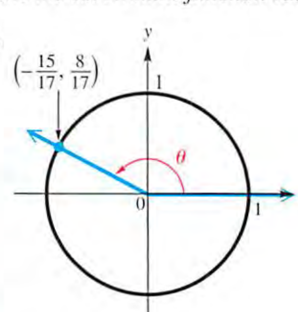
48. cos 6 **49.** tan 6.29 **50.** tan(−6.29)

Concept Check *Each figure in Exercises 51–54 shows an angle θ in standard position with its terminal side intersecting the unit circle. Evaluate the six circular function values of θ.*

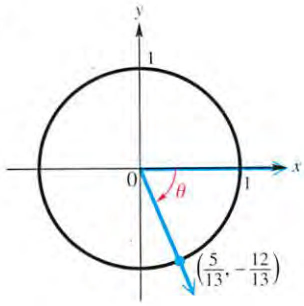
51.



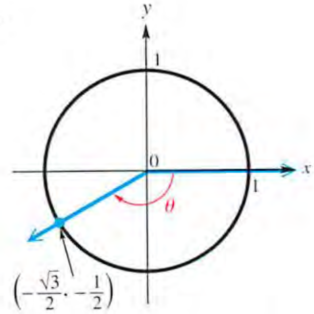
52.



53.



54.



Find the value of s in the interval $\left[0, \frac{\pi}{2}\right]$ *that makes each statement true. See Example 4(a).*

55. $\tan s = .2126$ **56.** $\cos s = .7826$ **57.** $\sin s = .9918$

58. $\cot s = .2994$ **59.** $\sec s = 1.0806$ **60.** $\csc s = 1.0219$

Find the exact value of s in the given interval that has the given circular function value. Do not use a calculator. See Example 4(b).

61. $\left[\frac{\pi}{2}, \pi\right]$; $\sin s = \frac{1}{2}$ **62.** $\left[\frac{\pi}{2}, \pi\right]$; $\cos s = -\frac{1}{2}$

63. $\left[\pi, \frac{3\pi}{2}\right]$; $\tan s = \sqrt{3}$ **64.** $\left[\pi, \frac{3\pi}{2}\right]$; $\sin s = -\frac{1}{2}$

65. $\left[\frac{3\pi}{2}, 2\pi\right]$; $\tan s = -1$ **66.** $\left[\frac{3\pi}{2}, 2\pi\right]$; $\cos s = \frac{\sqrt{3}}{2}$

Suppose an arc of length s lies on the unit circle $x^2 + y^2 = 1$*, starting at the point* $(1, 0)$ *and terminating at the point* (x, y)*. (See Figure 11.) Use a calculator to find the approximate coordinates for* (x, y)*. (Hint:* $x = \cos s$ *and* $y = \sin s$*.)*

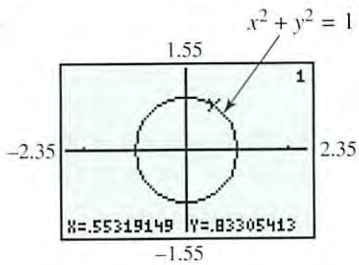
67. $s = 2.5$ **68.** $s = 3.4$ **69.** $s = -7.4$ **70.** $s = -3.9$

Concept Check *For each value of s, use a calculator to find* sin *s and* cos *s and then use the results to decide in which quadrant an angle of s radians lies.*

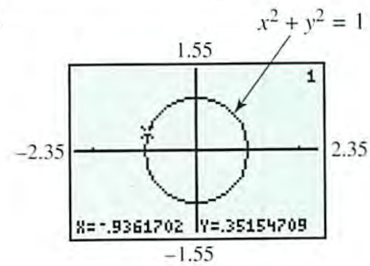
71. $s = 51$ **72.** $s = 49$ **73.** $s = 65$ **74.** $s = 79$

Concept Check *In Exercises 75 and 76, each graphing calculator screen shows a point on the unit circle. What is the length of the shortest arc of the circle from* $(1, 0)$ *to the point?*

75.



76.



(Modeling) *Solve each problem. See Example 5.*

77. *Elevation of the Sun* Refer to Example 5.

(a) Repeat the example for New Orleans, which has latitude $L = 30°$.

(b) Compare your answers. Do they agree with your intuition?

78. ***Length of a Day*** The number of daylight hours H at any location can be calculated using the formula

$$\cos(.1309H) = -\tan D \tan L,$$

where D and L are defined in Example 5. Use this trigonometric equation to calculate the shortest and longest days in Minneapolis, Minnesota, if its latitude $L = 44.88°$, the shortest day occurs when $D = -23.44°$, and the longest day occurs when $D = 23.44°$. Remember to convert degrees to radians. (*Source:* Winter, C., R. Sizmann, and L. L. Vant-Hull, Editors, *Solar Power Plants,* Springer-Verlag, 1991.)

79. ***Maximum Temperatures*** Because the values of the circular functions repeat every 2π, they are used to describe things that repeat periodically. For example, the maximum afternoon temperature in a given city might be modeled by

$$t = 60 - 30 \cos \frac{x\pi}{6},$$

where t represents the maximum afternoon temperature in month x, with $x = 0$ representing January, $x = 1$ representing February, and so on. Find the maximum afternoon temperature for each of the following months.

(a) January **(b)** April **(c)** May
(d) June **(e)** August **(f)** October

80. ***Temperature in Fairbanks*** The temperature in Fairbanks is modeled by

$$T(x) = 37 \sin\left[\frac{2\pi}{365}(x - 101)\right] + 25,$$

where $T(x)$ is the temperature in degrees Fahrenheit on day x, with $x = 1$ corresponding to January 1 and $x = 365$ corresponding to December 31. Use a calculator to estimate the temperature on the following days. (*Source:* Lando, B. and C. Lando, "Is the Graph of Temperature Variation a Sine Curve?", *The Mathematics Teacher,* 70, September 1977.)

(a) March 1 (day 60) **(b)** April 1 (day 91) **(c)** Day 150
(d) June 15 **(e)** September 1 **(f)** October 31

CHAPTER 3 ▶ Quiz (Sections 3.1–3.3)

Convert each degree measure to radians.

1. 225° **2.** −330°

Convert each radian measure to degrees.

3. $\frac{5\pi}{3}$ **4.** $-\frac{7\pi}{6}$

A central angle of a circle with radius 300 in. *intercepts an arc of* 450 in. *(These measures are accurate to the nearest inch.) Find each measure.*

5. the radian measure of the angle **6.** the area of the sector

Find each circular function value. Give exact values.

7. $\cos \frac{7\pi}{4}$ **8.** $\sin\left(-\frac{5\pi}{6}\right)$ **9.** $\tan 3\pi$

10. Find the exact value of s in the interval $\left[\frac{\pi}{2}, \pi\right]$ if $\sin s = \frac{\sqrt{3}}{2}$.

3.4 Linear and Angular Speed

Linear Speed ▪ Angular Speed

Linear Speed

In many situations we need to know how fast a point on a circular disk is moving or how fast the central angle of such a disk is changing. Some examples occur with machinery involving gears or pulleys or the speed of a car around a curved portion of highway.

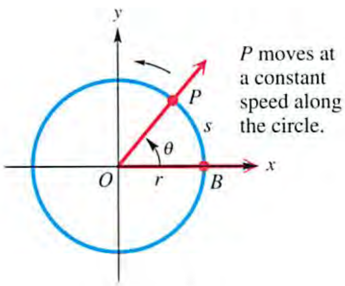


Figure 19

Suppose that point P moves at a constant speed along a circle of radius r and center O. See Figure 19. The measure of how fast the position of P is changing is called **linear speed.** If v represents linear speed, then

$$\text{speed} = \frac{\text{distance}}{\text{time}}, \qquad \text{or} \qquad v = \frac{s}{t},$$

where s is the length of the arc traced by point P at time t. (This formula is just a restatement of $d = rt$ with s as distance, v as rate (speed), and t as time.)

Angular Speed

As point P in Figure 19 moves along the circle, ray OP rotates around the origin. Since ray OP is the terminal side of angle POB, the measure of the angle changes as P moves along the circle. The measure of how fast angle POB is changing is called **angular speed.** Angular speed, symbolized ω, is given as

$$\omega = \frac{\theta}{t}, \qquad \theta \text{ in radians,}$$

where θ is the measure of angle POB at time t. As with earlier formulas in this chapter, θ must be measured in radians, with ω expressed as radians per unit of time. Angular speed is used in physics and engineering, among other applications.

In **Section 3.2,** the length s of the arc intercepted on a circle of radius r by a central angle of measure θ radians was found to be $s = r\theta$. Using this formula, the formula for linear speed, $v = \frac{s}{t}$, becomes

$$v = \frac{s}{t} = \frac{r\theta}{t} = r \cdot \frac{\theta}{t} = r\omega. \qquad s = r\theta;\ \omega = \frac{\theta}{t}$$

The formula $v = r\omega$ relates linear and angular speeds.

As an example of linear and angular speeds, consider the following. The human joint that can be flexed the fastest is the wrist, which can rotate through 90°, or $\frac{\pi}{2}$ radians, in .045 sec while holding a tennis racket. The angular speed of a human wrist swinging a tennis racket is

$$\omega = \frac{\theta}{t} = \frac{\frac{\pi}{2}}{.045} \approx 35 \text{ radians per sec.}$$

If the radius (distance) from the tip of the racket to the wrist joint is 2 ft, then the speed at the tip of the racket is

$$v = r\omega \approx 2(35) = 70 \text{ ft per sec,} \qquad \text{or about 48 mph.}$$

In a tennis serve the arm rotates at the shoulder, so the final speed of the racket is considerably faster. (*Source:* Cooper, J. and R. Glassow, *Kinesiology,* Second Edition, C.V. Mosby, 1968.)

The formulas for angular and linear speed are summarized in the table.

Angular Speed	Linear Speed
$\omega = \frac{\theta}{t}$ (ω in radians per unit time, θ in radians)	$v = \frac{s}{t}$ $v = \frac{r\theta}{t}$ $v = r\omega$

▶ EXAMPLE 1 USING LINEAR AND ANGULAR SPEED FORMULAS

Suppose that point P is on a circle with radius 10 cm, and ray OP is rotating with angular speed $\frac{\pi}{18}$ radian per sec.

(a) Find the angle generated by P in 6 sec.

(b) Find the distance traveled by P along the circle in 6 sec.

(c) Find the linear speed of P in centimeters per second.

Solution

(a) The speed of ray OP is $\omega = \frac{\pi}{18}$ radian per sec. Since $\omega = \frac{\theta}{t}$, then in 6 sec

$$\frac{\pi}{18} = \frac{\theta}{6}$$ Let $\omega = \frac{\pi}{18}$ and $t = 6$ in the angular speed formula.

$$\theta = \frac{6\pi}{18} = \frac{\pi}{3} \text{ radians.}$$ Solve for θ.

(b) From part (a), P generates an angle of $\frac{\pi}{3}$ radians in 6 sec. The distance traveled by P along the circle is

$$s = r\theta = 10\left(\frac{\pi}{3}\right) = \frac{10\pi}{3} \text{ cm.}$$ (Section 3.2)

(c) From part (b), $s = \frac{10\pi}{3}$ for 6 sec, so for 1 sec we divide by 6.

$$v = \frac{s}{t} = \frac{\frac{10\pi}{3}}{6} = \frac{10\pi}{3} \div 6 = \frac{10\pi}{3} \cdot \frac{1}{6} = \frac{5\pi}{9} \text{ cm per sec}$$

Be careful simplifying this complex fraction.

NOW TRY EXERCISE 3. ◀

▶ EXAMPLE 2 FINDING ANGULAR SPEED OF A PULLEY AND LINEAR SPEED OF A BELT

A belt runs a pulley of radius 6 cm at 80 revolutions per min.

(a) Find the angular speed of the pulley in radians per second.

(b) Find the linear speed of the belt in centimeters per second.

Solution

(a) In 1 min, the pulley makes 80 revolutions. Each revolution is 2π radians, so

$$80(2\pi) = 160\pi \text{ radians per min.}$$

Since there are 60 sec in 1 min, we find ω, the angular speed in radians per second, by dividing 160π by 60.

$$\omega = \frac{160\pi}{60} = \frac{8\pi}{3} \text{ radians per sec}$$

(b) The linear speed of the belt will be the same as that of a point on the circumference of the pulley. Thus,

$$v = r\omega = 6\left(\frac{8\pi}{3}\right) = 16\pi \approx 50 \text{ cm per sec.}$$

NOW TRY EXERCISE 39.

▶ EXAMPLE 3 FINDING LINEAR SPEED AND DISTANCE TRAVELED BY A SATELLITE

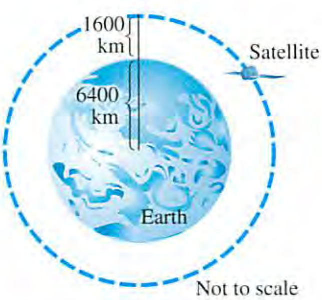


Figure 20

A satellite traveling in a circular orbit 1600 km above the surface of Earth takes 2 hr to make an orbit. The radius of Earth is approximately 6400 km. See Figure 20.

(a) Approximate the linear speed of the satellite in kilometers per hour.

(b) Approximate the distance the satellite travels in 4.5 hr.

Solution

(a) The distance of the satellite from the center of Earth is approximately

$$r = 1600 + 6400 = 8000 \text{ km.}$$

For one orbit, $\theta = 2\pi$, and

$$s = r\theta = 8000(2\pi) \text{ km.}$$

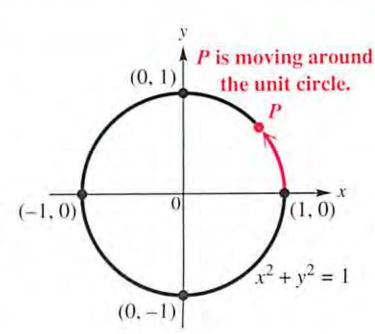
Since it takes 2 hr to complete an orbit, the linear speed is approximately

$$v = \frac{s}{t} = \frac{8000(2\pi)}{2} = 8000\pi \approx 25{,}000 \text{ km per hr.}$$

(b) $s = vt = 8000\pi(4.5) = 36{,}000\pi \approx 110{,}000$ km

NOW TRY EXERCISE 37.

3.4 Exercises



1. *Concept Check* If the point P moves around the circumference of the unit circle at an angular velocity of 1 radian per sec, how long will it take for P to move around the entire circle?

2. *Concept Check* If the point P moves around the circumference of the unit circle at a speed of 1 unit per sec, how long will it take for P to move around the entire circle?

Suppose that point P is on a circle with radius r, and ray OP is rotating with angular speed ω. For the given values of r, ω, and t, find each of the following.

(a) *the angle generated by P in time t*
(b) *the distance traveled by P along the circle in time t*
(c) *the linear speed of P*

See Example 1.

3. $r = 20$ cm, $\omega = \dfrac{\pi}{12}$ radian per sec, $t = 6$ sec

4. $r = 30$ cm, $\omega = \dfrac{\pi}{10}$ radian per sec, $t = 4$ sec

Use the formula $\omega = \frac{\theta}{t}$ *to find the value of the missing variable.*

5. $\omega = \dfrac{2\pi}{3}$ radians per sec, $t = 3$ sec

6. $\omega = \dfrac{\pi}{4}$ radian per min, $t = 5$ min

7. $\theta = \dfrac{3\pi}{4}$ radians, $t = 8$ sec

8. $\theta = \dfrac{2\pi}{5}$ radians, $t = 10$ sec

9. $\theta = \dfrac{2\pi}{9}$ radian, $\omega = \dfrac{5\pi}{27}$ radian per min

10. $\theta = \dfrac{3\pi}{8}$ radians, $\omega = \dfrac{\pi}{24}$ radian per min

11. $\theta = 3.871142$ radians, $t = 21.4693$ sec

12. $\omega = .90674$ radian per min, $t = 11.876$ min

Use the formula $v = r\omega$ *to find the value of the missing variable.*

13. $r = 12$ m, $\omega = \dfrac{2\pi}{3}$ radians per sec

14. $r = 8$ cm, $\omega = \dfrac{9\pi}{5}$ radians per sec

15. $v = 9$ m per sec, $r = 5$ m

16. $v = 18$ ft per sec, $r = 3$ ft

17. $v = 107.692$ m per sec, $r = 58.7413$ m

18. $r = 24.93215$ cm, $\omega = .372914$ radian per sec

The formula $\omega = \frac{\theta}{t}$ *can be rewritten as* $\theta = \omega t$*. Using* ωt *for* θ *changes* $s = r\theta$ *to* $s = r\omega t$*. Use the formula* $s = r\omega t$ *to find the value of the missing variable.*

19. $r = 6$ cm, $\omega = \dfrac{\pi}{3}$ radians per sec, $t = 9$ sec

20. $r = 9$ yd, $\omega = \dfrac{2\pi}{5}$ radians per sec, $t = 12$ sec

21. $s = 6\pi$ cm, $r = 2$ cm, $\omega = \dfrac{\pi}{4}$ radian per sec

22. $s = \dfrac{12\pi}{5}$ m, $r = \dfrac{3}{2}$ m, $\omega = \dfrac{2\pi}{5}$ radians per sec

23. $s = \frac{3\pi}{4}$ km, $r = 2$ km, $t = 4$ sec

24. $s = \frac{8\pi}{9}$ m, $r = \frac{4}{3}$ m, $t = 12$ sec

Find ω for each of the following.

25. the hour hand of a clock

26. a line from the center to the edge of a CD revolving 300 times per min

27. the minute hand of a clock

28. the second hand of a clock

Find v for each of the following.

29. the tip of the minute hand of a clock, if the hand is 7 cm long

30. the tip of the second hand of a clock, if the hand is 28 mm long

31. a point on the edge of a flywheel of radius 2 m, rotating 42 times per min

32. a point on the tread of a tire of radius 18 cm, rotating 35 times per min

33. the tip of an airplane propeller 3 m long, rotating 500 times per min (*Hint:* $r = 1.5$ m)

34. a point on the edge of a gyroscope of radius 83 cm, rotating 680 times per min

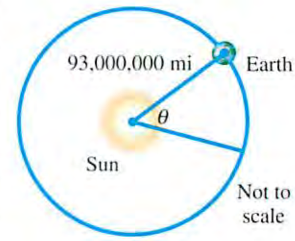
Solve each problem. See Examples 1–3.

35. ***Speed of a Bicycle*** The tires of a bicycle have radius 13 in. and are turning at the rate of 215 revolutions per min. See the figure. How fast is the bicycle traveling in miles per hour? (*Hint:* 5280 ft = 1 mi)

36. ***Hours in a Martian Day*** Mars rotates on its axis at the rate of about .2552 radian per hr. Approximately how many hours are in a Martian day (or *sol*)? (*Source:* Wright, J. W., General Editor, *The Universal Almanac,* Andrews and McMeel, 1997.)

Opposite sides of Mars

37. ***Angular and Linear Speeds of Earth*** Earth travels about the sun in an orbit that is almost circular. Assume that the orbit is a circle with radius 93,000,000 mi. Its angular and linear speeds are used in designing solar-power facilities.

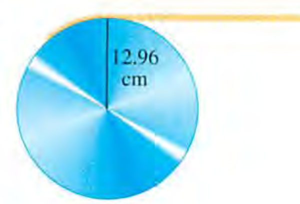


(a) Assume that a year is 365 days, and find the angle formed by Earth's movement in one day.
(b) Give the angular speed in radians per hour.
(c) Find the linear speed of Earth in miles per hour.

38. ***Angular and Linear Speeds of Earth*** Earth revolves on its axis once every 24 hr. Assuming that Earth's radius is 6400 km, find the following.

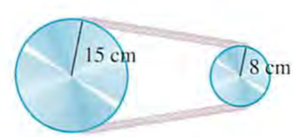
(a) angular speed of Earth in radians per day and radians per hour
(b) linear speed at the North Pole or South Pole
(c) linear speed at Quito, Ecuador, a city on the equator
(d) linear speed at Salem, Oregon (halfway from the equator to the North Pole)

39. ***Speeds of a Pulley and a Belt*** The pulley shown has a radius of 12.96 cm. Suppose it takes 18 sec for 56 cm of belt to go around the pulley.



(a) Find the linear speed of the belt in centimeters per second.
(b) Find the angular speed of the pulley in radians per second.

40. ***Angular Speeds of Pulleys*** The two pulleys in the figure have radii of 15 cm and 8 cm, respectively. The larger pulley rotates 25 times in 36 sec. Find the angular speed of each pulley in radians per second.



41. ***Radius of a Spool of Thread*** A thread is being pulled off a spool at the rate of 59.4 cm per sec. Find the radius of the spool if it makes 152 revolutions per min.

42. ***Time to Move Along a Railroad Track*** A railroad track is laid along the arc of a circle of radius 1800 ft. The circular part of the track subtends a central angle of 40°. How long (in seconds) will it take a point on the front of a train traveling 30.0 mph to go around this portion of the track?

43. ***Angular Speed of a Motor Propeller*** A 90-horsepower outboard motor at full throttle will rotate its propeller at exactly 5000 revolutions per min. Find the angular speed of the propeller in radians per second.

44. ***Linear Speed of a Golf Club*** The shoulder joint can rotate at 25.0 radians per sec. If a golfer's arm is straight and the distance from the shoulder to the club head is 5.00 ft, find the linear speed of the club head from shoulder rotation. (*Source:* Cooper, J. and R. Glassow, *Kinesiology*, Second Edition, C.V. Mosby, 1968.)

Chapter 3 Summary

KEY TERMS

3.1 radian
circumference
3.2 sector of a circle
latitude
longitude
subtend
degree of curvature
nautical mile
statute mile
3.3 unit circle
circular functions
reference arc
3.4 linear speed v
angular speed ω

QUICK REVIEW

CONCEPTS	EXAMPLES

3.1 Radian Measure

An angle with its vertex at the center of a circle that intercepts an arc on the circle equal in length to the radius of the circle has a measure of **1 radian.**

See Figure 1 on page 102.

Degree/Radian Relationship $\mathbf{180° = \pi}$ **radians**

Converting Between Degrees and Radians

1. Multiply a degree measure by $\frac{\pi}{180}$ radian and simplify to convert to radians.
2. Multiply a radian measure by $\frac{180°}{\pi}$ and simplify to convert to degrees.

Convert 135° to radians.

$$135° = 135\left(\frac{\pi}{180} \text{ radian}\right) = \frac{3\pi}{4} \text{ radians}$$

Convert $-\frac{5\pi}{3}$ radians to degrees.

$$-\frac{5\pi}{3} \text{ radians} = -\frac{5\pi}{3}\left(\frac{180°}{\pi}\right) = -300°$$

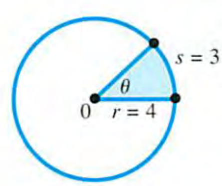
3.2 Applications of Radian Measure

Arc Length
The length s of the arc intercepted on a circle of radius r by a central angle of measure θ radians is given by the product of the radius and the radian measure of the angle, or

$$s = r\theta, \qquad \theta \text{ in radians.}$$

In the figure, $s = r\theta$, so

$$\theta = \frac{s}{r} = \frac{3}{4} \text{ radian.}$$



Area of a Sector
The area of a sector of a circle of radius r and central angle θ is given by

$$A = \frac{1}{2}r^2\theta, \qquad \theta \text{ in radians.}$$

The area of the sector in the figure above is

$$A = \frac{1}{2}(4)^2\left(\frac{3}{4}\right) = 6 \text{ sq units.}$$

(continued)

CONCEPTS | EXAMPLES

3.3 The Unit Circle and Circular Functions

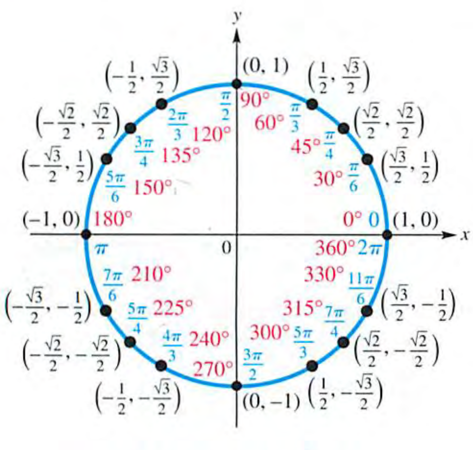
Circular Functions

Start at the point $(1, 0)$ on the unit circle $x^2 + y^2 = 1$ and lay off an arc of length $|s|$ along the circle, going counterclockwise if s is positive and clockwise if s is negative. Let the endpoint of the arc be at the point (x, y). The six circular functions of s are defined as follows. (Assume that no denominators are 0.)

$$\sin s = y \qquad \cos s = x \qquad \tan s = \frac{y}{x}$$

$$\csc s = \frac{1}{y} \qquad \sec s = \frac{1}{x} \qquad \cot s = \frac{x}{y}$$

The Unit Circle



The unit circle $x^2 + y^2 = 1$

Use the unit circle to find each value.

$$\sin\frac{5\pi}{6} = \frac{1}{2}$$

$$\cos\frac{3\pi}{2} = 0$$

$$\tan\frac{\pi}{4} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$$

$$\csc\frac{7\pi}{4} = \frac{1}{-\frac{\sqrt{2}}{2}} = -\sqrt{2}$$

$$\sec\frac{7\pi}{6} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2\sqrt{3}}{3}$$

$$\cot\frac{\pi}{3} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{3}$$

$$\sin 0 = 0$$

$$\cos\frac{\pi}{2} = 0$$

$$\tan \pi = \frac{0}{-1} = 0$$

$$\sin\left(-\frac{\pi}{2}\right) = \sin\frac{3\pi}{2} = -1$$

$$\cot\left(-\frac{3\pi}{2}\right) = \cot\frac{\pi}{2} = \frac{0}{1} = 0$$

$\tan\frac{\pi}{2}$ is undefined.

3.4 Linear and Angular Speed

Formulas for Angular and Linear Speed

Angular Speed	Linear Speed
$\omega = \frac{\theta}{t}$	$v = \frac{s}{t}$
(ω in radians per unit time, θ in radians)	$v = \frac{r\theta}{t}$
	$v = r\omega$

A belt runs a pulley of radius 8 in. at 60 revolutions per min. Find

(a) the angular speed ω in radians per minute, and

(b) the linear speed v of the belt in inches per minute.

(a) $\omega = 60(2\pi) = 120\pi$ radians per min

(b) $v = r\omega = 8(120\pi) = 960\pi$ in. per min

CHAPTER 3 ▶ Review Exercises

1. *Concept Check* Which is larger—an angle of 1° or an angle of 1 radian?
2. *Concept Check* Consider each angle in standard position having the given radian measure. In what quadrant does the terminal side lie?
   (a) 3 (b) 4 (c) −2 (d) 7
3. Find three angles coterminal with an angle of 1 radian.
4. Give an expression that generates all angles coterminal with an angle of $\frac{\pi}{6}$ radian. Let n represent any integer.

Convert each degree measure to radians. Leave answers as multiples of π.

5. 45° 6. 120° 7. 175°
8. 330° 9. 800° 10. 1020°

Convert each radian measure to degrees.

11. $\frac{5\pi}{4}$ 12. $\frac{9\pi}{10}$ 13. $\frac{8\pi}{3}$
14. $-\frac{6\pi}{5}$ 15. $-\frac{11\pi}{18}$ 16. $\frac{21\pi}{5}$

Suppose the tip of the minute hand of a clock is 2 in. *from the center of the clock. For each duration, determine the distance traveled by the tip of the minute hand.*

17. 15 min 18. 20 min
19. 3 hr 20. $10\frac{1}{2}$ hr

Solve each problem. Use a calculator as necessary.

21. *Arc Length* The radius of a circle is 15.2 cm. Find the length of an arc of the circle intercepted by a central angle of $\frac{3\pi}{4}$ radians.
22. *Arc Length* Find the length of an arc intercepted by a central angle of .769 radian on a circle with radius 11.4 cm.
23. *Arc Length* A circle has radius 8.973 cm. Find the length of an arc on this circle intercepted by a central angle of 49.06°.
24. *Area of a Sector* A central angle of $\frac{7\pi}{4}$ radians forms a sector of a circle. Find the area of the sector if the radius of the circle is 28.69 in.
25. *Area of a Sector* Find the area of a sector of a circle having a central angle of 21° 40′ in a circle of radius 38.0 m.
26. *Height of a Tree* A tree 2000 yd away subtends an angle of 1° 10′. Find the height of the tree to two significant digits.

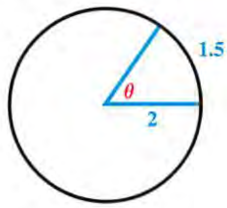
Distance Between Cities *Assume that the radius of Earth is* 6400 km *in Exercises 27 and 28.*

27. Find the distance in kilometers between cities on a north-south line that are on latitudes 28° N and 12° S, respectively.

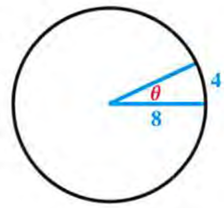
28. Two cities on the equator have longitudes of 72° E and 35° W, respectively. Find the distance between the cities.

Concept Check *In Exercises 29 and 30, find the measure of the central angle θ (in radians) and the area of the sector.*

29.



30.



31. ***Concept Check*** The hour hand of a wall clock measures 6 in. from its tip to the center of the clock.

(a) Through what angle (in radians) does the hour hand pass between 1 o'clock and 3 o'clock?

(b) What distance does the tip of the hour hand travel during the time period from 1 o'clock to 3 o'clock?

32. Describe what would happen to the central angle for a given arc length of a circle if the circle's radius were doubled. (Assume everything else is unchanged.)

Find each exact function value. Do not use a calculator.

33. $\tan \frac{\pi}{3}$ **34.** $\cos \frac{2\pi}{3}$ **35.** $\sin\left(-\frac{5\pi}{6}\right)$

36. $\tan\left(-\frac{7\pi}{3}\right)$ **37.** $\csc\left(-\frac{11\pi}{6}\right)$ **38.** $\cot\left(-\frac{17\pi}{3}\right)$

Without using a calculator, determine which of the following is greater.

39. tan 1 or tan 2 **40.** sin 1 or tan 1

41. cos 2 or sin 2 **42.** cos(sin 0) or sin(cos 0)

Use a calculator to find an approximation for each circular function value. Be sure your calculator is set in radian mode.

43. sin 1.0472 **44.** tan 1.2275 **45.** cos(−.2443)

46. cot 3.0543 **47.** sec 7.3159 **48.** csc 4.8386

Find the value of s in the interval $\left[0, \frac{\pi}{2}\right]$ that makes each statement true.

49. $\cos s = .9250$ **50.** $\tan s = 4.0112$ **51.** $\sin s = .4924$

52. $\csc s = 1.2361$ **53.** $\cot s = .5022$ **54.** $\sec s = 4.5600$

Find the exact value of s in the given interval that has the given circular function value. Do not use a calculator.

55. $\left[0, \frac{\pi}{2}\right]$; $\cos s = \frac{\sqrt{2}}{2}$ **56.** $\left[\frac{\pi}{2}, \pi\right]$; $\tan s = -\sqrt{3}$

57. $\left[\pi, \frac{3\pi}{2}\right]$; $\sec s = -\frac{2\sqrt{3}}{3}$

58. $\left[\frac{3\pi}{2}, 2\pi\right]$; $\sin s = -\frac{1}{2}$

Solve each problem, where t, ω, θ, and s are as defined in ***Section 3.4.***

59. Find t if $\theta = \frac{5\pi}{12}$ radians and $\omega = \frac{8\pi}{9}$ radians per sec.

60. Find θ if $t = 12$ sec and $\omega = 9$ radians per sec.

61. Find ω if $t = 8$ sec and $\theta = \frac{2\pi}{5}$ radians.

62. Find ω if $s = \frac{12\pi}{25}$ ft, $r = \frac{3}{5}$ ft, and $t = 15$ sec.

63. Find s if $r = 11.46$ cm, $\omega = 4.283$ radians per sec, and $t = 5.813$ sec.

64. ***Linear Speed of a Flywheel*** Find the linear speed of a point on the edge of a flywheel of radius 7 cm if the flywheel is rotating 90 times per sec.

65. ***Angular Speed of a Ferris Wheel*** A Ferris wheel has radius 25 ft. If it takes 30 sec for the wheel to turn $\frac{5\pi}{6}$ radians, what is the angular speed of the wheel?

(Modeling) *Solve each problem.*

66. ***Phase Angle of the Moon*** Because the moon orbits Earth, we observe different phases of the moon during the period of a month. In the figure, t is called the **phase angle.**

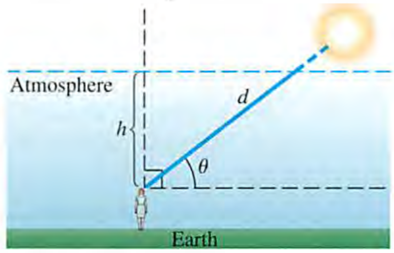
The **phase** F of the moon is modeled by

$$F(t) = \frac{1}{2}(1 - \cos t),$$

and gives the fraction of the moon's face that is illuminated by the sun. (*Source:* Duffet-Smith, P., *Practical Astronomy with Your Calculator,* Cambridge University Press, 1988.) Evaluate each expression and interpret the result.

(a) $F(0)$ **(b)** $F\left(\frac{\pi}{2}\right)$ **(c)** $F(\pi)$ **(d)** $F\left(\frac{3\pi}{2}\right)$

67. ***Atmospheric Effect on Sunlight*** The shortest path for the sun's rays through Earth's atmosphere occurs when the sun is directly overhead. Disregarding the curvature of Earth, as the sun moves lower on the horizon, the distance that sunlight passes through the atmosphere increases by a factor of $\csc \theta$, where θ is the angle of elevation of the sun. This increased distance reduces both the intensity of the sun and the amount of ultraviolet light that reaches Earth's surface. See the figure below. (*Source:* Winter, C., R. Sizmann, and L. L. Vant-Hull, Editors, *Solar Power Plants,* Springer-Verlag, 1991.)



(a) Verify that $d = h \csc \theta$.

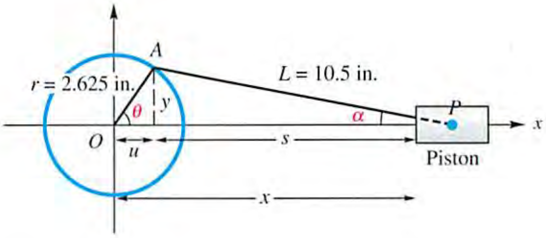
(b) Determine θ when $d = 2h$.

(c) The atmosphere filters out the ultraviolet light that causes skin to burn. Compare the difference between sunbathing when $\theta = \frac{\pi}{2}$ and when $\theta = \frac{\pi}{3}$. Which measure gives less ultraviolet light?

68. ***Engine Specifications*** The figure shows a rotating wheel (crankshaft) with radius r and a connecting rod AP with length L. The piston slides back and forth along the x-axis as the wheel rotates counterclockwise at a rate of R revolutions per minute (RPM). The 1937 John Deere B engine, used in all models from 1935–1938, had the following specifications.

Maximum RPM: 1340 (no load) $\approx$ 505,168.1 radians per hr
Connecting rod: 10.500000 in.
Stroke: 5.250000 in.

(*Source:* Drost, J. P. and R. H. Kunferman, "Related Rates Challenge Problem: Calculate the Velocity of a Piston," *The AMATYC Review,* Vol. 21 No. 1, Fall 1999.)



Find an expression for each measure.

(a) y **(b)** u **(c)** s **(d)** x

(e) The velocity v of the piston at the maximum RPM is given by

$$v = -2.625 \sin \theta \left(1 + \frac{\cos \theta}{\sqrt{15 + \cos^2 \theta}}\right)(505{,}168.1)\left(\frac{1}{12 \cdot 5280}\right),$$

where the last quantity changes inches to miles, so the velocity will be in miles per hour. Use the maximum-finding feature of a graphing calculator, with the window $[0, 6]$ by $[-25, 25]$, to find the value of θ that maximizes velocity. Give this maximum velocity.

CHAPTER 3 ▶ Test

Convert each degree measure to radians.

1. $120°$
2. $-45°$
3. $5°$ (to the nearest hundredth)

Convert each radian measure to degrees.

4. $\frac{3\pi}{4}$
5. $-\frac{7\pi}{6}$
6. 4 (to the nearest hundredth)

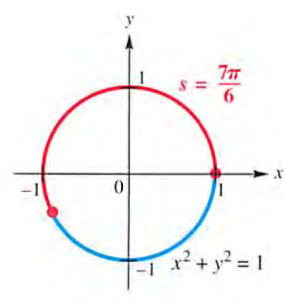
7. A central angle of a circle with radius 150 cm intercepts an arc of 200 cm. Find each measure.
   (a) the radian measure of the angle
   (b) the area of a sector with that central angle

8. ***Rotation of Gas Gauge Arrow*** The arrow on a car's gasoline gauge is $\frac{1}{2}$ in. long. See the figure. Through what angle does the arrow rotate when it moves 1 in. on the gauge?

Find each circular function value.

9. $\sin \frac{3\pi}{4}$
10. $\cos\left(-\frac{7\pi}{6}\right)$
11. $\tan \frac{3\pi}{2}$
12. $\sec \frac{8\pi}{3}$
13. $\tan \pi$
14. $\cos \frac{3\pi}{2}$

15. Give the sine, cosine, and tangent of s.

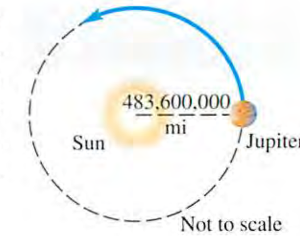


16. Give the domains of the six circular functions.

17. (a) Use a calculator to approximate s in the interval $\left[0, \frac{\pi}{2}\right]$, if $\sin s = .8258$.
    (b) Find the exact value of s in the interval $\left[0, \frac{\pi}{2}\right]$, if $\cos s = \frac{1}{2}$.

18. ***Angular and Linear Speed of a Point*** Suppose that point P is on a circle with radius 60 cm, and ray OP is rotating with angular speed $\frac{\pi}{12}$ radian per sec.
    (a) Find the angle generated by P in 8 sec.
    (b) Find the distance traveled by P along the circle in 8 sec.
    (c) Find the linear speed of P.

19. ***Orbital Speed of Jupiter*** It takes Jupiter 11.64 yr to complete one orbit around the sun. See the figure. If Jupiter's average distance from the sun is 483,600,000 mi, find its orbital speed (speed along its orbital path) in miles per second. (*Source:* Wright, J. W., General Editor, *The Universal Almanac*, Andrews and McMeel, 1997.)



20. ***Ferris Wheel*** A Ferris wheel has radius 50.0 ft. A person takes a seat and then the wheel turns $\frac{2\pi}{3}$ radians.

(a) How far is the person above the ground?

(b) If it takes 30 sec for the wheel to turn $\frac{2\pi}{3}$ radians, what is the angular speed of the wheel?

CHAPTER 3 ▶ Quantitative Reasoning

Can you find the radius of an Indian artifact given an arc of a circle?

Suppose you find an interesting Indian pottery fragment. The archaeology professor at your college believes it is a piece of the edge of a ceremonial plate and shows you a formula that will give the radius of the original plate using measurements from your fragment, shown in Figure A. Measurements are in inches.

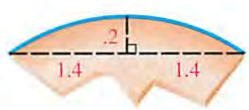


Figure A

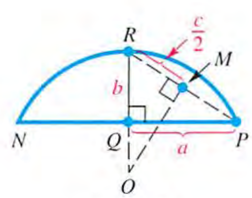


Figure B

In Figure B, a is $\frac{1}{2}$ the length of chord NP and b is the distance from the midpoint of chord NP to the circle. According to the professor's formula, the radius of the circle, OR, is given by

$$r = \frac{a^2 + b^2}{2b}.$$

Why does this formula work? See if you can use the trigonometry you've studied so far to explain it by answering the following.

1. Extend line segment RQ to a radius of the arc, and draw line segment RP and line segment MO, which is the perpendicular bisector of line segment RP. See Figure B. Find a pair of similar triangles in the figure. How do you know they are similar?
2. Find two pairs of corresponding sides, and set their ratios equal. Solve the equation for r.
3. How are a, b, and c related? Use this relationship to substitute for c in terms of a and b.
4. What is the radius of the original plate from which your fragment came?