4 Graphs of the Circular Functions

Phenomena that repeat in a regular pattern, such as rotation of a planet on its axis, high and low tides, and average monthly temperature, can be modeled by *periodic functions.* In Example 6 of Section 4.2, we model the average monthly temperature in New Orleans, a city famous for its Mardi Gras celebration, using a graph of the sine function. Since the 18th century, residents of New Orleans have hosted this annual carnival, featuring festive parades and pageants, elaborate costumes, and dancing in the streets, to mark the final days before the Christian season of Lent. (*Source: Microsoft Encarta Encyclopedia.*)

4.1 Graphs of the Sine and Cosine Functions

Periodic Functions ▪ Graph of the Sine Function ▪ Graph of the Cosine Function ▪ Graphing Techniques, Amplitude, and Period ▪ Using a Trigonometric Model

Periodic Functions Many things in daily life repeat with a predictable pattern, such as weather, tides, and hours of daylight. Because the sine and cosine functions repeat their values in a regular pattern, they are *periodic functions.* Figure 1 shows a periodic graph that represents a normal heartbeat.

Figure 1

▼ LOOKING AHEAD TO CALCULUS
Periodic functions are used throughout calculus, so you will need to know their characteristics. One use of these functions is to describe the location of a point in the plane using **polar coordinates,** an alternative to rectangular coordinates. (See **Chapter 8.**)

PERIODIC FUNCTION

A **periodic function** is a function f such that

$$f(x) = f(x + np),$$

for every real number x in the domain of f, every integer n, and some positive real number p. The least possible positive value of p is the **period** of the function.

The circumference of the unit circle is 2π, so the least value of p for which the sine and cosine functions repeat is 2π. ***Therefore, the sine and cosine functions are periodic functions with period 2π.***

Graph of the Sine Function In **Section 3.3** we saw that for a real number s, the point on the unit circle corresponding to s has coordinates $(\cos s, \sin s)$. See Figure 2. Trace along the circle to verify the results shown in the table.

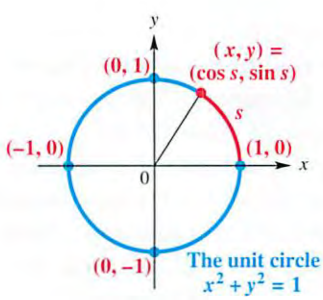


Figure 2

As s Increases from	$\sin s$	$\cos s$
0 to $\frac{\pi}{2}$	Increases from 0 to 1	Decreases from 1 to 0
$\frac{\pi}{2}$ to π	Decreases from 1 to 0	Decreases from 0 to −1
π to $\frac{3\pi}{2}$	Decreases from 0 to −1	Increases from −1 to 0
$\frac{3\pi}{2}$ to 2π	Increases from −1 to 0	Increases from 0 to 1

To avoid confusion when graphing the sine function, we use x rather than s; this corresponds to the letters in the xy-coordinate system. Selecting key values of x and finding the corresponding values of $\sin x$ leads to the table in Figure 3.

To obtain the traditional graph in Figure 3, we plot the points from the table, use symmetry, and join them with a smooth curve. Since $y = \sin x$ is periodic with period 2π and has domain $(-\infty, \infty)$, the graph continues in the same pattern in both directions. This graph is called a **sine wave,** or **sinusoid.**

SINE FUNCTION $f(x) = \sin x$

Domain: $(-\infty, \infty)$ Range: $[-1, 1]$

x	y
0	0
$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	1
π	0
$\frac{3\pi}{2}$	-1
2π	0

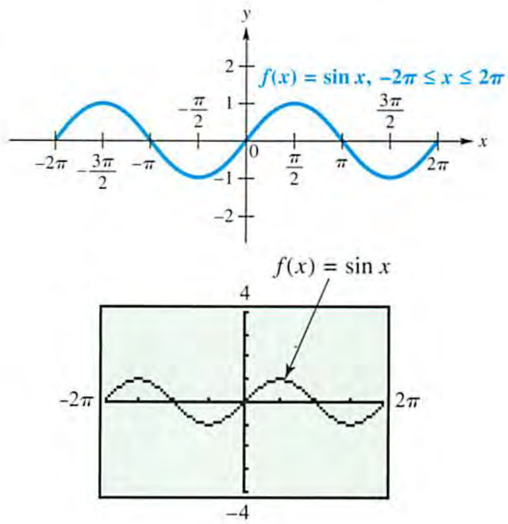


Figure 3

- The graph is continuous over its entire domain, $(-\infty, \infty)$.
- Its x-intercepts are of the form $n\pi$, where n is an integer.
- Its period is 2π.
- The graph is symmetric with respect to the origin, so the function is an odd function. For all x in the domain, $\sin(-x) = -\sin x$.

▶ **Note** In algebra, we say that a function f is an **odd function** if for all x in the domain of f, $f(-x) = -f(x)$. The graph of an odd function is symmetric with respect to the origin (Appendix D); that is, if (x, y) belongs to the function, then $(-x, -y)$ also belongs to the function. For example, $\left(\frac{\pi}{2}, 1\right)$ and $\left(-\frac{\pi}{2}, -1\right)$ are points on the graph of $y = \sin x$, illustrating the property $\sin(-x) = -\sin x$.

Sine graphs occur in many practical applications. For example, look back at Figure 2 and assume that the line from the origin to some point (p, q) on the circle is part of the pedal of a bicycle, with a foot placed at (p, q). Since $q = \sin x$, the height of the pedal from the horizontal axis in Figure 2 is given by $\sin x$. By choosing various angles for the pedal and calculating q for each angle, the height of the pedal leads to the sine curve shown in Figure 4 on the next page.

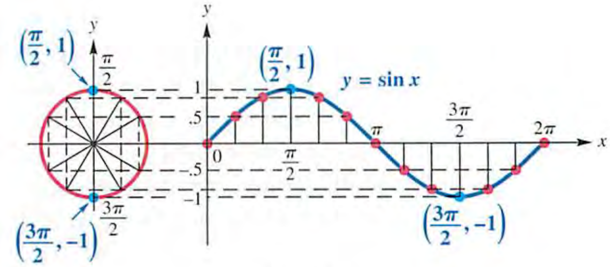


Figure 4

▼ LOOKING AHEAD TO CALCULUS

The discussion of the derivative of a function in calculus shows that for the sine function, the slope of the tangent line at any point x is given by $\cos x$. For example, look at the graph of $y = \sin x$ and notice that a tangent line at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$ will be horizontal and thus have slope 0. Now look at the graph of $y = \cos x$ and see that for these values, $\cos x = 0$.

Graph of the Cosine Function

The graph of $y = \cos x$ in Figure 5 has the same shape as the graph of $y = \sin x$. ***The graph of the cosine function is, in fact, the graph of the sine function shifted, or translated, $\frac{\pi}{2}$ units to the left.***

COSINE FUNCTION $f(x) = \cos x$

Domain: $(-\infty, \infty)$ Range: $[-1, 1]$

x	y
0	1
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	0
π	-1
$\frac{3\pi}{2}$	0
2π	1

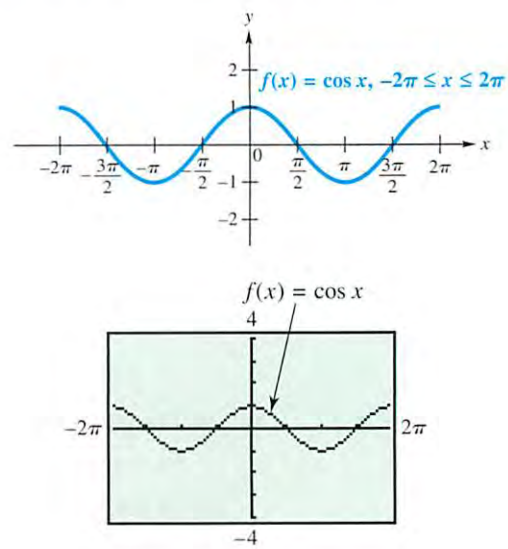


Figure 5

- The graph is continuous over its entire domain, $(-\infty, \infty)$.
- Its x-intercepts are of the form $(2n + 1)\frac{\pi}{2}$, where n is an integer.
- Its period is 2π.
- The graph is symmetric with respect to the y-axis, so the function is an even function. For all x in the domain, $\cos(-x) = \cos x$.

▶ **Note** A function f is an **even function** if for all x in the domain of f, $f(-x) = f(x)$. The graph of an even function is symmetric with respect to the y-axis (Appendix D); that is, if (x, y) belongs to the function, then $(-x, y)$ also belongs to the function. For example, $\left(\frac{\pi}{2}, 0\right)$ and $\left(-\frac{\pi}{2}, 0\right)$ are points on the graph of $y = \cos x$, illustrating the property $\cos(-x) = \cos x$.

The calculator graphs of $f(x) = \sin x$ in Figure 3 and $f(x) = \cos x$ in Figure 5 are graphed in the window $[-2\pi, 2\pi]$ by $[-4, 4]$, with Xscl $= \frac{\pi}{2}$ and Yscl $= 1$. This is called the **trig viewing window.** (Your model may use a different "standard" trigonometric viewing window. Consult your owner's manual.) ■

Graphing Techniques, Amplitude, and Period

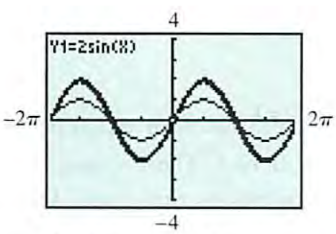
The examples that follow show graphs that are "stretched" or "compressed" (shrunk) either vertically, horizontally, or both when compared with the graphs of $y = \sin x$ or $y = \cos x$.

▶ EXAMPLE 1 GRAPHING $y = a \sin x$

Graph $y = 2 \sin x$, and compare to the graph of $y = \sin x$.

Solution For a given value of x, the value of y is twice as large as it would be for $y = \sin x$, as shown in the table of values. The only change in the graph is the range, which becomes $[-2, 2]$. See Figure 6, which includes a graph of $y = \sin x$ for comparison.

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin x$	0	1	0	-1	0
$2 \sin x$	0	2	0	-2	0



The thick graph style represents the function $y = 2 \sin x$ in Example 1.

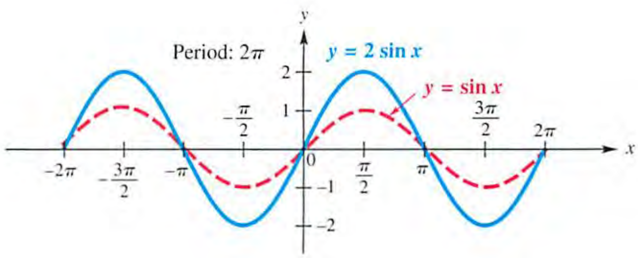


Figure 6

◀

The **amplitude** of a periodic function is half the difference between the maximum and minimum values. It describes the height of the graph both above and below a horizontal line passing through the "middle" of the graph. Thus, for both the basic sine and cosine functions, the amplitude is

$$\frac{1}{2}[1 - (-1)] = \frac{1}{2}(2) = 1.$$

Generalizing from Example 1 gives the following.

AMPLITUDE

The graph of $\boldsymbol{y = a \sin x}$ or $\boldsymbol{y = a \cos x}$, with $a \neq 0$, will have the same shape as the graph of $y = \sin x$ or $y = \cos x$, respectively, except with range $[-|a|, |a|]$. The amplitude is $|a|$.

NOW TRY EXERCISE 15. ◀

No matter what the value of the amplitude, the periods of $y = a \sin x$ and $y = a \cos x$ are still 2π. Consider $y = \sin 2x$. We can complete a table of values for the interval $[0, 2\pi]$.

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$\sin 2x$	0	1	0	−1	0	1	0	−1	0

Note that one complete cycle occurs in π units, not 2π units. Therefore, the period here is π, which equals $\frac{2\pi}{2}$. Now consider $y = \sin 4x$. Look at the next table.

x	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$\frac{5\pi}{8}$	$\frac{3\pi}{4}$	$\frac{7\pi}{8}$	π
$\sin 4x$	0	1	0	−1	0	1	0	−1	0

These values suggest that one complete cycle is achieved in $\frac{\pi}{2}$ or $\frac{2\pi}{4}$ units, which is reasonable since

$$\sin\left(4 \cdot \frac{\pi}{2}\right) = \sin 2\pi = 0.$$

In general, the graph of a function of the form $y = \sin bx$ or $y = \cos bx$, for $b > 0$, will have a period different from 2π when $b \neq 1$. To see why this is so, remember that the values of $\sin bx$ or $\cos bx$ will take on all possible values as bx ranges from 0 to 2π. Therefore, to find the period of either of these functions, we must solve the three-part inequality

$$0 \le bx \le 2\pi \quad \text{(Appendix A)}$$

$$0 \le x \le \frac{2\pi}{b}. \quad \text{Divide each part by the positive number } b.$$

Thus, the period is $\frac{2\pi}{b}$. By dividing the interval $\left[0, \frac{2\pi}{b}\right]$ into four equal parts, we obtain the values for which $\sin bx$ or $\cos bx$ is -1, 0, or 1. These values will give minimum points, x-intercepts, and maximum points on the graph. Once these points are determined, we can sketch the graph by joining the points with a smooth sinusoidal curve. (If a function has $b < 0$, then the identities of the next chapter can be used to rewrite the function so that $b > 0$.)

▶ **Note** One method to divide an interval into four equal parts is as follows.

Step 1 Find the midpoint of the interval by adding the x-values of the endpoints and dividing by 2. (Appendix B)

Step 2 Find the midpoints of the two intervals found in Step 1, using the same procedure.

▶ EXAMPLE 2 GRAPHING $y = \sin bx$

Graph $y = \sin 2x$, and compare to the graph of $y = \sin x$.

Solution In this function the coefficient of x is 2, so $b = 2$ and the period is $\frac{2\pi}{2} = \pi$. Therefore, the graph will complete one period over the interval $[0, \pi]$.

The endpoints are 0 and π, and the three points between the endpoints are

$$\frac{1}{4}(0 + \pi), \qquad \frac{1}{2}(0 + \pi), \qquad \text{and} \qquad \frac{3}{4}(0 + \pi),$$

which give the following x-values:

$0,$	$\frac{\pi}{4},$	$\frac{\pi}{2},$	$\frac{3\pi}{4},$	$\pi.$
↑	↑	↑	↑	↑
Left endpoint	First-quarter point	Midpoint	Third-quarter point	Right endpoint

We plot the points from the table of values given on page 148, and join them with a smooth sinusoidal curve. More of the graph can be sketched by repeating this cycle, as shown in Figure 7. The amplitude is not changed.

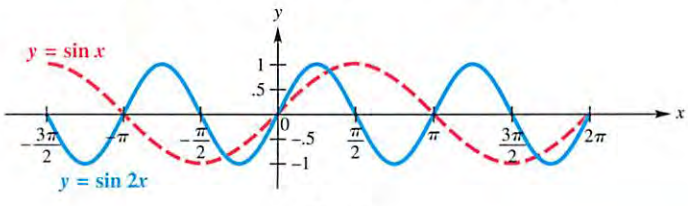


Figure 7

NOW TRY EXERCISE 27.

We can think of the graph of $y = \sin bx$ as a horizontal stretching of the graph of $y = \sin x$ when $0 < b < 1$ and a horizontal shrinking when $b > 1$.

PERIOD

For $b > 0$, the graph of $\mathbf{y = \sin bx}$ will resemble that of $y = \sin x$, but with period $\frac{2\pi}{b}$. Also, the graph of $\mathbf{y = \cos bx}$ will resemble that of $y = \cos x$, but with period $\frac{2\pi}{b}$.

EXAMPLE 3 GRAPHING $y = \cos bx$

Graph $y = \cos \frac{2}{3}x$ over one period.

Solution The period is $\frac{2\pi}{\frac{2}{3}} = 2\pi \div \frac{2}{3} = 2\pi \cdot \frac{3}{2} = 3\pi$. We divide the interval $[0, 3\pi]$ into four equal parts to get the x-values $0, \frac{3\pi}{4}, \frac{3\pi}{2}, \frac{9\pi}{4}$, and 3π that yield minimum points, maximum points, and x-intercepts. We use these values to obtain a table of key points for one period.

x	0	$\frac{3\pi}{4}$	$\frac{3\pi}{2}$	$\frac{9\pi}{4}$	3π
$\frac{2}{3}x$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos \frac{2}{3}x$	1	0	-1	0	1

The amplitude is 1 because the maximum value is 1, the minimum value is -1, and $\frac{1}{2}(1 - (-1)) = \frac{1}{2}(2) = 1$. We plot these points and join them with a smooth curve. The graph is shown in Figure 8.

NOW TRY EXERCISE 25.

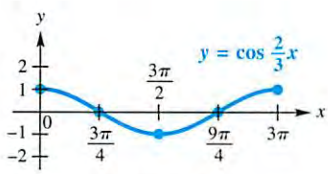
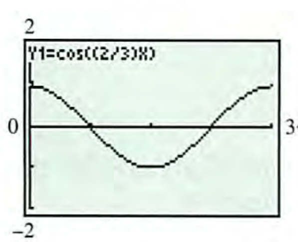


Figure 8



This screen shows a graph of the function in Example 3. By choosing Xscl = $\frac{3\pi}{4}$, the x-intercepts, maxima, and minima coincide with tick marks on the x-axis.

▶ **Note** Look back at the middle row of the table in Example 3. Dividing the interval $\left[0, \frac{2\pi}{b}\right]$ into four equal parts will always give the values $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π for this row, in this case resulting in values of -1, 0, or 1. These values lead to key points on the graph, which can then be easily sketched.

GUIDELINES FOR SKETCHING GRAPHS OF SINE AND COSINE FUNCTIONS

To graph $y = a \sin bx$ or $y = a \cos bx$, with $b > 0$, follow these steps.

Step 1 Find the period, $\frac{2\pi}{b}$. Start at 0 on the x-axis, and lay off a distance of $\frac{2\pi}{b}$.

Step 2 Divide the interval into four equal parts. (See the Note preceding Example 2 on page 148.)

Step 3 Evaluate the function for each of the five x-values resulting from Step 2. The points will be maximum points, minimum points, and x-intercepts.

Step 4 Plot the points found in Step 3, and join them with a sinusoidal curve having amplitude $|a|$.

Step 5 Draw the graph over additional periods as needed.

▶ EXAMPLE 4 GRAPHING $y = a \sin bx$

Graph $y = -2 \sin 3x$ over one period using the preceding guidelines.

Solution

Step 1 For this function, $b = 3$, so the period is $\frac{2\pi}{3}$. The function will be graphed over the interval $\left[0, \frac{2\pi}{3}\right]$.

Step 2 Divide the interval $\left[0, \frac{2\pi}{3}\right]$ into four equal parts to get the x-values $0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$, and $\frac{2\pi}{3}$.

Step 3 Make a table of values determined by the x-values from Step 2.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
$3x$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin 3x$	0	1	0	-1	0
$-2 \sin 3x$	0	-2	0	2	0

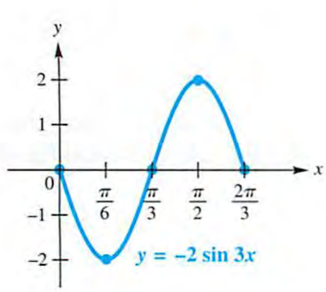


Figure 9

Step 4 Plot the points $(0, 0)$, $\left(\frac{\pi}{6}, -2\right)$, $\left(\frac{\pi}{3}, 0\right)$, $\left(\frac{\pi}{2}, 2\right)$, and $\left(\frac{2\pi}{3}, 0\right)$, and join them with a sinusoidal curve with amplitude 2. See Figure 9.

Step 5 The graph can be extended by repeating the cycle.

Notice that when a is negative, the graph of $y = a \sin bx$ is the reflection across the x-axis of the graph of $y = |a| \sin bx$.

NOW TRY EXERCISE 29. ◀

▶ EXAMPLE 5 **GRAPHING $y = a \cos bx$ FOR b EQUAL TO A MULTIPLE OF π**

Graph $y = -3 \cos \pi x$ over one period.

Solution

Step 1 Since $b = \pi$, the period is $\frac{2\pi}{\pi} = 2$, so we will graph the function over the interval $[0, 2]$.

Step 2 Dividing $[0, 2]$ into four equal parts yields the x-values $0, \frac{1}{2}, 1, \frac{3}{2}$, and 2.

Step 3 Make a table using these x-values.

x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
πx	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos \pi x$	1	0	-1	0	1
$-3 \cos \pi x$	-3	0	3	0	-3

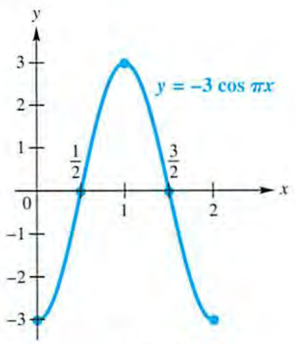


Figure 10

Step 4 Plot the points $(0, -3)$, $\left(\frac{1}{2}, 0\right)$, $(1, 3)$, $\left(\frac{3}{2}, 0\right)$, and $(2, -3)$, and join them with a sinusoidal curve having amplitude $|-3| = 3$. See Figure 10.

Step 5 The graph can be extended by repeating the cycle.

Notice that when b is an integer multiple of π, the x-intercepts of the graph are rational numbers.

NOW TRY EXERCISE 37. ◀

Using a Trigonometric Model

▶ EXAMPLE 6 **INTERPRETING A SINE FUNCTION MODEL**

The average temperature (in °F) at Mould Bay, Canada, can be approximated by the function defined by

$$f(x) = 34 \sin\left[\frac{\pi}{6}(x - 4.3)\right],$$

where x is the month and $x = 1$ corresponds to January, $x = 2$ to February, and so on.

(a) To observe the graph over a two-year interval and to see the maximum and minimum points, graph f in the window $[0, 25]$ by $[-45, 45]$.

(b) What is the average temperature during the month of May?

(c) What would be an approximation for the average *yearly* temperature at Mould Bay?

Solution

(a) The graph of $f(x) = 34 \sin\left[\frac{\pi}{6}(x - 4.3)\right]$ is shown in Figure 11 on the next page. Its amplitude is 34, and the period is $\frac{2\pi}{\frac{\pi}{6}} = 2\pi \div \frac{\pi}{6} = 2\pi \cdot \frac{6}{\pi} = 12$. The function f has a period of 12 months, or 1 year, which agrees with the changing of the seasons.

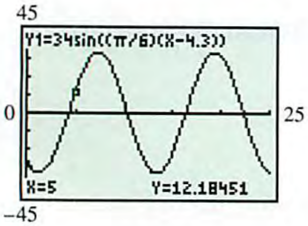


Figure 11

(b) May is the fifth month, so the average temperature during May is

$$f(5) = 34 \sin\left[\frac{\pi}{6}(5 - 4.3)\right] \approx 12°\text{F}. \quad \text{Let } x = 5. \text{ (Appendix C)}$$

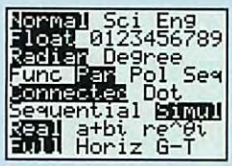
See the display at the bottom of the screen in Figure 11.

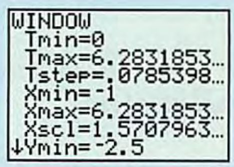
(c) From the graph, it appears that the average yearly temperature is about 0°F since the graph is centered vertically about the line $y = 0$.

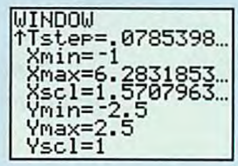
NOW TRY EXERCISE 55.

CONNECTIONS

Using a TI-83/84 Plus calculator, adjust the settings to correspond to the following screens.

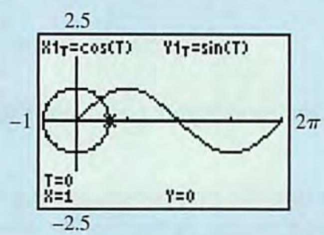


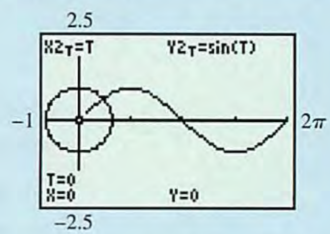


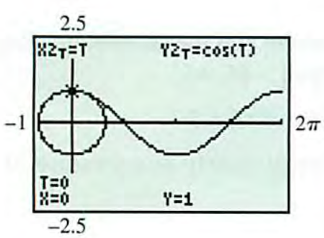


Tmax is 2π,
Tstep is $\frac{\pi}{40}$,
Xmax is 2π,
Xscl is $\frac{\pi}{2}$.

Graph the two equations (which are in **parametric form**), and watch as the unit circle and the sine function are graphed simultaneously. Press the TRACE key once to get the screen shown on the left below, and then press the up-arrow key to get the screen shown on the right below. The screen on the left gives a unit circle interpretation of cos 0 = 1 and sin 0 = 0. The screen on the right gives a rectangular coordinate graph interpretation of sin 0 = 0.







Now go back and redefine Y_{2T} as cos(T). Graph both equations again; the second screen will look like the one in the margin, after the TRACE and up-arrow keys are pressed. This screen indicates that cos 0 = 1.

FOR DISCUSSION OR WRITING

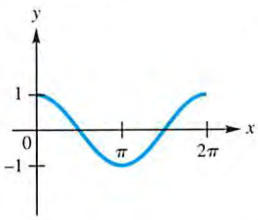
1. On the unit circle, let T = 2. Find X and Y, and interpret their values.
2. On the sine graph, trace to the point where T = 1.9. What values of X and Y are displayed? Interpret these values with an equation in X and Y. Then repeat, but use the cosine graph.

4.1 Exercises

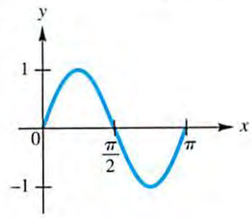
Concept Check *In Exercises 1–8, match each function with its graph.*

1. $y = \sin x$
2. $y = \cos x$
3. $y = -\sin x$
4. $y = -\cos x$
5. $y = \sin 2x$
6. $y = \cos 2x$
7. $y = 2 \sin x$
8. $y = 2 \cos x$

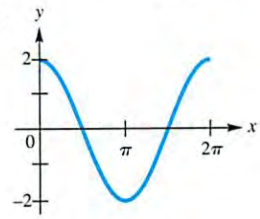
A.



B.



C.



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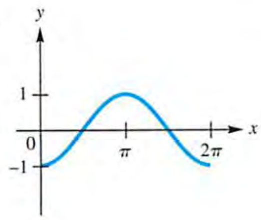

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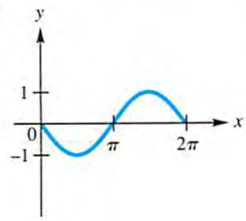

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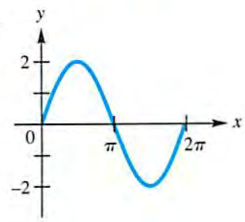
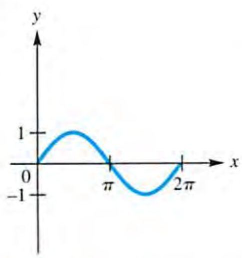
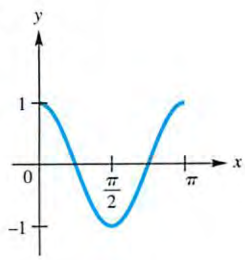


G.



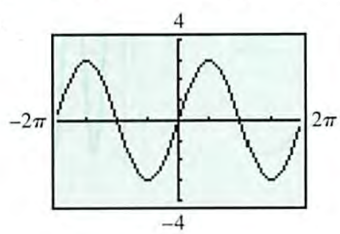
H.



Concept Check *In Exercises 9–12, match each function with its calculator graph.*

9. $y = \sin 3x$
10. $y = \cos 3x$
11. $y = 3 \cos x$
12. $y = 3 \sin x$

A.



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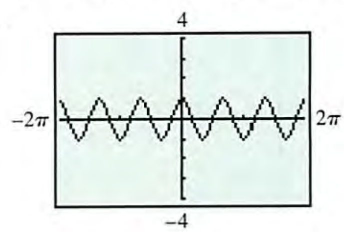
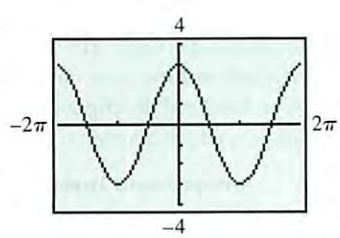


C.



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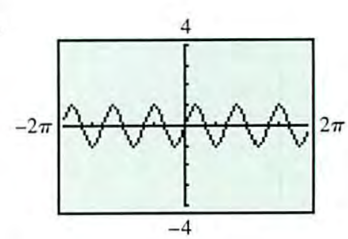


Graph each function over the interval $[-2\pi, 2\pi]$. Give the amplitude. See Example 1.

13. $y = 2 \cos x$
14. $y = 3 \sin x$
15. $y = \frac{2}{3} \sin x$
16. $y = \frac{3}{4} \cos x$
17. $y = -\cos x$
18. $y = -\sin x$
19. $y = -2 \sin x$
20. $y = -3 \cos x$
21. $y = \sin (-x)$

22. ***Concept Check*** In Exercise 21, why is the graph the same as that of $y = -\sin x$?

Graph each function over a two-period interval. Give the period and amplitude. See Examples 2–5.

23. $y = \sin \frac{1}{2}x$ **24.** $y = \sin \frac{2}{3}x$ **25.** $y = \cos \frac{3}{4}x$

26. $y = \cos \frac{1}{3}x$ **27.** $y = \sin 3x$ **28.** $y = \cos 2x$

29. $y = 2 \sin \frac{1}{4}x$ **30.** $y = 3 \sin 2x$ **31.** $y = -2 \cos 3x$

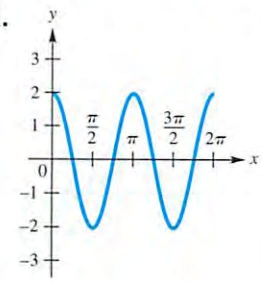
32. $y = -5 \cos 2x$ **33.** $y = \cos \pi x$ **34.** $y = -\sin \pi x$

35. $y = -2 \sin 2\pi x$ **36.** $y = 3 \cos 2\pi x$ **37.** $y = \frac{1}{2} \cos \frac{\pi}{2}x$

38. $y = -\frac{2}{3} \sin \frac{\pi}{4}x$ **39.** $y = \pi \sin \pi x$ **40.** $y = -\pi \cos \pi x$

Connecting Graphs with Equations *Each function graphed is of the form $y = a \sin bx$ or $y = a \cos bx$, where $b > 0$. Determine the equation of the graph.*

41.



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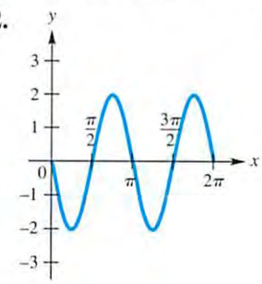

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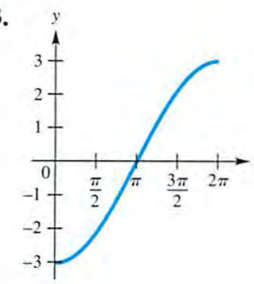

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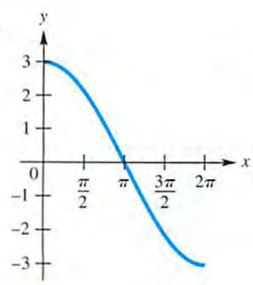

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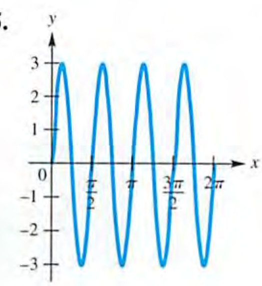

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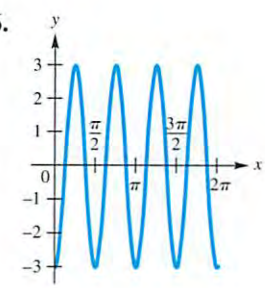
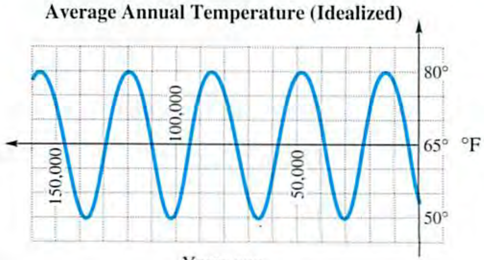


(Modeling) *Solve each problem.*

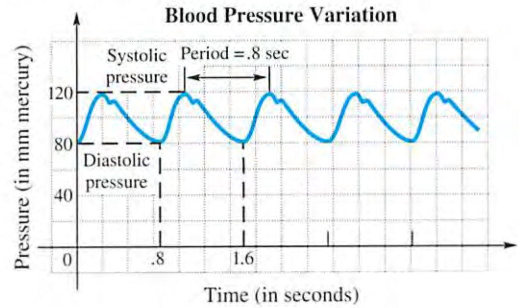
47. ***Average Annual Temperature*** Scientists believe that the average annual temperature in a given location is periodic. The average temperature at a given place during a given season fluctuates as time goes on, from colder to warmer, and back to colder. The graph shows an idealized description of the temperature (in °F) for approximately the last 150 thousand years of a location at the same latitude as Anchorage, Alaska.

Average Annual Temperature (Idealized)



(a) Find the highest and lowest temperatures recorded.
(b) Use these two numbers to find the amplitude.
(c) Find the period of the function.
(d) What is the trend of the temperature now?

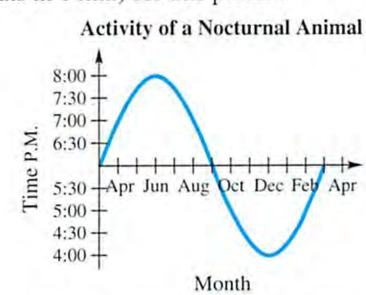
48. ***Blood Pressure Variation*** The graph gives the variation in blood pressure for a typical person. **Systolic** and **diastolic pressures** are the upper and lower limits of the periodic changes in pressure that produce the pulse. The length of time between peaks is called the period of the pulse.



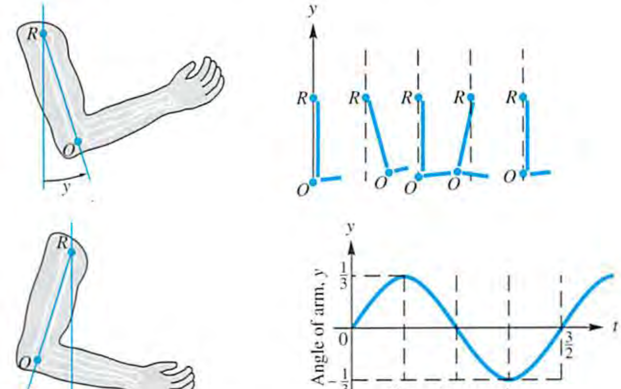
(a) Find the amplitude of the graph.
(b) Find the pulse rate (the number of pulse beats in 1 min) for this person.

49. ***Activity of a Nocturnal Animal*** Many of the activities of living organisms are periodic. For example, the graph at the right shows the time that a certain nocturnal animal begins its evening activity.

(a) Find the amplitude of this graph.
(b) Find the period.



50. ***Position of a Moving Arm*** The figure shows schematic diagrams of a rhythmically moving arm. The upper arm RO rotates back and forth about the point R; the position of the arm is measured by the angle y between the actual position and the downward vertical position. (*Source:* De Sapio, R., *Calculus for the Life Sciences*. Copyright © 1978 by W. H. Freeman and Company. Reprinted by permission.)



This graph shows the relationship between angle y and time t in seconds.

(a) Find an equation of the form $y = a \sin kt$ for the graph shown.
(b) How long does it take for a complete movement of the arm?

51. *Voltage of an Electrical Circuit* The voltage E in an electrical circuit is modeled by

$$E = 5 \cos 120\pi t,$$

where t is time measured in seconds.

(a) Find the amplitude and the period.

(b) How many cycles are completed in 1 sec? (The number of cycles (periods) completed in 1 sec is the **frequency** of the function.)

(c) Find E when $t = 0, .03, .06, .09, .12$.

(d) Graph E for $0 \leq t \leq \frac{1}{30}$.

52. *Voltage of an Electrical Circuit* For another electrical circuit, the voltage E is modeled by

$$E = 3.8 \cos 40\pi t,$$

where t is time measured in seconds.

(a) Find the amplitude and the period.

(b) Find the frequency. See Exercise 51(b).

(c) Find E when $t = .02, .04, .08, .12, .14$.

(d) Graph one period of E.

53. *Atmospheric Carbon Dioxide* At Mauna Loa, Hawaii, atmospheric carbon dioxide levels in parts per million (ppm) have been measured regularly since 1958. The function defined by

$$L(x) = .022x^2 + .55x + 316 + 3.5 \sin 2\pi x$$

can be used to model these levels, where x is in years and $x = 0$ corresponds to 1960. (*Source:* Nilsson, A., *Greenhouse Earth,* John Wiley and Sons, 1992.)

(a) Graph L in the window $[15, 35]$ by $[325, 365]$.

(b) When do the seasonal maximum and minimum carbon dioxide levels occur?

(c) L is the sum of a quadratic function and a sine function. What is the significance of each of these functions? Discuss what physical phenomena may be responsible for each function.

54. *Atmospheric Carbon Dioxide* Refer to Exercise 53. The carbon dioxide content in the atmosphere at Barrow, Alaska, in parts per million (ppm) can be modeled using the function defined by

$$C(x) = .04x^2 + .6x + 330 + 7.5 \sin 2\pi x,$$

where $x = 0$ corresponds to 1970. (*Source:* Zeilik, M. and S. Gregory, *Introductory Astronomy and Astrophysics,* Brooks/Cole, 1998.)

(a) Graph C in the window $[5, 25]$ by $[320, 380]$.

(b) Discuss possible reasons why the amplitude of the oscillations in the graph of C is larger than the amplitude of the oscillations in the graph of L in Exercise 53, which models Hawaii.

(c) Define a new function C that is valid if x represents the actual year, where $1970 \leq x \leq 1995$. (See horizontal translations in **Appendix D.**)

55. *Average Annual Temperature* The temperature in a certain city in Alaska is modeled by

$$T(x) = 25 + 37 \sin\left[\frac{2\pi}{365}(x - 101)\right],$$

where $T(x)$ is the temperature in degrees Fahrenheit on day x, with $x = 1$ corresponding to January 1 and $x = 365$ corresponding to December 31. Use a calculator to estimate the temperature on the following days. (*Source:* Lando, B. and C. Lando, "Is the Graph of Temperature Variation a Sine Curve?", *The Mathematics Teacher*, 70, September 1977.)

(a) March 15 (day 74) **(b)** April 5 (day 95) **(c)** Day 200
(d) June 25 **(e)** October 1 **(f)** December 31

56. *Fluctuation in the Solar Constant* The **solar constant** S is the amount of energy per unit area that reaches Earth's atmosphere from the sun. It is equal to 1367 watts per sq m but varies slightly throughout the seasons. This fluctuation ΔS in S can be calculated using the formula

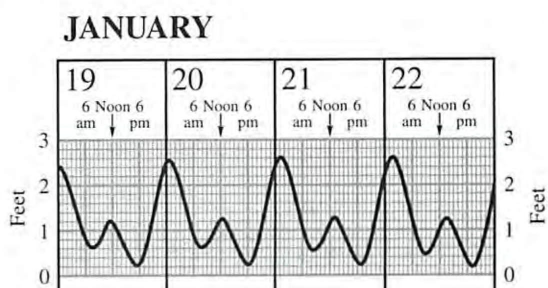
$$\Delta S = .034S \sin\left[\frac{2\pi(82.5 - N)}{365.25}\right].$$

In this formula, N is the day number covering a four-year period, where $N = 1$ corresponds to January 1 of a leap year and $N = 1461$ corresponds to December 31 of the fourth year. (*Source:* Winter, C., R. Sizmann, and L. L. Vant-Hull, Editors, *Solar Power Plants*, Springer-Verlag, 1991.)

(a) Calculate ΔS for $N = 80$, which is the spring equinox in the first year.
(b) Calculate ΔS for $N = 1268$, which is the summer solstice in the fourth year.
(c) What is the maximum value of ΔS?
(d) Find a value for N where ΔS is equal to 0.

Tides for Kahului Harbor *The chart shows the tides for Kahului Harbor (on the island of Maui, Hawaii). To identify high and low tides and times for other Maui areas, the following adjustments must be made.*

Hana:	High, +40 min, +.1 ft;	Makena:	High, +1:21, −.5 ft;
	Low, +18 min, −.2 ft		Low, +1:09, −.2 ft
Maalaea:	High, +1:52, −.1 ft;	Lahaina:	High, +1:18, −.2 ft;
	Low, +1:19, −.2 ft		Low, +1:01, −.1 ft



Source: Maui News. Original chart prepared by Edward K. Noda and Associates.

Use the graph to work Exercises 57–62.

57. The graph is an example of a periodic function. What is the period (in hours)?

58. What is the amplitude?

59. At what time on January 20 was low tide at Kahului? What was the height?

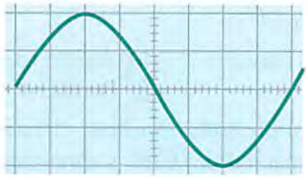
60. Repeat Exercise 59 for Maalaea.

61. At what time on January 22 was high tide at Kahului? What was the height?

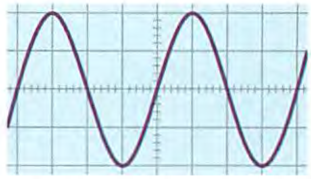
62. Repeat Exercise 61 for Lahaina.

Musical Sound Waves *Pure sounds produce single sine waves on an oscilloscope. Find the amplitude and period of each sine wave graph in Exercises 63 and 64. On the vertical scale, each square represents .5; on the horizontal scale, each square represents* 30° *or* $\frac{\pi}{6}$.

63.



64.



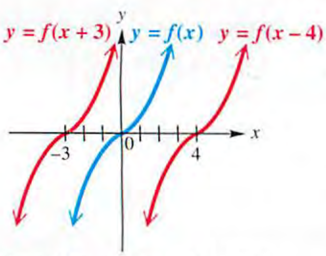
65. (a) Compare the graphs of $y = \sin 2x$ and $y = 2 \sin x$ over the interval $[0, 2\pi]$. Can we say that, in general, $\sin bx = b \sin x$? Explain.

(b) Compare the graphs of $y = \cos 3x$ and $y = 3 \cos x$ over the interval $[0, 2\pi]$. Can we say that, in general, $\cos bx = b \cos x$? Explain.

66. Refer to the graph of $y = \sin x$ in Figure 3. The graph completes one cycle between $x = 0$ and $x = 2\pi$. Consider the statement, "The function $y = \sin bx$ completes b cycles between 0 and 2π." Use your graphing calculator to confirm the statement for some positive integer values of b, such as 3, 4, and 5. Interpret and confirm the statement for $b = \frac{1}{2}$ and $b = \frac{3}{2}$.

4.2 Translations of the Graphs of the Sine and Cosine Functions

Horizontal Translations ▪ Vertical Translations ▪ Combinations of Translations ▪ Determining a Trigonometric Model Using Curve Fitting



Horizontal translations of $y = f(x)$

(Appendix D)

Figure 12

Horizontal Translations

The graph of the function defined by $y = f(x - d)$ is translated *horizontally* compared to the graph of $y = f(x)$. The translation is d units to the right if $d > 0$ and $|d|$ units to the left if $d < 0$. See Figure 12. With circular functions, a horizontal translation is called a **phase shift.** In the function $y = f(x - d)$, the expression $x - d$ is called the **argument.** In Examples 1–3, we give two methods that can be used to sketch the graph of a circular function involving a phase shift.

▶ EXAMPLE 1 GRAPHING $y = \sin(x - d)$

Graph $y = \sin(x - \frac{\pi}{3})$ over one period.

Solution *Method 1* For the argument $x - \frac{\pi}{3}$ to result in all possible values throughout one period, it must take on all values between 0 and 2π, inclusive. Therefore, to find an interval of one period, we solve the three-part inequality

$$0 \le x - \frac{\pi}{3} \le 2\pi \quad \text{(Appendix A)}$$

$$\frac{\pi}{3} \le x \le \frac{7\pi}{3}. \quad \text{Add } \tfrac{\pi}{3} \text{ to each part.}$$

Use the method described in the Note on page 148 to divide the interval $\left[\frac{\pi}{3}, \frac{7\pi}{3}\right]$ into four equal parts, obtaining the following x-values.

$$\frac{\pi}{3}, \quad \frac{5\pi}{6}, \quad \frac{4\pi}{3}, \quad \frac{11\pi}{6}, \quad \frac{7\pi}{3}$$

These are *key* *x*-values.

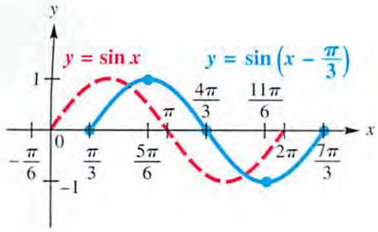


Figure 13

A table of values using these x-values follows.

x	$\frac{\pi}{3}$	$\frac{5\pi}{6}$	$\frac{4\pi}{3}$	$\frac{11\pi}{6}$	$\frac{7\pi}{3}$
$x - \frac{\pi}{3}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin(x - \frac{\pi}{3})$	0	1	0	−1	0

We join the corresponding points with a smooth curve to get the solid blue graph shown in Figure 13. The period is 2π, and the amplitude is 1.

Method 2 We can also graph $y = \sin\left(x - \frac{\pi}{3}\right)$ by using a horizontal translation of the graph of $y = \sin x$. The argument $x - \frac{\pi}{3}$ indicates that the graph will be translated $\frac{\pi}{3}$ units to the *right* (the phase shift) as compared to the graph of $y = \sin x$. See Figure 13.

Therefore, to graph a function using this method, first graph the basic circular function, and then graph the desired function by using the appropriate translation.

NOW TRY EXERCISE 35.

Note The graph in Figure 13 of Example 1 can be extended through additional periods by repeating the given portion of the graph, as necessary.

EXAMPLE 2 GRAPHING $y = a\cos(x - d)$

Graph $y = 3\cos(x + \frac{\pi}{4})$ over one period.

Solution *Method 1* The graph can be sketched over one period by first solving the three-part inequality

$$0 \le x + \frac{\pi}{4} \le 2\pi$$

$$-\frac{\pi}{4} \le x \le \frac{7\pi}{4}. \quad \text{Subtract } \tfrac{\pi}{4} \text{ from each part.}$$

Dividing this interval into four equal parts gives x-values of

$$-\frac{\pi}{4}, \quad \frac{\pi}{4}, \quad \frac{3\pi}{4}, \quad \frac{5\pi}{4}, \quad \frac{7\pi}{4}. \quad \text{Key } x\text{-values}$$

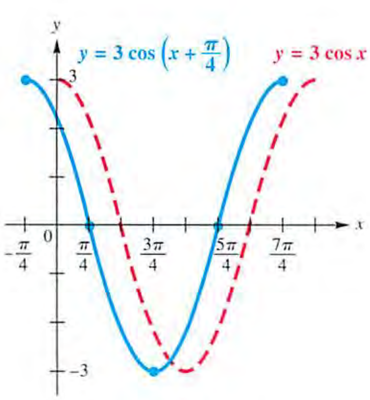
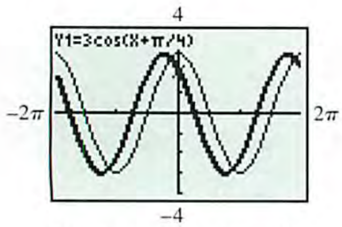


Figure 14

A table of points for these x-values leads to maximum points, minimum points, and x-intercepts.

x	$-\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$
$x + \frac{\pi}{4}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos(x + \frac{\pi}{4})$	1	0	−1	0	1
$3\cos(x + \frac{\pi}{4})$	3	0	−3	0	3

We join the corresponding points with a smooth curve to get the solid blue graph shown in Figure 14. The period is 2π, and the amplitude is 3.



This screen shows the graph of

$$Y_1 = 3\cos\left(X + \frac{\pi}{4}\right)$$

in Example 2 as a thick line. The graph of

$$Y_2 = 3\cos X$$

is shown as a thin line for comparison.

Method 2 Write $3\cos\left(x + \frac{\pi}{4}\right)$ in the form $a\cos(x - d)$.

$$y = 3\cos\left(x + \frac{\pi}{4}\right) = 3\cos\left[x - \left(-\frac{\pi}{4}\right)\right]$$

This result shows that $d = -\frac{\pi}{4}$. Since $-\frac{\pi}{4}$ is negative, the phase shift is $\left|-\frac{\pi}{4}\right| = \frac{\pi}{4}$ unit to the left. The graph is the same as that of $y = 3\cos x$ (the dashed red graph in Figure 14 on the preceding page), except that it is translated $\frac{\pi}{4}$ unit to the left (the solid blue graph in Figure 14).

NOW TRY EXERCISE 37.

▶ EXAMPLE 3 GRAPHING $y = a\cos b(x - d)$

Graph $y = -2\cos(3x + \pi)$ over two periods.

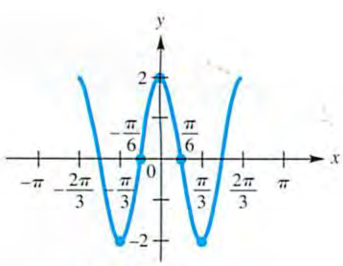
Solution *Method 1* The function can be sketched over one period by solving the three-part inequality

$$0 \leq 3x + \pi \leq 2\pi$$

to get the interval $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$. Divide this interval into four equal parts to get the points $\left(-\frac{\pi}{3}, -2\right)$, $\left(-\frac{\pi}{6}, 0\right)$, $(0, 2)$, $\left(\frac{\pi}{6}, 0\right)$, and $\left(\frac{\pi}{3}, -2\right)$. Plot these points and join them with a smooth curve. By graphing an additional half period to the left and to the right, we obtain the graph shown in Figure 15.

Method 2 First write the expression in the form $a\cos b(x - d)$.

$$y = -2\cos(3x + \pi) = -2\cos 3\left(x + \frac{\pi}{3}\right) \quad \text{Rewrite } 3x + \pi \text{ as } 3\left(x + \tfrac{\pi}{3}\right).$$

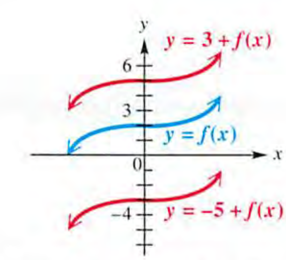


$y = -2\cos(3x + \pi)$

Figure 15

Then $a = -2$, $b = 3$, and $d = -\frac{\pi}{3}$. The amplitude is $|-2| = 2$, and the period is $\frac{2\pi}{3}$ (since the value of b is 3). The phase shift is $\left|-\frac{\pi}{3}\right| = \frac{\pi}{3}$ units to the left as compared to the graph of $y = -2\cos 3x$. Again, see Figure 15.

NOW TRY EXERCISE 43.



Vertical translations of $y = f(x)$
(Appendix D)

Figure 16

Vertical Translations The graph of a function of the form $y = c + f(x)$ is translated *vertically* as compared with the graph of $y = f(x)$. See Figure 16. The translation is c units up if $c > 0$ and $|c|$ units down if $c < 0$.

▶ EXAMPLE 4 GRAPHING $y = c + a\cos bx$

Graph $y = 3 - 2\cos 3x$ over two periods.

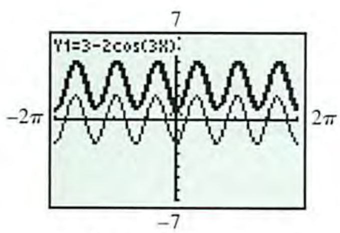
Solution The values of y will be 3 greater than the corresponding values of y in $y = -2\cos 3x$. This means that the graph of $y = 3 - 2\cos 3x$ is the same as the graph of $y = -2\cos 3x$, vertically translated 3 units up. Since the period of $y = -2\cos 3x$ is $\frac{2\pi}{3}$, the key points have x-values

$$0, \quad \frac{\pi}{6}, \quad \frac{\pi}{3}, \quad \frac{\pi}{2}, \quad \frac{2\pi}{3}. \quad \text{Key } x\text{-values}$$

Use these x-values to make a table of points.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
$\cos 3x$	1	0	−1	0	1
$2 \cos 3x$	2	0	−2	0	2
$3 - 2 \cos 3x$	1	3	5	3	1

The key points are shown on the graph in Figure 17, along with more of the graph, sketched using the fact that the function is periodic.



The function in Example 4 is shown using the thick graph style. Notice also the thin graph style for $y = -2 \cos 3x$.

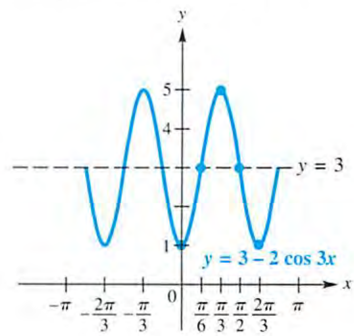


Figure 17

NOW TRY EXERCISE 47.

Combinations of Translations

A function of the form

$$y = c + a \sin b(x - d) \quad \text{or} \quad y = c + a \cos b(x - d), \quad b > 0,$$

which involves stretching, shrinking, and translating, can be graphed according to the following guidelines.

FURTHER GUIDELINES FOR SKETCHING GRAPHS OF SINE AND COSINE FUNCTIONS

Method 1 Follow these steps.

Step 1 Find an interval whose length is one period $\frac{2\pi}{b}$ by solving the three-part inequality $0 \leq b(x - d) \leq 2\pi$. (Appendix A)

Step 2 Divide the interval into four equal parts. (See the Note on page 148.)

Step 3 Evaluate the function for each of the five x-values resulting from Step 2. The points will be maximum points, minimum points, and points that intersect the line $y = c$ ("middle" points of the wave).

Step 4 Plot the points found in Step 3, and join them with a sinusoidal curve having amplitude $|a|$.

Step 5 Draw the graph over additional periods, as needed.

Method 2 First graph $y = a \sin bx$ or $y = a \cos bx$. The amplitude of the function is $|a|$, and the period is $\frac{2\pi}{b}$. Then use translations to graph the desired function. The vertical translation is c units up if $c > 0$ and $|c|$ units down if $c < 0$. The horizontal translation (phase shift) is d units to the right if $d > 0$ and $|d|$ units to the left if $d < 0$.

▶ EXAMPLE 5 GRAPHING $y = c + a \sin b(x - d)$

Graph $y = -1 + 2\sin(4x + \pi)$ over two periods.

Solution We use Method 1. First write the expression on the right side in the form $c + a \sin b(x - d)$.

$$y = -1 + 2\sin(4x + \pi) = -1 + 2\sin\left[4\left(x + \frac{\pi}{4}\right)\right]$$

Step 1 Find an interval whose length is one period.

$$0 \le 4\left(x + \frac{\pi}{4}\right) \le 2\pi$$

$$0 \le x + \frac{\pi}{4} \le \frac{\pi}{2} \qquad \text{Divide each part by 4.}$$

$$-\frac{\pi}{4} \le x \le \frac{\pi}{4} \qquad \text{Subtract } \tfrac{\pi}{4} \text{ from each part.}$$

Step 2 Divide the interval $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ into four equal parts to get the x-values

$$-\frac{\pi}{4}, \quad -\frac{\pi}{8}, \quad 0, \quad \frac{\pi}{8}, \quad \frac{\pi}{4}. \qquad \text{Key } x\text{-values}$$

Step 3 Make a table of values.

x	$-\frac{\pi}{4}$	$-\frac{\pi}{8}$	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$
$x + \frac{\pi}{4}$	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$
$4(x + \frac{\pi}{4})$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin 4(x + \frac{\pi}{4})$	0	1	0	-1	0
$2\sin 4(x + \frac{\pi}{4})$	0	2	0	-2	0
$-1 + 2\sin(4x + \pi)$	-1	1	-1	-3	-1

Steps 4 and 5 Plot the points found in the table and join them with a sinusoidal curve. Figure 18 shows the graph, extended to the right and left to include two full periods.

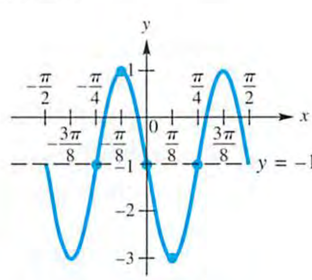


Figure 18

NOW TRY EXERCISE 53. ◀

Determining a Trigonometric Model Using Curve Fitting

A sinusoidal function is often a good approximation of a set of real data points.

▶ EXAMPLE 6 MODELING TEMPERATURE WITH A SINE FUNCTION

The maximum average monthly temperature in New Orleans is 82°F and the minimum is 54°F. The table shows the average monthly temperatures. The scatter diagram for a two-year interval in Figure 19 strongly suggests that the temperatures can be modeled with a sine curve.

Month	°F	Month	°F
Jan	54	July	82
Feb	55	Aug	81
Mar	61	Sept	77
Apr	69	Oct	71
May	73	Nov	59
June	79	Dec	55

Source: Miller, A., J. Thompson, and R. Peterson, *Elements of Meteorology, Fourth Edition,* Charles E. Merrill Publishing Co., 1983.

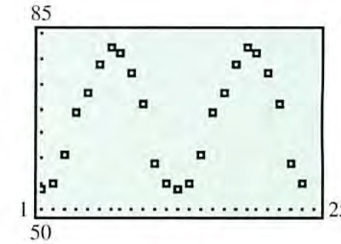


Figure 19

(a) Using only the maximum and minimum temperatures, determine a function of the form $f(x) = a \sin[b(x - d)] + c$, where a, b, c, and d are constants, that models the average monthly temperature in New Orleans. Let x represent the month, with January corresponding to $x = 1$.

(b) On the same coordinate axes, graph f for a two-year period together with the actual data values found in the table.

(c) Use the **sine regression** feature of a graphing calculator to determine a second model for these data.

Solution

(a) We use the maximum and minimum average monthly temperatures to find the amplitude a.

$$a = \frac{82 - 54}{2} = 14$$

The average of the maximum and minimum temperatures is a good choice for c. The average is

$$\frac{82 + 54}{2} = 68.$$

Since temperatures repeat every 12 months, b is $\frac{2\pi}{12} = \frac{\pi}{6}$. The coldest month is January, when $x = 1$, and the hottest month is July, when $x = 7$, so we should choose d to be about 4. The table shows that temperatures are actually a little warmer after July than before, so we experiment with values just greater than 4 to find d. Trial and error using a calculator leads to $d = 4.2$. Thus,

$$f(x) = a \sin[b(x - d)] + c = 14 \sin\left[\frac{\pi}{6}(x - 4.2)\right] + 68.$$

(b) Figure 20 shows the data points from the table, the graph of $y = 14\sin\left[\frac{\pi}{6}(x - 4.2)\right] + 68$, and the graph of $y = 14\sin\frac{\pi}{6}x + 68$ for comparison.

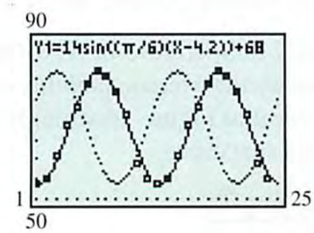
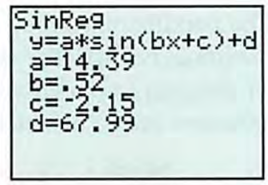
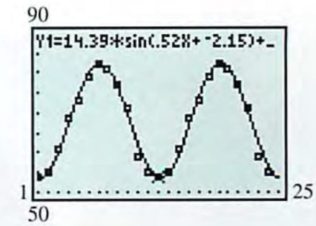


Figure 20



Values are rounded to the nearest hundredth.

(a)



(b)

Figure 21

(c) We used the given data for a two-year period to produce the model described in Figure 21(a). Figure 21(b) shows its graph along with the data points.

NOW TRY EXERCISE 57.

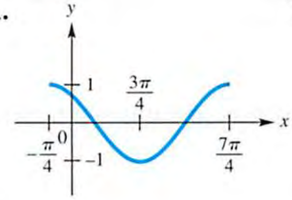
4.2 Exercises

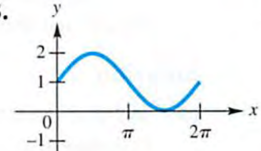
Concept Check *In Exercises 1–8, match each function with its graph in A–H.*

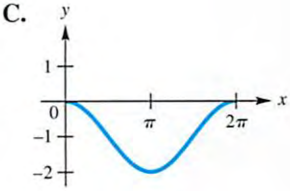
1. $y = \sin\left(x - \frac{\pi}{4}\right)$ **2.** $y = \sin\left(x + \frac{\pi}{4}\right)$ **3.** $y = \cos\left(x - \frac{\pi}{4}\right)$

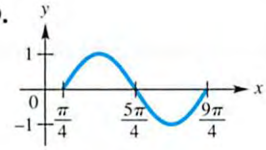
4. $y = \cos\left(x + \frac{\pi}{4}\right)$ **5.** $y = 1 + \sin x$ **6.** $y = -1 + \sin x$

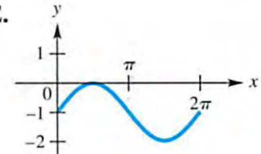
7. $y = 1 + \cos x$ **8.** $y = -1 + \cos x$

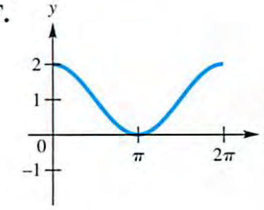
A.


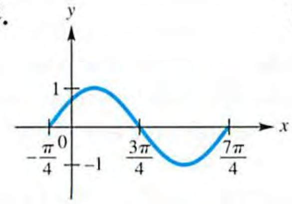
B.


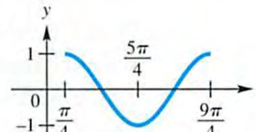
C.


D.


E.


F.


G.


H.


Concept Check *In Exercises 9–12, match each function with its calculator graph in the standard trig window.*

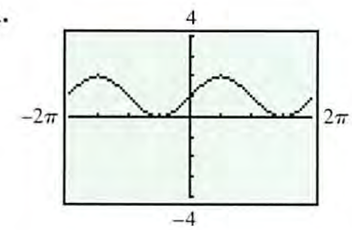
9. $y = \cos\left(x - \frac{\pi}{4}\right)$

10. $y = \sin\left(x - \frac{\pi}{4}\right)$

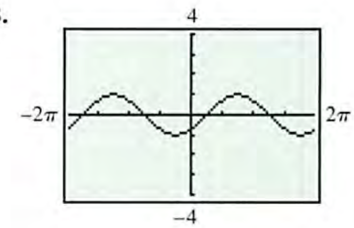
11. $y = 1 + \sin x$

12. $y = -1 + \cos x$

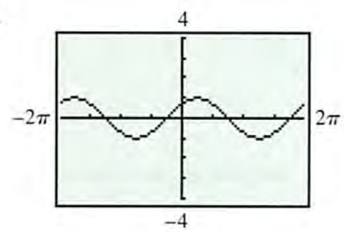
A.



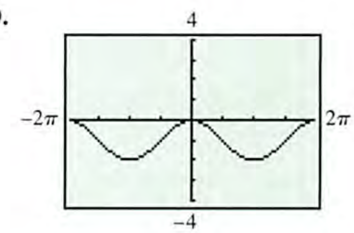
B.



C.



D.



13. The graphs of $y = \sin x + 1$ and $y = \sin(x + 1)$ are **NOT** the same. Explain why this is so.

14. ***Concept Check*** Refer to Exercise 13. Which one of the two graphs is the same as that of $y = 1 + \sin x$?

Concept Check *Match each function in Column I with the appropriate description in Column II.*

I	II
15. $y = 3\sin(2x - 4)$	**A.** amplitude = 2, period = $\frac{\pi}{2}$, phase shift = $\frac{3}{4}$
16. $y = 2\sin(3x - 4)$	**B.** amplitude = 3, period = π, phase shift = 2
17. $y = 4\sin(3x - 2)$	**C.** amplitude = 4, period = $\frac{2\pi}{3}$, phase shift = $\frac{2}{3}$
18. $y = 2\sin(4x - 3)$	**D.** amplitude = 2, period = $\frac{2\pi}{3}$, phase shift = $\frac{4}{3}$

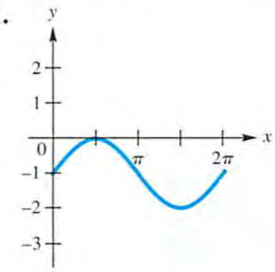
Concept Check *In Exercises 19 and 20, fill in the blanks with the word* right *or the word* left.

19. If the graph of $y = \cos x$ is translated $\frac{\pi}{2}$ units horizontally to the __________, it will coincide with the graph of $y = \sin x$.

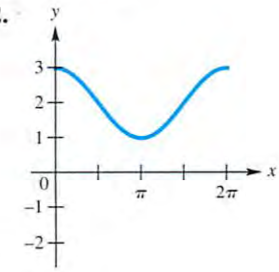
20. If the graph of $y = \sin x$ is translated $\frac{\pi}{2}$ units horizontally to the __________, it will coincide with the graph of $y = \cos x$.

Connecting Graphs with Equations *Each function graphed is of the form $y = c + \cos x$, $y = c + \sin x$, $y = \cos(x - d)$, or $y = \sin(x - d)$, where d is the least possible positive value. Determine the equation of the graph.*

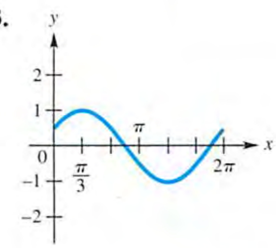
21.



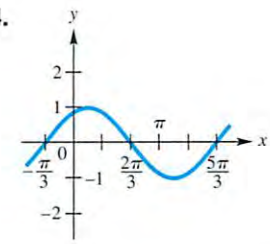
22.



23.



24.



Find the amplitude, the period, any vertical translation, and any phase shift of the graph of each function. See Examples 1–5.

25. $y = 2\sin(x - \pi)$

26. $y = \frac{2}{3}\sin\left(x + \frac{\pi}{2}\right)$

27. $y = 4\cos\left(\frac{1}{2}x + \frac{\pi}{2}\right)$

28. $y = \frac{1}{2}\sin\left(\frac{1}{2}x + \pi\right)$

29. $y = 3\cos\frac{\pi}{2}\left(x - \frac{1}{2}\right)$

30. $y = -\cos\pi\left(x - \frac{1}{3}\right)$

31. $y = 2 - \sin\left(3x - \frac{\pi}{5}\right)$

32. $y = -1 + \frac{1}{2}\cos(2x - 3\pi)$

Graph each function over a two-period interval. See Examples 1 and 2.

33. $y = \cos\left(x - \frac{\pi}{2}\right)$

34. $y = \sin\left(x - \frac{\pi}{4}\right)$

35. $y = \sin\left(x + \frac{\pi}{4}\right)$

36. $y = \cos\left(x - \frac{\pi}{3}\right)$

37. $y = 2\cos\left(x - \frac{\pi}{3}\right)$

38. $y = 3\sin\left(x - \frac{3\pi}{2}\right)$

Graph each function over a one-period interval. See Example 3.

39. $y = \frac{3}{2}\sin 2\left(x + \frac{\pi}{4}\right)$

40. $y = -\frac{1}{2}\cos 4\left(x + \frac{\pi}{2}\right)$

41. $y = -4\sin(2x - \pi)$

42. $y = 3\cos(4x + \pi)$

43. $y = \frac{1}{2}\cos\left(\frac{1}{2}x - \frac{\pi}{4}\right)$

44. $y = -\frac{1}{4}\sin\left(\frac{3}{4}x + \frac{\pi}{8}\right)$

Graph each function over a two-period interval. See Example 4.

45. $y = -3 + 2\sin x$

46. $y = 2 - 3\cos x$

47. $y = -1 - 2\cos 5x$

48. $y = 1 - \frac{2}{3}\sin\frac{3}{4}x$

49. $y = 1 - 2\cos\frac{1}{2}x$

50. $y = -3 + 3\sin\frac{1}{2}x$

51. $y = -2 + \frac{1}{2}\sin 3x$

52. $y = 1 + \frac{2}{3}\cos\frac{1}{2}x$

Graph each function over a one-period interval. See Example 5.

53. $y = -3 + 2\sin\left(x + \frac{\pi}{2}\right)$

54. $y = 4 - 3\cos(x - \pi)$

55. $y = \frac{1}{2} + \sin 2\left(x + \frac{\pi}{4}\right)$

56. $y = -\frac{5}{2} + \cos 3\left(x - \frac{\pi}{6}\right)$

(Modeling) *Solve each problem. See Example 6.*

57. ***Average Monthly Temperature*** The average monthly temperature (in °F) in Vancouver, Canada, is shown in the table.

Month	°F	Month	°F
Jan	36	July	64
Feb	39	Aug	63
Mar	43	Sept	57
Apr	48	Oct	50
May	55	Nov	43
June	59	Dec	39

Source: Miller, A. and J. Thompson, *Elements of Meteorology, Fourth Edition*, Charles E. Merrill Publishing Co., 1983.

(a) Plot the average monthly temperature over a two-year period letting $x = 1$ correspond to the month of January during the first year. Do the data seem to indicate a translated sine graph?

(b) The highest average monthly temperature is 64°F in July, and the lowest average monthly temperature is 36°F in January. Their average is 50°F. Graph the data together with the line $y = 50$. What does this line represent with regard to temperature in Vancouver?

(c) Approximate the amplitude, period, and phase shift of the translated sine wave.

(d) Determine a function of the form $f(x) = a\sin b(x - d) + c$, where a, b, c, and d are constants, that models the data.

(e) Graph f together with the data on the same coordinate axes. How well does f model the given data?

(f) Use the sine regression capability of a graphing calculator to find the equation of a sine curve that fits these data.

58. ***Average Monthly Temperature*** The average monthly temperature (in °F) in Phoenix, Arizona, is shown in the table.

Month	°F	Month	°F
Jan	51	July	90
Feb	55	Aug	90
Mar	63	Sept	84
Apr	67	Oct	71
May	77	Nov	59
June	86	Dec	52

Source: Miller, A. and J. Thompson, *Elements of Meteorology, Fourth Edition*, Charles E. Merrill Publishing Co., 1983.

(a) Predict the average yearly temperature and compare it to the actual value of 70°F.

(b) Plot the average monthly temperature over a two-year period by letting $x = 1$ correspond to January of the first year.

(c) Determine a function of the form $f(x) = a\cos b(x - d) + c$, where a, b, c, and d are constants, that models the data.

(d) Graph f together with the data on the same coordinate axes. How well does f model the data?

(e) Use the sine regression capability of a graphing calculator to find the equation of a sine curve that fits these data (two years).

CHAPTER 4 ▶ Quiz (Sections 4.1–4.2)

Graph each function over a two-period interval. Give the period and amplitude.

1. $y = -4 \sin x$

2. $y = -\frac{1}{2} \cos 2x$

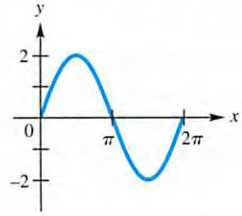
3. $y = 3 \sin \pi x$

4. $y = -2 \cos\left(x + \frac{\pi}{4}\right)$

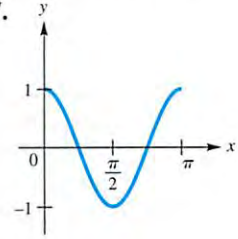
5. $y = 2 + \sin(2x - \pi)$

Connecting Graphs with Equations *Each function graphed is of the form $y = a \cos bx$ or $y = a \sin bx$, where $b > 0$. Determine the equation of the graph.*

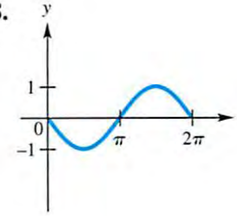
6.



7.



8.



Average Monthly Temperature *The average temperature (in °F) at a certain location can be approximated by the function defined by*

$$f(x) = 12 \sin\left[\frac{\pi}{6}(x - 3.9)\right] + 72,$$

where $x = 1$ represents January, $x = 2$ represents February, and so on.

9. What is the average temperature in April?

10. What is the lowest average monthly temperature? What is the highest?

4.3 Graphs of the Tangent and Cotangent Functions

Graph of the Tangent Function ▪ Graph of the Cotangent Function ▪ Graphing Techniques

Graph of the Tangent Function

Consider the table of selected points accompanying the graph of the tangent function in Figure 22 on the next page. These points include special values between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. The tangent function is undefined for odd multiples of $\frac{\pi}{2}$ and, thus, has *vertical asymptotes* for such values. A **vertical asymptote** is a vertical line that the graph approaches but does not intersect, while function values increase or decrease without bound as x-values get closer and closer to the line. Furthermore, since $\tan(-x) = -\tan x$ (see Exercise 43), the graph of the tangent function is symmetric with respect to the origin.

x	$y = \tan x$
$-\frac{\pi}{3}$	$-\sqrt{3} \approx -1.7$
$-\frac{\pi}{4}$	-1
$-\frac{\pi}{6}$	$-\frac{\sqrt{3}}{3} \approx -.6$
0	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{3} \approx .6$
$\frac{\pi}{4}$	1
$\frac{\pi}{3}$	$\sqrt{3} \approx 1.7$

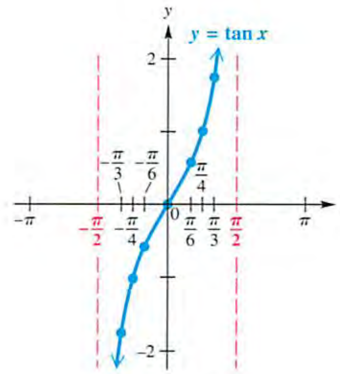


Figure 22

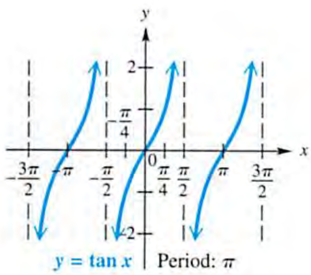


Figure 23

The tangent function has period π. Because $\tan x = \frac{\sin x}{\cos x}$, tangent values are 0 when sine values are 0, and undefined when cosine values are 0. As x-values go from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, tangent values go from $-\infty$ to ∞ and increase throughout the interval. Those same values are repeated as x goes from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$, $\frac{3\pi}{2}$ to $\frac{5\pi}{2}$, and so on. The graph of $y = \tan x$ from $-\frac{3\pi}{2}$ to $\frac{3\pi}{2}$ is shown in Figure 23.

TANGENT FUNCTION $f(x) = \tan x$

Domain: $\{x \mid x \neq (2n + 1)\frac{\pi}{2}$, where n is any integer$\}$ Range: $(-\infty, \infty)$

x	y
$-\frac{\pi}{2}$	undefined
$-\frac{\pi}{4}$	-1
0	0
$\frac{\pi}{4}$	1
$\frac{\pi}{2}$	undefined

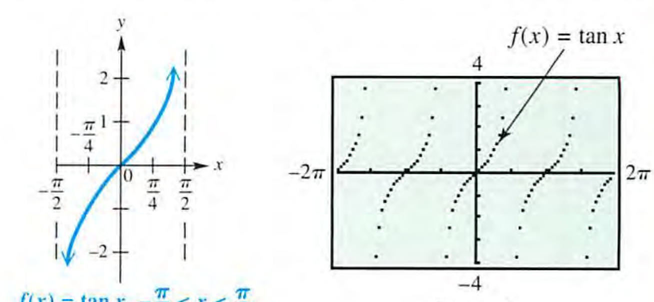


Figure 24

- The graph is discontinuous at values of x of the form $x = (2n + 1)\frac{\pi}{2}$ and has vertical asymptotes at these values.
- Its x-intercepts are of the form $x = n\pi$.
- Its period is π.
- Its graph has no amplitude, since there are no minimum or maximum values.
- The graph is symmetric with respect to the origin, so the function is an odd function. For all x in the domain, $\tan(-x) = -\tan x$.

Graph of the Cotangent Function A similar analysis for selected points between 0 and π for the graph of the cotangent function yields the graph in Figure 25 on the next page. Here the vertical asymptotes are at x-values that are integer multiples of π. Because $\cot(-x) = -\cot x$ (see Exercise 44), this graph is also symmetric with respect to the origin. (This can be seen when more of the graph is plotted.)

x	$y = \cot x$
$\frac{\pi}{6}$	$\sqrt{3} \approx 1.7$
$\frac{\pi}{4}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{3} \approx .6$
$\frac{\pi}{2}$	0
$\frac{2\pi}{3}$	$-\frac{\sqrt{3}}{3} \approx -.6$
$\frac{3\pi}{4}$	-1
$\frac{5\pi}{6}$	$-\sqrt{3} \approx -1.7$

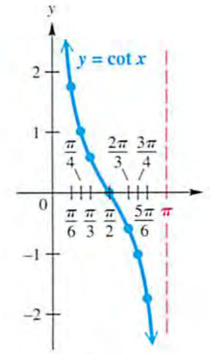


Figure 25

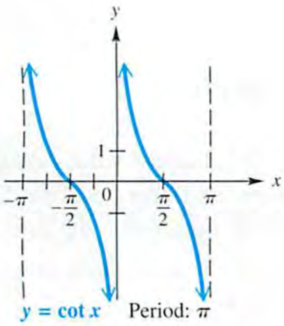


Figure 26

The cotangent function also has period π. Cotangent values are 0 when cosine values are 0, and undefined when sine values are 0. As x-values go from 0 to π, cotangent values go from ∞ to $-\infty$ and decrease throughout the interval. Those same values are repeated as x goes from π to 2π, 2π to 3π, and so on. The graph of $y = \cot x$ from $-\pi$ to π is shown in Figure 26. The graph continues in this pattern.

COTANGENT FUNCTION $f(x) = \cot x$

Domain: $\{x \mid x \neq n\pi$, where n is any integer$\}$ Range: $(-\infty, \infty)$

x	y
0	undefined
$\frac{\pi}{4}$	1
$\frac{\pi}{2}$	0
$\frac{3\pi}{4}$	-1
π	undefined

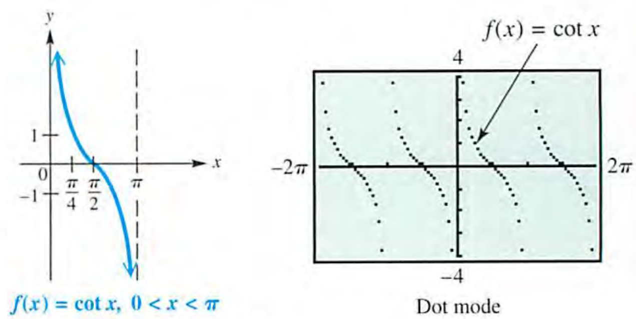


Figure 27

- The graph is discontinuous at values of x of the form $x = n\pi$ and has vertical asymptotes at these values.
- Its x-intercepts are of the form $x = (2n + 1)\frac{\pi}{2}$.
- Its period is π.
- Its graph has no amplitude, since there are no minimum or maximum values.
- The graph is symmetric with respect to the origin, so the function is an odd function. For all x in the domain, $\cot(-x) = -\cot x$.

The tangent function can be graphed directly with a graphing calculator, using the tangent key. To graph the cotangent function, however, we must use one of the identities $\cot x = \frac{1}{\tan x}$ or $\cot x = \frac{\cos x}{\sin x}$, since graphing calculators generally do not have cotangent keys. ■

Graphing Techniques

GUIDELINES FOR SKETCHING GRAPHS OF TANGENT AND COTANGENT FUNCTIONS

To graph $y = a \tan bx$ or $y = a \cot bx$, with $b > 0$, follow these steps.

Step 1 Determine the period, $\frac{\pi}{b}$. To locate two adjacent vertical asymptotes, solve the following equations for x:

For $y = a \tan bx$: $bx = -\frac{\pi}{2}$ and $bx = \frac{\pi}{2}$.

For $y = a \cot bx$: $bx = 0$ and $bx = \pi$.

Step 2 Sketch the two vertical asymptotes found in Step 1.

Step 3 Divide the interval formed by the vertical asymptotes into four equal parts. (See the Note on page 148.)

Step 4 Evaluate the function for the first-quarter point, midpoint, and third-quarter point, using the x-values found in Step 3.

Step 5 Join the points with a smooth curve, approaching the vertical asymptotes. Indicate additional asymptotes and periods of the graph as necessary.

▶ EXAMPLE 1 GRAPHING $y = \tan bx$

Graph $y = \tan 2x$.

Solution

Step 1 The period of this function is $\frac{\pi}{2}$. To locate two adjacent vertical asymptotes, solve $2x = -\frac{\pi}{2}$ and $2x = \frac{\pi}{2}$ (since this is a tangent function). The two asymptotes have equations $x = -\frac{\pi}{4}$ and $x = \frac{\pi}{4}$.

Step 2 Sketch the two vertical asymptotes $x = \pm\frac{\pi}{4}$, as shown in Figure 28.

Step 3 Divide the interval $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ into four equal parts. This gives the following key x-values.

first-quarter value: $-\frac{\pi}{8}$, middle value: 0, third-quarter value: $\frac{\pi}{8}$ Key x-values

Step 4 Evaluate the function for the x-values found in Step 3.

x	$-\frac{\pi}{8}$	0	$\frac{\pi}{8}$
$2x$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$
$\tan 2x$	-1	0	1

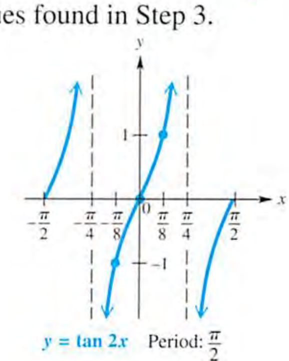


Figure 28

Step 5 Join these points with a smooth curve, approaching the vertical asymptotes. See Figure 28. Another period has been graphed, one half period to the left and one half period to the right.

NOW TRY EXERCISE 7. ◀

▶ EXAMPLE 2 GRAPHING $y = a \tan bx$

Graph $y = -3 \tan \frac{1}{2}x$.

Solution The period is $\frac{\pi}{\frac{1}{2}} = \pi \div \frac{1}{2} = \pi \cdot \frac{2}{1} = 2\pi$. Adjacent asymptotes are at $x = -\pi$ and $x = \pi$. Dividing the interval $(-\pi, \pi)$ into four equal parts gives key x-values of $-\frac{\pi}{2}$, 0, and $\frac{\pi}{2}$. Evaluating the function at these x-values gives the following key points.

$$\left(-\frac{\pi}{2}, 3\right), \quad (0, 0), \quad \left(\frac{\pi}{2}, -3\right) \quad \text{Key points}$$

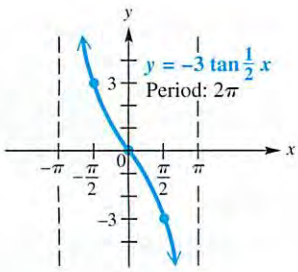


Figure 29

By plotting these points and joining them with a smooth curve, we obtain the graph shown in Figure 29. Because the coefficient -3 is negative, the graph is reflected across the x-axis compared to the graph of $y = 3 \tan \frac{1}{2}x$.

NOW TRY EXERCISE 15. ◀

▶ **Note** The function defined by $y = -3 \tan \frac{1}{2}x$ in Example 2, graphed in Figure 29, has a graph that compares to the graph of $y = \tan x$ as follows.

1. The period is larger because $b = \frac{1}{2}$, and $\frac{1}{2} < 1$.
2. The graph is "stretched" because $a = -3$, and $|-3| > 1$.
3. Each branch of the graph goes down from left to right (that is, the function decreases) between each pair of adjacent asymptotes because $a = -3$, and $-3 < 0$. When $a < 0$, the graph is reflected across the x-axis compared to the graph of $y = |a| \tan bx$.

▶ EXAMPLE 3 GRAPHING $y = a \cot bx$

Graph $y = \frac{1}{2} \cot 2x$.

Solution Because this function involves the cotangent, we can locate two adjacent asymptotes by solving the equations $2x = 0$ and $2x = \pi$. The lines $x = 0$ (the y-axis) and $x = \frac{\pi}{2}$ are two such asymptotes. Divide the interval $\left(0, \frac{\pi}{2}\right)$ into four equal parts, getting key x-values of $\frac{\pi}{8}, \frac{\pi}{4}$, and $\frac{3\pi}{8}$. Evaluating the function at these x-values gives the key points $\left(\frac{\pi}{8}, \frac{1}{2}\right), \left(\frac{\pi}{4}, 0\right), \left(\frac{3\pi}{8}, -\frac{1}{2}\right)$. Joining these points with a smooth curve approaching the asymptotes gives the graph shown in Figure 30.

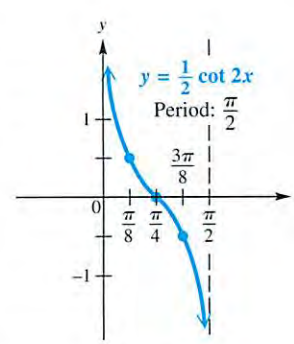


Figure 30

NOW TRY EXERCISE 17. ◀

Like the other circular functions, the graphs of the tangent and cotangent functions may be translated horizontally and vertically.

▶ EXAMPLE 4 GRAPHING A TANGENT FUNCTION WITH A VERTICAL TRANSLATION

Graph $y = 2 + \tan x$.

Analytic Solution

Every value of y for this function will be 2 units more than the corresponding value of y in $y = \tan x$, causing the graph of $y = 2 + \tan x$ to be translated 2 units up compared with the graph of $y = \tan x$. See Figure 31.

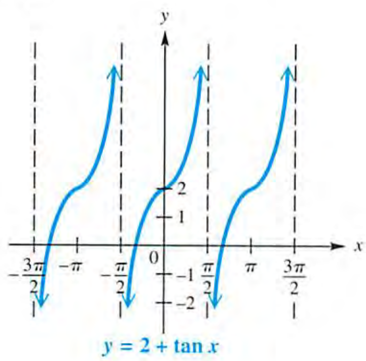


Figure 31

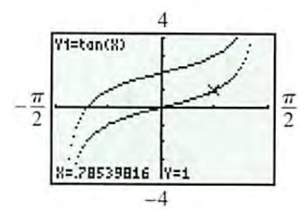
Graphing Calculator Solution

To see the vertical translation, observe the coordinates displayed at the bottoms of the screens in Figures 32 and 33. For $X = \frac{\pi}{4} \approx .78539816$,

$$Y_1 = \tan X = 1,$$

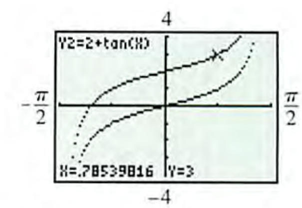
while for the same X-value,

$$Y_2 = 2 + \tan X = 2 + 1 = 3.$$



Dot mode

Figure 32



Dot mode

Figure 33

NOW TRY EXERCISE 23. ◀

▶ EXAMPLE 5 GRAPHING A COTANGENT FUNCTION WITH VERTICAL AND HORIZONTAL TRANSLATIONS

Graph $y = -2 - \cot(x - \frac{\pi}{4})$.

Solution Here $b = 1$, so the period is π. The graph will be translated down 2 units (because $c = -2$), reflected across the x-axis (because of the negative sign in front of the cotangent), and will have a phase shift (horizontal translation) $\frac{\pi}{4}$ unit to the right $\left(\text{because of the argument } \left(x - \frac{\pi}{4}\right)\right)$. To locate adjacent asymptotes, since this function involves the cotangent, we solve the following equations:

$$x - \frac{\pi}{4} = 0, \quad \text{so } x = \frac{\pi}{4} \qquad \text{and} \qquad x - \frac{\pi}{4} = \pi, \quad \text{so } x = \frac{5\pi}{4}.$$

Dividing the interval $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ into four equal parts and evaluating the function at the three key x-values within the interval gives these points.

$$\left(\frac{\pi}{2}, -3\right), \quad \left(\frac{3\pi}{4}, -2\right), \quad (\pi, -1) \quad \text{Key points}$$

Join these points with a smooth curve. This period of the graph, along with the one in the domain interval $\left(-\frac{3\pi}{4}, \frac{\pi}{4}\right)$, is shown in Figure 34.

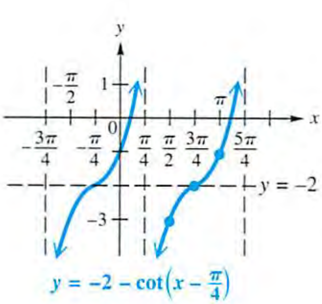


Figure 34

NOW TRY EXERCISE 31. ◀

4.3 Exercises

Concept Check *In Exercises 1–6, match each function with its graph from choices A–F.*

1. $y = -\tan x$

2. $y = -\cot x$

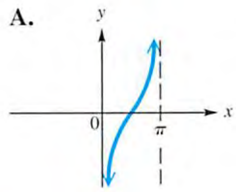
3. $y = \tan\left(x - \frac{\pi}{4}\right)$

4. $y = \cot\left(x - \frac{\pi}{4}\right)$

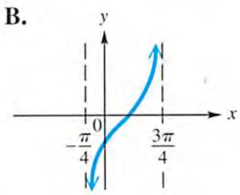
5. $y = \cot\left(x + \frac{\pi}{4}\right)$

6. $y = \tan\left(x + \frac{\pi}{4}\right)$

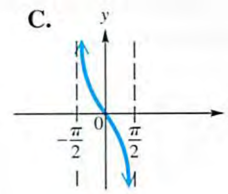
A.



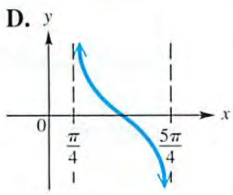
B.



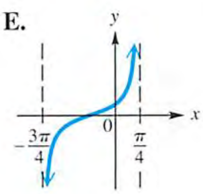
C.



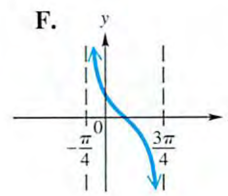
D.



E.



F.



Graph each function over a one-period interval. See Examples 1–3.

7. $y = \tan 4x$

8. $y = \tan \frac{1}{2}x$

9. $y = 2 \tan x$

10. $y = 2 \cot x$

11. $y = 2 \tan \frac{1}{4}x$

12. $y = \frac{1}{2} \cot x$

13. $y = \cot 3x$

14. $y = -\cot \frac{1}{2}x$

15. $y = -2 \tan \frac{1}{4}x$

16. $y = 3 \tan \frac{1}{2}x$

17. $y = \frac{1}{2} \cot 4x$

18. $y = -\frac{1}{2} \cot 2x$

Graph each function over a two-period interval. See Examples 4 and 5.

19. $y = \tan(2x - \pi)$

20. $y = \tan\left(\frac{x}{2} + \pi\right)$

21. $y = \cot\left(3x + \frac{\pi}{4}\right)$

22. $y = \cot\left(2x - \frac{3\pi}{2}\right)$

23. $y = 1 + \tan x$

24. $y = 1 - \tan x$

25. $y = 1 - \cot x$

26. $y = -2 - \cot x$

27. $y = -1 + 2 \tan x$

28. $y = 3 + \frac{1}{2} \tan x$

29. $y = -1 + \frac{1}{2} \cot(2x - 3\pi)$

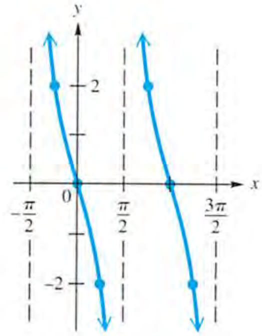
30. $y = -2 + 3 \tan(4x + \pi)$

31. $y = 1 - 2 \cot 2\left(x + \frac{\pi}{2}\right)$

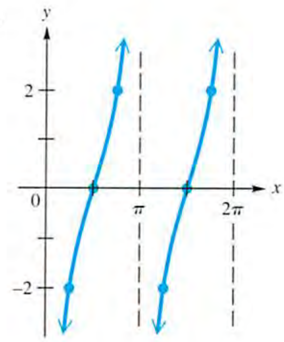
32. $y = \frac{2}{3} \tan\left(\frac{3}{4}x - \pi\right) - 2$

Connecting Graphs with Equations *Each function graphed is of the form* $y = a \tan bx$ *or* $y = a \cot bx$, *where* $b > 0$. *Determine the equation of the graph. (Half- and quarter-points are identified by dots.)*

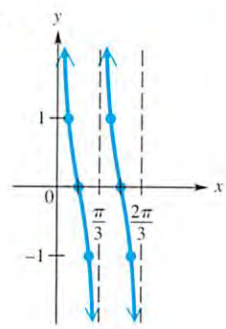
33.



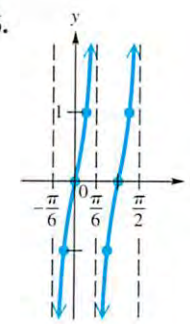
34.



35.



36.



Concept Check *In Exercises 37–40, tell whether each statement is* true *or* false. *If* false, *tell why.*

37. The least positive number k for which $x = k$ is an asymptote for the tangent function is $\frac{\pi}{2}$.

38. The least positive number k for which $x = k$ is an asymptote for the cotangent function is $\frac{\pi}{2}$.

39. The graph of $y = \tan x$ in Figure 23 suggests that $\tan(-x) = \tan x$ for all x in the domain of $\tan x$.

40. The graph of $y = \cot x$ in Figure 26 suggests that $\cot(-x) = -\cot x$ for all x in the domain of $\cot x$.

41. ***Concept Check*** If c is any number, then how many solutions does the equation $c = \tan x$ have in the interval $(-2\pi, 2\pi]$?

42. Consider the function defined by $f(x) = -4 \tan(2x + \pi)$. What is the domain of f? What is its range?

43. Show that $\tan(-x) = -\tan x$ by writing $\tan(-x)$ as $\frac{\sin(-x)}{\cos(-x)}$ and then using the relationships for $\sin(-x)$ and $\cos(-x)$.

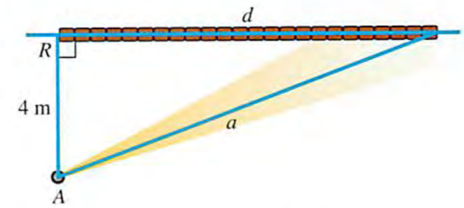
44. Show that $\cot(-x) = -\cot x$ by writing $\cot(-x)$ as $\frac{\cos(-x)}{\sin(-x)}$ and then using the relationships for $\cos(-x)$ and $\sin(-x)$.

45. ***(Modeling) Distance of a Rotating Beacon*** A rotating beacon is located at point A next to a long wall. (See the figure on the next page.) The beacon is 4 m from the wall. The distance d is given by

$$d = 4 \tan 2\pi t,$$

where t is time measured in seconds since the beacon started rotating. (When $t = 0$, the beacon is aimed at point R. When the beacon is aimed to the right of R, the value of d is positive; d is negative if the beacon is aimed to the left of R.) Find d for each time.

(a) $t = 0$
(b) $t = .4$
(c) $t = .8$
(d) $t = 1.2$
(e) Why is .25 a meaningless value for t?



46. Simultaneously graph $y = \tan x$ and $y = x$ in the window $[-1, 1]$ by $[-1, 1]$ with a graphing calculator. Write a short description of the relationship between $\tan x$ and x for small x-values.

RELATING CONCEPTS

***For individual or collaborative investigation* (Exercises 47–52)**

Consider the function defined by

$$y = -2 - \cot\left(x - \frac{\pi}{4}\right)$$

from Example 5. ***Work these exercises in order.***

47. What is the least positive number for which $y = \cot x$ is undefined?

48. Let k represent the number you found in Exercise 47. Set $x - \frac{\pi}{4}$ equal to k, and solve to find a positive number for which $\cot\left(x - \frac{\pi}{4}\right)$ is undefined.

49. Based on your answer in Exercise 48 and the fact that the cotangent function has period π, give the general form of the equations of the asymptotes of the graph of $y = -2 - \cot\left(x - \frac{\pi}{4}\right)$. Let n represent any integer.

50. Use the capabilities of your calculator to find the least positive x-intercept of the graph of this function.

51. Use the fact that the period of this function is π to find the next positive x-intercept.

52. Give the solution set of the equation $-2 - \cot\left(x - \frac{\pi}{4}\right) = 0$ over all real numbers. Let n represent any integer.

4.4 Graphs of the Secant and Cosecant Functions

Graph of the Secant Function ▪ Graph of the Cosecant Function ▪ Graphing Techniques ▪ Addition of Ordinates ▪ Connecting Graphs with Equations

Graph of the Secant Function Consider the table of selected points accompanying the graph of the secant function in Figure 35 on the next page. These points include special values between $-\pi$ and π. The secant function is undefined for odd multiples of $\frac{\pi}{2}$ and, thus, like the tangent function has vertical asymptotes for such values. Furthermore, since $\sec(-x) = \sec x$ (see Exercise 31), the graph of the secant function is symmetric with respect to the y-axis.

x	$y = \sec x$	x	$y = \sec x$
$\pm\frac{\pi}{3}$	2	$\pm\frac{2\pi}{3}$	-2
$\pm\frac{\pi}{4}$	$\sqrt{2} \approx 1.4$	$\pm\frac{3\pi}{4}$	$-\sqrt{2} \approx -1.4$
$\pm\frac{\pi}{6}$	$\frac{2\sqrt{3}}{3} \approx 1.2$	$\pm\frac{5\pi}{6}$	$-\frac{2\sqrt{3}}{3} \approx -1.2$
0	1	$\pm\pi$	-1

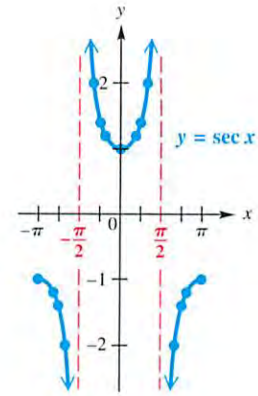


Figure 35

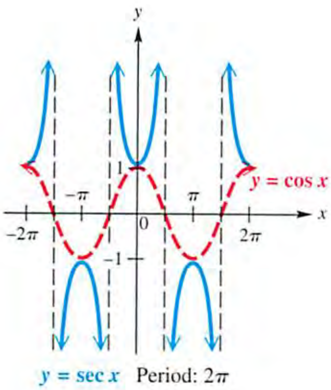


Figure 36

Because secant values are reciprocals of corresponding cosine values, the period of the secant function is 2π, the same as for $y = \cos x$. When $\cos x = 1$, the value of $\sec x$ is also 1; likewise, when $\cos x = -1$, $\sec x = -1$ as well. For all x, $-1 \leq \cos x \leq 1$, and thus, $|\sec x| \geq 1$ for all x in its domain. Figure 36 shows how the graphs of $y = \cos x$ and $y = \sec x$ are related.

SECANT FUNCTION $f(x) = \sec x$

Domain: $\{x \mid x \neq (2n + 1)\frac{\pi}{2}$, where n is any integer$\}$ Range: $(-\infty, -1] \cup [1, \infty)$

x	y
$-\frac{\pi}{2}$	undefined
$-\frac{\pi}{4}$	$\sqrt{2}$
0	1
$\frac{\pi}{4}$	$\sqrt{2}$
$\frac{\pi}{2}$	undefined
$\frac{3\pi}{4}$	$-\sqrt{2}$
π	-1
$\frac{3\pi}{2}$	undefined

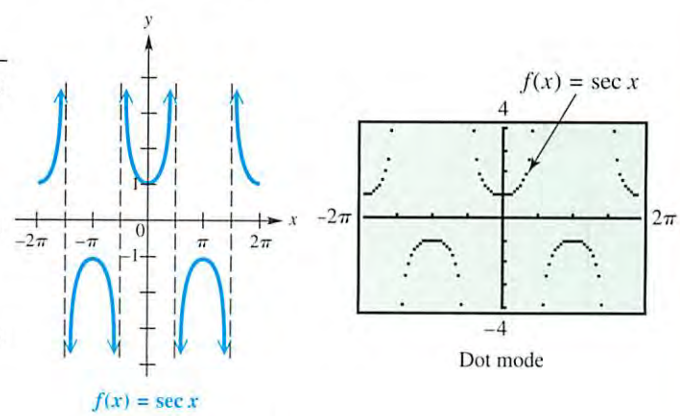


Figure 37

- The graph is discontinuous at values of x of the form $x = (2n + 1)\frac{\pi}{2}$ and has vertical asymptotes at these values.
- There are no x-intercepts.
- Its period is 2π.
- Its graph has no amplitude, since there are no maximum or minimum values.
- The graph is symmetric with respect to the y-axis, so the function is an even function. For all x in the domain, $\sec(-x) = \sec x$.

Graph of the Cosecant Function A similar analysis for selected points between $-\pi$ and π for the graph of the cosecant function yields the graph in Figure 38. The vertical asymptotes are at x-values that are integer multiples of π. Because $\csc(-x) = -\csc x$ (see Exercise 32), this graph is symmetric with respect to the origin.

x	$y = \csc x$	x	$y = \csc x$
$\frac{\pi}{6}$	2	$-\frac{\pi}{6}$	-2
$\frac{\pi}{4}$	$\sqrt{2} \approx 1.4$	$-\frac{\pi}{4}$	$-\sqrt{2} \approx -1.4$
$\frac{\pi}{3}$	$\frac{2\sqrt{3}}{3} \approx 1.2$	$-\frac{\pi}{3}$	$-\frac{2\sqrt{3}}{3} \approx -1.2$
$\frac{\pi}{2}$	1	$-\frac{\pi}{2}$	-1
$\frac{2\pi}{3}$	$\frac{2\sqrt{3}}{3} \approx 1.2$	$-\frac{2\pi}{3}$	$-\frac{2\sqrt{3}}{3} \approx -1.2$
$\frac{3\pi}{4}$	$\sqrt{2} \approx 1.4$	$-\frac{3\pi}{4}$	$-\sqrt{2} \approx -1.4$
$\frac{5\pi}{6}$	2	$-\frac{5\pi}{6}$	-2

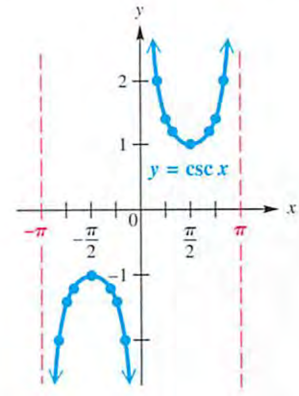


Figure 38

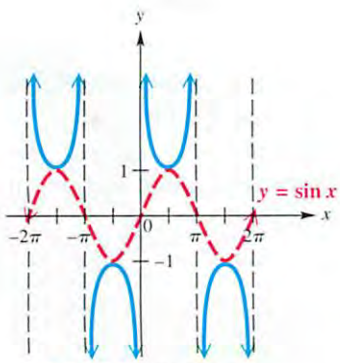


Figure 39

Because cosecant values are reciprocals of corresponding sine values, the period of the cosecant function is 2π, the same as for $y = \sin x$. When $\sin x = 1$, the value of $\csc x$ is also 1; likewise, when $\sin x = -1$, $\csc x = -1$. For all x, $-1 \le \sin x \le 1$, and thus $|\csc x| \ge 1$ for all x in its domain. Figure 39 shows how the graphs of $y = \sin x$ and $y = \csc x$ are related.

COSECANT FUNCTION $f(x) = \csc x$

Domain: $\{x \mid x \neq n\pi$, where n is any integer$\}$ Range: $(-\infty, -1] \cup [1, \infty)$

x	y
0	undefined
$\frac{\pi}{6}$	2
$\frac{\pi}{3}$	$\frac{2\sqrt{3}}{3}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{2\sqrt{3}}{3}$
π	undefined
$\frac{3\pi}{2}$	-1
2π	undefined

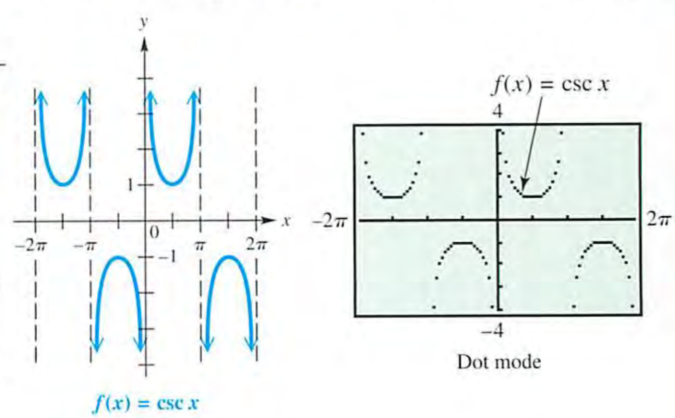
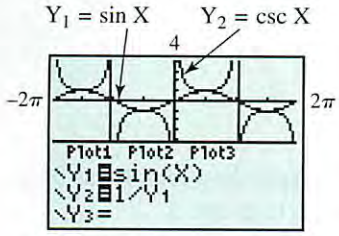


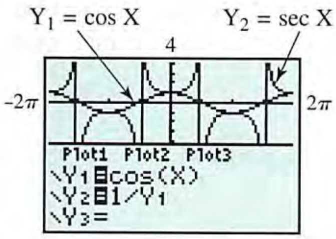
Figure 40

- The graph is discontinuous at values of x of the form $x = n\pi$ and has vertical asymptotes at these values.
- There are no x-intercepts.
- Its period is 2π.
- Its graph has no amplitude, since there are no maximum or minimum values.
- The graph is symmetric with respect to the origin, so the function is an odd function. For all x in the domain, $\csc(-x) = -\csc x$.



Trig window; connected mode

Figure 41



Trig window; connected mode

Figure 42

Typically, calculators do not have keys for the cosecant and secant functions. To graph $y = \csc x$ with a graphing calculator, use the fact that

$$\csc x = \frac{1}{\sin x}.$$

The graphs of $Y_1 = \sin X$ and $Y_2 = \csc X$ are shown in Figure 41. The calculator is in split screen and connected modes. Similarly, the secant function is graphed by using the identity

$$\sec x = \frac{1}{\cos x},$$

as shown in Figure 42.

Using dot mode for graphing will eliminate the vertical lines that appear in Figures 41 and 42. While they suggest asymptotes and are sometimes called **pseudo-asymptotes,** they are not actually parts of the graphs. ■

Graphing Techniques

In the previous section, we gave guidelines for sketching graphs of tangent and cotangent functions. We now present similar guidelines for graphing cosecant and secant functions.

GUIDELINES FOR SKETCHING GRAPHS OF COSECANT AND SECANT FUNCTIONS

To graph $\boldsymbol{y = a \csc bx}$ or $\boldsymbol{y = a \sec bx}$, with $b > 0$, follow these steps.

Step 1 Graph the corresponding reciprocal function as a guide, using a dashed curve.

To Graph	Use as a Guide
$y = a \csc bx$	$y = a \sin bx$
$y = a \sec bx$	$y = a \cos bx$

Step 2 Sketch the vertical asymptotes. They will have equations of the form $x = k$, where k is an x-intercept of the graph of the guide function.

Step 3 Sketch the graph of the desired function by drawing the typical U-shaped branches between the adjacent asymptotes. The branches will be above the graph of the guide function when the guide function values are positive and below the graph of the guide function when the guide function values are negative. The graph will resemble those in Figures 37 and 40 in the function boxes on pages 177 and 178.

Like graphs of the sine and cosine functions, graphs of the secant and cosecant functions may be translated vertically and horizontally. The period of both basic functions is 2π.

▶ EXAMPLE 1 GRAPHING $y = a \sec bx$

Graph $y = 2 \sec \frac{1}{2}x$.

Solution

Step 1 This function involves the secant, so the corresponding reciprocal function will involve the cosine. The guide function to graph is

$$y = 2 \cos \frac{1}{2}x.$$

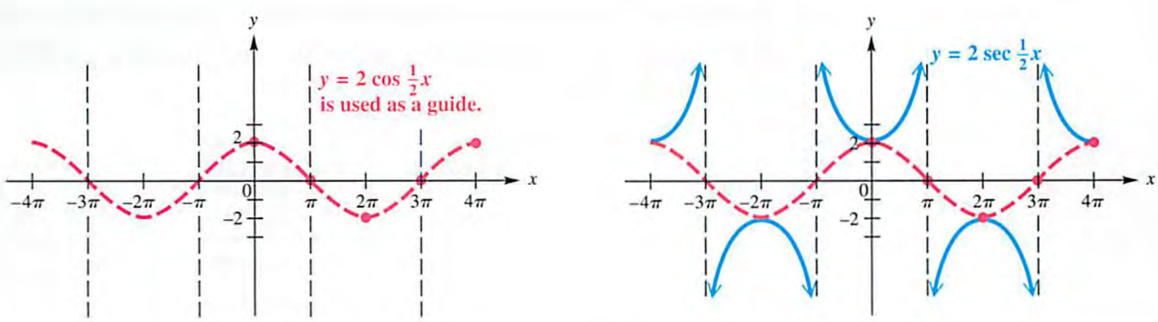
Using the guidelines of **Section 4.1,** we find that this guide function has amplitude 2 and one period of the graph lies along the interval that satisfies the inequality

$$0 \le \frac{1}{2}x \le 2\pi, \qquad \text{or} \qquad [0, 4\pi]. \quad \text{(Appendix A)}$$

Dividing this interval into four equal parts gives the key points

$$(0, 2), \quad (\pi, 0), \quad (2\pi, -2), \quad (3\pi, 0), \quad (4\pi, 2), \quad \text{Key points}$$

which are joined with a dashed red curve to indicate that this graph is only a guide. An additional period is graphed as seen in Figure 43(a).



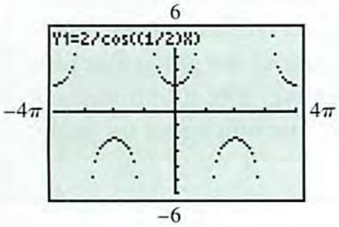
(a) (b)

Figure 43

Step 2 Sketch the vertical asymptotes. These occur at x-values for which the guide function equals 0, such as

$$x = -3\pi, \qquad x = -\pi, \qquad x = \pi, \qquad x = 3\pi.$$

See Figure 43(a).



Dot mode

This is a calculator graph of the function in Example 1.

Step 3 Sketch the graph of $y = 2 \sec \frac{1}{2}x$ by drawing the typical U-shaped branches, approaching the asymptotes. See the solid blue graph in Figure 43(b).

NOW TRY EXERCISE 5. ◀

EXAMPLE 2 GRAPHING $y = a\csc(x - d)$

Graph $y = \frac{3}{2}\csc(x - \frac{\pi}{2})$.

Solution

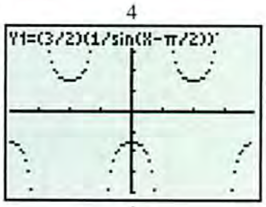
Step 1 Use the guidelines of **Section 4.2** to graph the corresponding reciprocal function defined by

$$y = \frac{3}{2}\sin\left(x - \frac{\pi}{2}\right),$$

shown as a red dashed curve in Figure 44.

Step 2 Sketch the vertical asymptotes through the x-intercepts of the graph of $y = \frac{3}{2}\sin\left(x - \frac{\pi}{2}\right)$. These have the form $x = (2n + 1)\frac{\pi}{2}$, where n is any integer. See the black dashed lines in Figure 44.

Step 3 Sketch the graph of $y = \frac{3}{2}\csc\left(x - \frac{\pi}{2}\right)$ by drawing the typical U-shaped branches between adjacent asymptotes. See the solid blue graph in Figure 44.



Dot mode

This is a calculator graph of the function in Example 2.

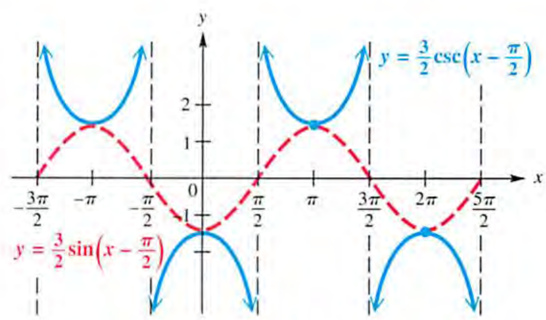


Figure 44

NOW TRY EXERCISE 7.

Addition of Ordinates

New functions can be formed by adding or subtracting other functions. A function formed by combining two other functions, such as

$$y = \cos x + \sin x,$$

has historically been graphed using a method known as **addition of ordinates.** (The x-value of a point is sometimes called its **abscissa,** while its y-value is called its **ordinate.**) To apply this method to this function, we graph the functions $y = \cos x$ and $y = \sin x$. Then, for selected values of x, we add $\cos x$ and $\sin x$, and plot the points $(x, \cos x + \sin x)$. Joining the resulting points with a sinusoidal curve gives the graph of the desired function. While this method illustrates some valuable concepts involving the arithmetic of functions, it is time-consuming.

With graphing calculators, this technique is easily illustrated. Let $Y_1 = \cos X$, $Y_2 = \sin X$, and $Y_3 = Y_1 + Y_2$. Figure 45 on the next page shows the result when Y_1 and Y_2 are graphed in thin graph style, and $Y_3 = \cos X + \sin X$ is graphed in thick graph style. Notice that for $X = \frac{\pi}{6} \approx .52359878$, $Y_1 + Y_2 = Y_3$.

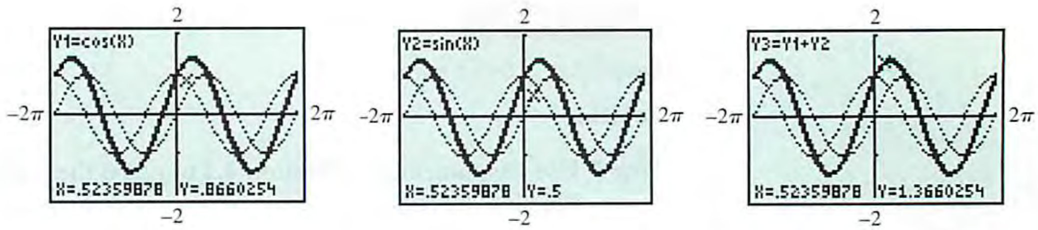


Figure 45

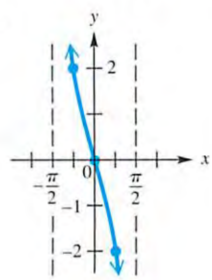
NOW TRY EXERCISE 35.

Connecting Graphs with Equations Now that the graphs of the six circular functions have been introduced, we can apply the concepts to give an equation of a given graph.

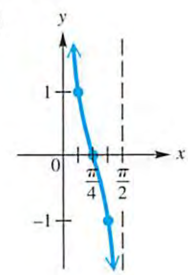
EXAMPLE 3 DETERMINING AN EQUATION FOR A GRAPH

Determine an equation for each graph.

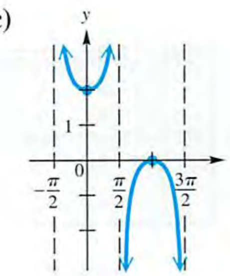
(a)



(b)



(c)



Solution

(a) This graph is that of $y = \tan x$ but reflected across the x-axis and stretched vertically by a factor of 2. Therefore, an equation for this graph is

$$y = -2 \tan x.$$

x-axis reflection (the $-$ sign); Vertical stretch (the 2)

(b) This graph is that of $y = \cot x$, but the period is $\frac{\pi}{2}$ rather than π. Therefore, if $y = \cot bx$, where $b > 0$, we must have $b = 2$. So an equation for this graph is

$$y = \cot 2x.$$

(c) This is the graph of $y = \sec x$, translated one unit upward. An equation is

$$y = 1 + \sec x.$$

NOW TRY EXERCISES 19, 21, AND 23.

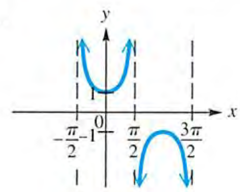
Note Because the circular functions are periodic, there are actually infinitely many equations that correspond to each graph in Example 3. We have given only one such equation in each case. For instance, confirm that $y = -\tan\left(2x - \frac{\pi}{2}\right)$ is another equation for the graph in Example 3(b).

4.4 Exercises

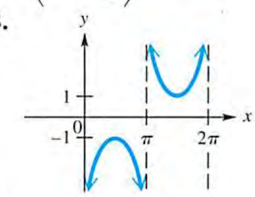
Concept Check *In Exercises 1–4, match each function with its graph from choices A–D.*

1. $y = -\csc x$ **2.** $y = -\sec x$ **3.** $y = \sec\left(x - \frac{\pi}{2}\right)$ **4.** $y = \csc\left(x + \frac{\pi}{2}\right)$

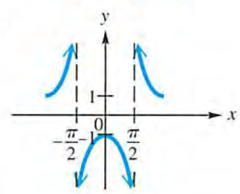
A.



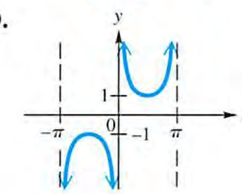
B.



C.



D.



Graph each function over a one-period interval. See Examples 1 and 2.

5. $y = 3 \sec \frac{1}{4}x$

6. $y = -2 \sec \frac{1}{2}x$

7. $y = -\frac{1}{2}\csc\left(x + \frac{\pi}{2}\right)$

8. $y = \frac{1}{2}\csc\left(x - \frac{\pi}{2}\right)$

9. $y = \csc\left(x - \frac{\pi}{4}\right)$

10. $y = \sec\left(x + \frac{3\pi}{4}\right)$

11. $y = \sec\left(x + \frac{\pi}{4}\right)$

12. $y = \csc\left(x + \frac{\pi}{3}\right)$

13. $y = \sec\left(\frac{1}{2}x + \frac{\pi}{3}\right)$

14. $y = \csc\left(\frac{1}{2}x - \frac{\pi}{4}\right)$

15. $y = 2 + 3\sec(2x - \pi)$

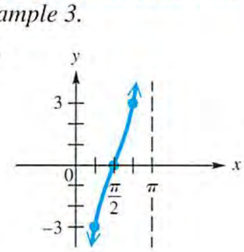
16. $y = 1 - 2\csc\left(x + \frac{\pi}{2}\right)$

17. $y = 1 - \frac{1}{2}\csc\left(x - \frac{3\pi}{4}\right)$

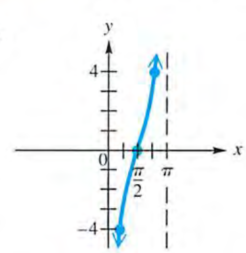
18. $y = 2 + \frac{1}{4}\sec\left(\frac{1}{2}x - \pi\right)$

Connecting Graphs with Equations *Determine an equation for each graph. See Example 3.*

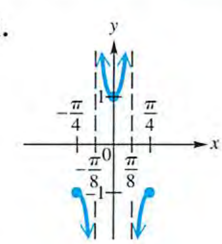
19.



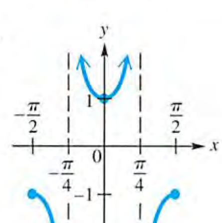
20.



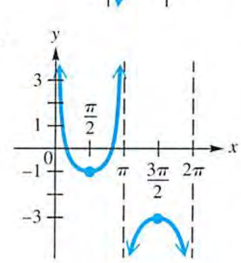
21.



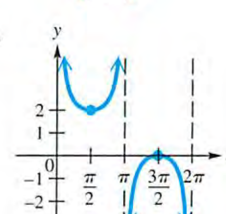
22.



23.



24.



Concept Check *In Exercises 25–28, tell whether each statement is* true *or* false. *If* false, *tell why.*

25. The tangent and secant functions are undefined for the same values.

26. The secant and cosecant functions are undefined for the same values.

27. The graph of $y = \sec x$ in Figure 37 suggests that $\sec(-x) = \sec x$ for all x in the domain of $\sec x$.

28. The graph of $y = \csc x$ in Figure 40 suggests that $\csc(-x) = -\csc x$ for all x in the domain of $\csc x$.

29. ***Concept Check*** If c is any number such that $-1 < c < 1$, then how many solutions does the equation $c = \sec x$ have over the entire domain of the secant function?

30. Consider the function defined by $g(x) = -2 \csc(4x + \pi)$. What is the domain of g? What is its range?

31. Show that $\sec(-x) = \sec x$ by writing $\sec(-x)$ as $\frac{1}{\cos(-x)}$ and then using the relationship between $\cos(-x)$ and $\cos x$.

32. Show that $\csc(-x) = -\csc x$ by writing $\csc(-x)$ as $\frac{1}{\sin(-x)}$ and then using the relationship between $\sin(-x)$ and $\sin x$.

33. ***(Modeling) Distance of a Rotating Beacon*** In the figure for Exercise 45 in **Section 4.3,** the distance a is given by

$$a = 4|\sec 2\pi t|.$$

Find a for each time.

(a) $t = 0$ **(b)** $t = .86$ **(c)** $t = 1.24$

34. Between each pair of successive asymptotes, a portion of the graph of $y = \sec x$ or $y = \csc x$ resembles a parabola. Can each of these portions actually be a parabola? Explain.

Use a graphing calculator to graph Y_1, Y_2, *and* $Y_1 + Y_2$ *on the same screen. Evaluate each of the three functions at* $X = \frac{\pi}{6}$, *and verify that* $Y_1(\frac{\pi}{6}) + Y_2(\frac{\pi}{6}) = (Y_1 + Y_2)(\frac{\pi}{6})$. *See the discussion on addition of ordinates.*

35. $Y_1 = \sin X$, $Y_2 = \sin 2X$

36. $Y_1 = \cos X$, $Y_2 = \sec X$

Summary Exercises on Graphing Circular Functions

These summary exercises provide practice with the various graphing techniques presented in this chapter. Graph each function over a one-period interval.

1. $y = 2 \sin \pi x$

2. $y = 4 \cos 1.5x$

3. $y = -2 + .5 \cos \frac{\pi}{4}x$

4. $y = 3 \sec \frac{\pi}{2}x$

5. $y = -4 \csc .5x$

6. $y = 3 \tan\left(\frac{\pi}{2}x + \pi\right)$

Graph each function over a two-period interval.

7. $y = -5 \sin \frac{x}{3}$

8. $y = 10 \cos\left(\frac{x}{4} + \frac{\pi}{2}\right)$

9. $y = 3 - 4 \sin(2.5x + \pi)$

10. $y = 2 - \sec[\pi(x - 3)]$

4.5 Harmonic Motion

Simple Harmonic Motion ▪ Damped Oscillatory Motion

Simple Harmonic Motion In part A of Figure 46, a spring with a weight attached to its free end is in equilibrium (or rest) position. If the weight is pulled down a units and released (part B of the figure), the spring's elasticity causes the weight to rise a units $(a > 0)$ above the equilibrium position, as seen in part C, and then oscillate about the equilibrium position. If friction is neglected, this oscillatory motion is described mathematically by a sinusoid. Other applications of this type of motion include sound, electric current, and electromagnetic waves.

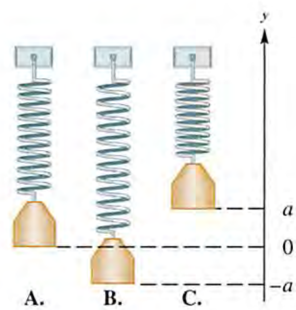


Figure 46

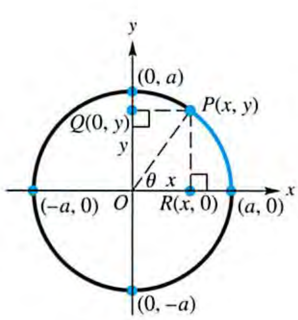


Figure 47

To develop a general equation for such motion, consider Figure 47. Suppose the point $P(x, y)$ moves around the circle counterclockwise at a uniform angular speed ω. Assume that at time $t = 0$, P is at $(a, 0)$. The angle swept out by ray OP at time t is given by $\theta = \omega t$. The coordinates of point P at time t are

$$x = a\cos\theta = a\cos\omega t \quad \text{and} \quad y = a\sin\theta = a\sin\omega t.$$

As P moves around the circle from the point $(a, 0)$, the point $Q(0, y)$ oscillates back and forth along the y-axis between the points $(0, a)$ and $(0, -a)$. Similarly, the point $R(x, 0)$ oscillates back and forth between $(a, 0)$ and $(-a, 0)$. This oscillatory motion is called **simple harmonic motion.**

The amplitude of the motion is $|a|$, and the period is $\frac{2\pi}{\omega}$. The moving points P and Q or P and R complete one oscillation or cycle per period. The number of cycles per unit of time, called the **frequency,** is the reciprocal of the period, $\frac{\omega}{2\pi}$, where $\omega > 0$.

SIMPLE HARMONIC MOTION

The position of a point oscillating about an equilibrium position at time t is modeled by either

$$s(t) = a\cos\omega t \quad \text{or} \quad s(t) = a\sin\omega t,$$

where a and ω are constants, with $\omega > 0$. The amplitude of the motion is $|a|$, the period is $\frac{2\pi}{\omega}$, and the frequency is $\frac{\omega}{2\pi}$ oscillations per time unit.

▶ EXAMPLE 1 MODELING THE MOTION OF A SPRING

Suppose that an object is attached to a coiled spring such as the one in Figure 46 on the preceding page. It is pulled down a distance of 5 in. from its equilibrium position, and then released. The time for one complete oscillation is 4 sec.

(a) Give an equation that models the position of the object at time t.

(b) Determine the position at $t = 1.5$ sec.

(c) Find the frequency.

Solution

(a) When the object is released at $t = 0$, the distance of the object from the equilibrium position is 5 in. below equilibrium. If $s(t)$ is to model the motion, then $s(0)$ must equal -5. We use

$$s(t) = a \cos \omega t,$$

with $a = -5$. We choose the cosine function because $\cos \omega(0) = \cos 0 = 1$, and $-5 \cdot 1 = -5$. (Had we chosen the sine function, a phase shift would have been required.) The period is 4, so

$$\frac{2\pi}{\omega} = 4, \quad \text{or} \quad \omega = \frac{\pi}{2}. \quad \text{Solve for } \omega. \text{ (Appendix A)}$$

Thus, the motion is modeled by

$$s(t) = -5 \cos \frac{\pi}{2} t.$$

(b) After 1.5 sec, the position is

$$s(1.5) = -5 \cos\left[\frac{\pi}{2}(1.5)\right] \approx 3.54 \text{ in.} \quad \text{(Appendix C)}$$

Since $3.54 > 0$, the object is above the equilibrium position.

(c) The frequency is the reciprocal of the period, or $\frac{1}{4}$ oscillation per sec.

NOW TRY EXERCISE 9. ◀

▶ EXAMPLE 2 ANALYZING HARMONIC MOTION

Suppose that an object oscillates according to the model

$$s(t) = 8 \sin 3t,$$

where t is in seconds and $s(t)$ is in feet. Analyze the motion.

Solution The motion is harmonic because the model is of the form $s(t) = a \sin \omega t$. Because $a = 8$, the object oscillates 8 ft in either direction from its starting point. The period $\frac{2\pi}{3} \approx 2.1$ is the time, in seconds, it takes for one complete oscillation. The frequency is the reciprocal of the period, so the object completes $\frac{3}{2\pi} \approx .48$ oscillation per sec.

NOW TRY EXERCISE 15. ◀

Damped Oscillatory Motion In the example of the stretched spring, we disregard the effect of friction. Friction causes the amplitude of the motion to diminish gradually until the weight comes to rest. In this situation, we say that the motion has been *damped* by the force of friction. Most oscillatory motions are damped, and the decrease in amplitude follows the pattern of exponential decay. A typical example of **damped oscillatory motion** is provided by the function defined by

$$s(t) = e^{-t} \sin t.$$

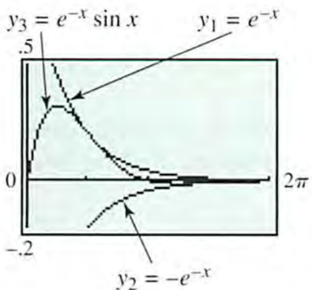


Figure 48

(The number $e \approx 2.718$ is the base of the natural logarithm function, first studied in college algebra courses.) Figure 48 shows how the graph of $y_3 = e^{-x} \sin x$ is bounded above by the graph of $y_1 = e^{-x}$ and below by the graph of $y_2 = -e^{-x}$. The damped motion curve dips below the x-axis at $x = \pi$ but stays above the graph of y_2. Figure 49 shows a traditional graph of $s(t) = e^{-t} \sin t$, along with the graph of $y = \sin t$.

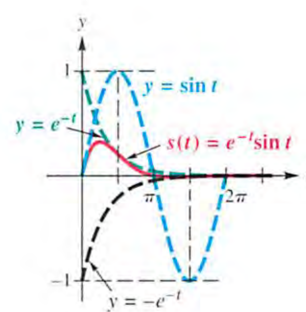


Figure 49

Shock absorbers are put on an automobile in order to damp oscillatory motion. Instead of oscillating up and down for a long while after hitting a bump or pothole, the oscillations of the car are quickly damped out for a smoother ride.

NOW TRY EXERCISE 21.

4.5 Exercises

(Modeling) Springs *Suppose that a weight on a spring has initial position $s(0)$ and period P.*

(a) *Find a function s given by $s(t) = a \cos \omega t$ that models the displacement of the weight.*

(b) *Evaluate $s(1)$. Is the weight moving upward, downward, or neither when $t = 1$? Support your results graphically or numerically.*

1. $s(0) = 2$ in.; $P = .5$ sec

2. $s(0) = 5$ in.; $P = 1.5$ sec

3. $s(0) = -3$ in.; $P = .8$ sec

4. $s(0) = -4$ in.; $P = 1.2$ sec

(Modeling) Music A note on the piano has given frequency F. Suppose the maximum displacement at the center of the piano wire is given by $s(0)$. Find constants a and ω so that the equation $s(t) = a \cos \omega t$ models this displacement. Graph s in the viewing window $[0, .05]$ by $[-.3, .3]$.

5. $F = 27.5$; $s(0) = .21$

6. $F = 110$; $s(0) = .11$

7. $F = 55$; $s(0) = .14$

8. $F = 220$; $s(0) = .06$

(Modeling) Solve each problem. See Examples 1 and 2.

9. ***Spring*** An object is attached to a coiled spring, as in Figure 46. It is pulled down a distance of 4 units from its equilibrium position, and then released. The time for one complete oscillation is 3 sec.

(a) Give an equation that models the position of the object at time t.
(b) Determine the position at $t = 1.25$ sec.
(c) Find the frequency.

10. ***Spring*** Repeat Exercise 9, but assume that the object is pulled down 6 units and the time for one complete oscillation is 4 sec.

11. ***Particle Movement*** Write the equation and then determine the amplitude, period, and frequency of the simple harmonic motion of a particle moving uniformly around a circle of radius 2 units, with angular speed

(a) 2 radians per sec **(b)** 4 radians per sec.

12. ***Pendulum*** What are the period P and frequency T of oscillation of a pendulum of length $\frac{1}{2}$ ft? (*Hint:* $P = 2\pi\sqrt{\frac{L}{32}}$, where L is the length of the pendulum in feet and P is in seconds.)

13. ***Pendulum*** In Exercise 12, how long should the pendulum be to have period 1 sec?

14. ***Spring*** The formula for the up and down motion of a weight on a spring is given by

$$s(t) = a \sin \sqrt{\frac{k}{m}}\, t.$$

If the spring constant k is 4, what mass m must be used to produce a period of 1 sec?

15. ***Spring*** The height attained by a weight attached to a spring set in motion is

$$s(t) = -4 \cos 8\pi t$$

inches after t seconds.

(a) Find the maximum height that the weight rises above the equilibrium position of $s(t) = 0$.
(b) When does the weight first reach its maximum height, if $t \geq 0$?
(c) What are the frequency and period?

16. ***Spring*** (See Exercise 14.) A spring with spring constant $k = 2$ and a 1-unit mass m attached to it is stretched and then allowed to come to rest.

(a) If the spring is stretched $\frac{1}{2}$ ft and released, what are the amplitude, period, and frequency of the resulting oscillatory motion?
(b) What is the equation of the motion?

17. ***Spring*** The position of a weight attached to a spring is

$$s(t) = -5 \cos 4\pi t$$

inches after t seconds.

(a) What is the maximum height that the weight rises above the equilibrium position?
(b) What are the frequency and period?
(c) When does the weight first reach its maximum height?
(d) Calculate and interpret $s(1.3)$.

18. ***Spring*** The position of a weight attached to a spring is

$$s(t) = -4 \cos 10t$$

inches after t seconds.

(a) What is the maximum height that the weight rises above the equilibrium position?
(b) What are the frequency and period?
(c) When does the weight first reach its maximum height?
(d) Calculate and interpret $s(1.466)$.

19. ***Spring*** A weight attached to a spring is pulled down 3 in. below the equilibrium position.

(a) Assuming that the frequency is $\frac{6}{\pi}$ cycles per sec, determine a model that gives the position of the weight at time t seconds.
(b) What is the period?

20. ***Spring*** A weight attached to a spring is pulled down 2 in. below the equilibrium position.

(a) Assuming that the period is $\frac{1}{3}$ sec, determine a model that gives the position of the weight at time t seconds.
(b) What is the frequency?

Use a graphing calculator to graph $y_1 = e^{-t} \sin t$, $y_2 = e^{-t}$, and $y_3 = -e^{-t}$ in the viewing window $[0, \pi]$ by $[-.5, .5]$.

21. Find the t-intercepts of the graph of y_1. Explain the relationship of these intercepts with the x-intercepts of the graph of $y = \sin x$.

22. Find any points of intersection of y_1 and y_2 or y_1 and y_3. How are these points related to the graph of $y = \sin x$?

Chapter 4 Summary

KEY TERMS

4.1 periodic function, period, sine wave (sinusoid), amplitude

4.2 phase shift, argument

4.3 vertical asymptote

4.4 addition of ordinates

4.5 simple harmonic motion, frequency, damped oscillatory motion

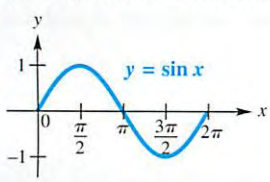
QUICK REVIEW

CONCEPTS	EXAMPLES

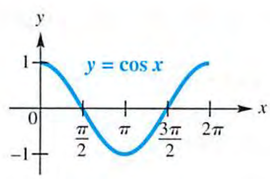
4.1 Graphs of the Sine and Cosine Functions

4.2 Translations of the Graphs of the Sine and Cosine Functions

Sine and Cosine Functions



Domain: $(-\infty, \infty)$
Range: $[-1, 1]$
Amplitude: 1
Period: 2π



Domain: $(-\infty, \infty)$
Range: $[-1, 1]$
Amplitude: 1
Period: 2π

The graph of

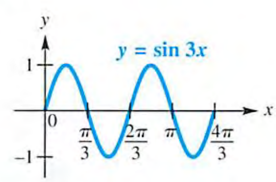
$$y = c + a \sin b(x - d) \quad \text{or} \quad y = c + a \cos b(x - d),$$

$b > 0$, has

1. amplitude $|a|$,
2. period $\frac{2\pi}{b}$,
3. vertical translation c units up if $c > 0$ or $|c|$ units down if $c < 0$, and
4. phase shift d units to the right if $d > 0$ or $|d|$ units to the left if $d < 0$.

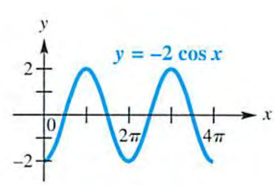
See pages 150 and 161 for a summary of graphing techniques.

Graph $y = \sin 3x$.



period: $\frac{2\pi}{3}$ amplitude: 1
domain: $(-\infty, \infty)$ range: $[-1, 1]$

Graph $y = -2 \cos x$.

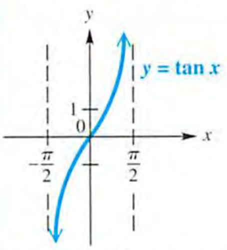


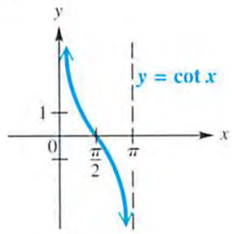
period: 2π amplitude: 2
domain: $(-\infty, \infty)$ range: $[-2, 2]$

CONCEPTS | EXAMPLES

4.3 Graphs of the Tangent and Cotangent Functions

Tangent and Cotangent Functions



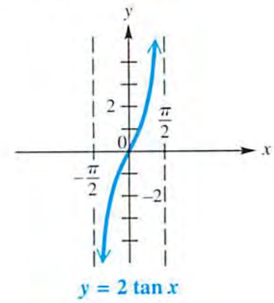


Domain: $\{x \mid x \neq (2n+1)\frac{\pi}{2}$, where n is any integer$\}$
Range: $(-\infty, \infty)$
Period: π

Domain: $\{x \mid x \neq n\pi$, where n is any integer$\}$
Range: $(-\infty, \infty)$
Period: π

See page 171 for a summary of graphing techniques.

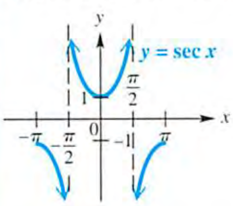
Graph one period of $y = 2\tan x$.

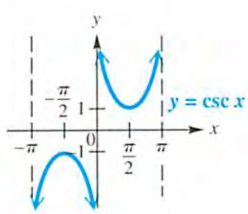


period: π
domain: $\{x \mid x \neq (2n+1)\frac{\pi}{2}$, where n is any integer$\}$
range: $(-\infty, \infty)$

4.4 Graphs of the Secant and Cosecant Functions

Secant and Cosecant Functions



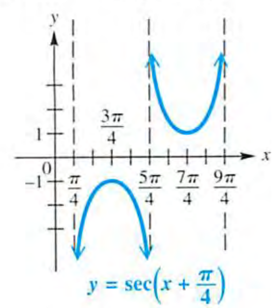


Domain: $\{x \mid x \neq (2n+1)\frac{\pi}{2}$, where n is any integer$\}$
Range: $(-\infty, -1] \cup [1, \infty)$
Period: 2π

Domain: $\{x \mid x \neq n\pi$, where n is any integer$\}$
Range: $(-\infty, -1] \cup [1, \infty)$
Period: 2π

See page 179 for a summary of graphing techniques.

Graph one period of $y = \sec\left(x + \frac{\pi}{4}\right)$.



period: 2π
phase shift: $-\frac{\pi}{4}$
domain: $\{x \mid x \neq \frac{\pi}{4} + n\pi$, where n is any integer$\}$
range: $(-\infty, -1] \cup [1, \infty)$

4.5 Harmonic Motion

Simple Harmonic Motion
The position of a point oscillating about an equilibrium position at time t is modeled by either

$$s(t) = a\cos\omega t \qquad \text{or} \qquad s(t) = a\sin\omega t,$$

where a and ω are constants, with $\omega > 0$. The amplitude of the motion is $|a|$, the period is $\frac{2\pi}{\omega}$, and the frequency is $\frac{\omega}{2\pi}$ oscillations per time unit.

A spring oscillates according to

$$s(t) = -5\cos 6t,$$

where t is in seconds and $s(t)$ is in inches. Find the amplitude, period, and frequency.

$$\text{amplitude} = |-5| = 5 \text{ in.}; \qquad \text{period} = \frac{2\pi}{6} = \frac{\pi}{3} \text{ sec};$$

$$\text{frequency} = \frac{3}{\pi} \text{ oscillation per sec}$$

CHAPTER 4 Review Exercises

1. *Concept Check* Which one of the following is true about the graph of $y = 4 \sin 2x$?

   A. It has amplitude 2 and period $\frac{\pi}{2}$.
   B. It has amplitude 4 and period π.
   C. Its range is $[0, 4]$.
   D. Its range is $[-4, 0]$.

2. *Concept Check* Which one of the following is false about the graph of $y = -3 \cos \frac{1}{2}x$?

   A. Its range is $[-3, 3]$.
   B. Its domain is $(-\infty, \infty)$.
   C. Its amplitude is 3, and its period is 4π.
   D. Its amplitude is 3, and its period is π.

3. *Concept Check* Which of the basic circular functions can have y-value $\frac{1}{2}$?
4. *Concept Check* Which of the basic circular functions can have y-value 2?

For each function, give the amplitude, period, vertical translation, and phase shift, as applicable.

5. $y = 2 \sin x$
6. $y = \tan 3x$
7. $y = -\frac{1}{2} \cos 3x$
8. $y = 2 \sin 5x$
9. $y = 1 + 2 \sin \frac{1}{4}x$
10. $y = 3 - \frac{1}{4} \cos \frac{2}{3}x$
11. $y = 3 \cos\left(x + \frac{\pi}{2}\right)$
12. $y = -\sin\left(x - \frac{3\pi}{4}\right)$
13. $y = \frac{1}{2} \csc\left(2x - \frac{\pi}{4}\right)$
14. $y = 2 \sec(\pi x - 2\pi)$
15. $y = \frac{1}{3} \tan\left(3x - \frac{\pi}{3}\right)$
16. $y = \cot\left(\frac{x}{2} + \frac{3\pi}{4}\right)$

Concept Check *Identify the circular function that satisfies each description.*

17. period is π, x-intercepts are of the form $n\pi$, where n is any integer
18. period is 2π, graph passes through the origin
19. period is 2π, graph passes through the point $\left(\frac{\pi}{2}, 0\right)$
20. period is 2π, domain is $\{x \mid x \neq n\pi$, where n is any integer$\}$
21. period is π, function is decreasing on the interval $(0, \pi)$
22. period is 2π, has vertical asymptotes of the form $x = (2n + 1)\frac{\pi}{2}$, where n is any integer
23. Suppose that f is a sine function with period 10 and $f(5) = 2$. Explain why $f(25) = 2$.
24. Suppose that f is a sine function with period π and $f\left(\frac{6\pi}{5}\right) = 1$. Explain why $f\left(-\frac{4\pi}{5}\right) = 1$.

Graph each function over a one-period interval.

25. $y = 3 \sin x$
26. $y = \frac{1}{2} \sec x$
27. $y = -\tan x$
28. $y = -2 \cos x$
29. $y = 2 + \cot x$
30. $y = -1 + \csc x$
31. $y = \sin 2x$
32. $y = \tan 3x$
33. $y = 3 \cos 2x$

34. $y = \frac{1}{2} \cot 3x$

35. $y = \cos\left(x - \frac{\pi}{4}\right)$

36. $y = \tan\left(x - \frac{\pi}{2}\right)$

37. $y = \sec\left(2x + \frac{\pi}{3}\right)$

38. $y = \sin\left(3x + \frac{\pi}{2}\right)$

39. $y = 1 + 2 \cos 3x$

40. $y = -1 - 3 \sin 2x$

41. $y = 2 \sin \pi x$

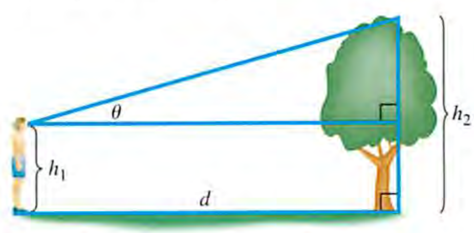
42. $y = -\frac{1}{2} \cos(\pi x - \pi)$

43. Explain why a function of the form $f(x) = 2 \sin(bx + c)$ has range $[-2, 2]$.

44. Explain why a function of the form $f(x) = 2 \csc(bx + c)$ has range $(-\infty, -2] \cup [2, \infty)$.

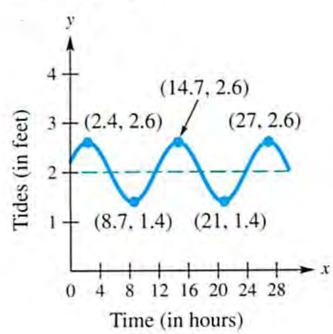
Solve each problem.

45. ***Viewing Angle to an Object*** Let a person whose eyes are h_1 feet from the ground stand d feet from an object h_2 feet tall, where $h_2 > h_1$. Let θ be the angle of elevation to the top of the object. See the figure.



(a) Show that $d = (h_2 - h_1) \cot \theta$.
(b) Let $h_2 = 55$ and $h_1 = 5$. Graph d for the interval $0 < \theta \leq \frac{\pi}{2}$.

46. ***(Modeling) Tides*** The figure shows a function f that models the tides in feet at Clearwater Beach, Florida, x hours after midnight starting on August 26, 2006. (*Source:* Pentcheff, D., *WWW Tide and Current Predictor.*)



(a) Find the time between high tides.
(b) What is the difference in water levels between high tide and low tide?
(c) The tides can be modeled by

$$f(x) = .6 \cos[.511(x - 2.4)] + 2.$$

Estimate the tides when $x = 10$.

47. ***(Modeling) Maximum Temperatures*** The maximum afternoon temperature in a given city might be modeled by

$$t = 60 - 30 \cos \frac{x\pi}{6},$$

where t represents the maximum afternoon temperature in month x, with $x = 0$ representing January, $x = 1$ representing February, and so on. Find the maximum afternoon temperature to the nearest degree for each month.

(a) January **(b)** April **(c)** May
(d) June **(e)** August **(f)** October

48. ***(Modeling) Average Monthly Temperature*** The average monthly temperature (in °F) in Chicago, Illinois, is shown in the table.

(a) Plot the average monthly temperature over a two-year period. Let $x = 1$ correspond to January of the first year.

(b) Determine a model function of the form $f(x) = a \sin b(x - d) + c$, where a, b, c, and d are constants.

(c) Explain the significance of each constant.

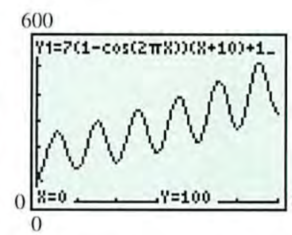
(d) Graph f together with the data on the same coordinate axes. How well does f model the data?

(e) Use the sine regression capability of a graphing calculator to find the equation of a sine curve that fits these data.

Month	°F	Month	°F
Jan	25	July	74
Feb	28	Aug	75
Mar	36	Sept	66
Apr	48	Oct	55
May	61	Nov	39
June	72	Dec	28

Source: Miller, A., J. Thompson, and R. Peterson, *Elements of Meteorology, Fourth Edition*, Charles E. Merrill Publishing Co., 1983.

49. ***(Modeling) Pollution Trends*** The amount of pollution in the air is lower after heavy spring rains and higher after periods of little rain. In addition to this seasonal fluctuation, the long-term trend is upward. An idealized graph of this situation is shown in the figure. Circular functions can be used to model the fluctuating part of the pollution levels. Powers of the number e (e is the base of the natural logarithm; $e \approx 2.718282$) can be used to model long-term growth. The pollution level in a certain area might be given by

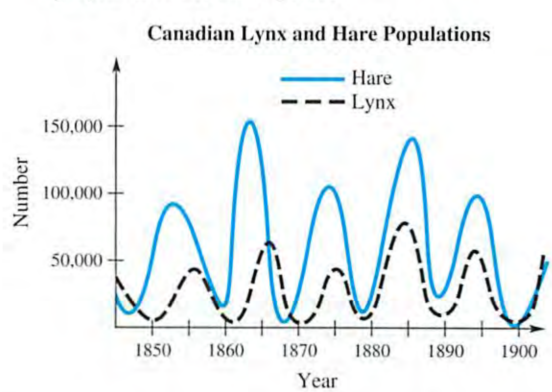


$$y = 7(1 - \cos 2\pi x)(x + 10) + 100e^{.2x},$$

where x is the time in years, with $x = 0$ representing January 1 of the base year. July 1 of the same year would be represented by $x = .5$, October 1 of the following year would be represented by $x = 1.75$, and so on. Find the pollution levels on each date.

(a) January 1, base year
(b) July 1, base year
(c) January 1, following year
(d) July 1, following year

50. ***(Modeling) Lynx and Hare Populations*** The figure shows the populations of lynx and hares in Canada for the years 1847–1903. The hares are food for the lynx. An increase in hare population causes an increase in lynx population some time later. The increasing lynx population then causes a decline in hare population. The two graphs have the same period.



(a) Estimate the length of one period.

(b) Estimate maximum and minimum hare populations.

An object in simple harmonic motion has position function s inches from an initial point, where t is the time in seconds. Find the amplitude, period, and frequency.

51. $s(t) = 4 \sin \pi t$

52. $s(t) = 3 \cos 2t$

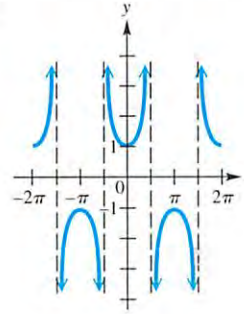
53. In Exercise 51, what does the frequency represent? Find the position of the object from the initial point at 1.5 sec, 2 sec, and 3.25 sec.

54. In Exercise 52, what does the period represent? What does the amplitude represent?

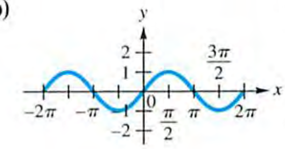
CHAPTER 4 ▶ Test

1. Identify each of the following basic circular function graphs.

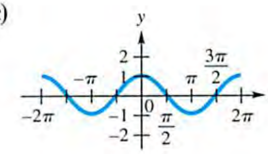
(a)



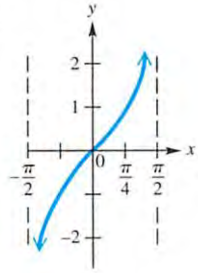
(b)



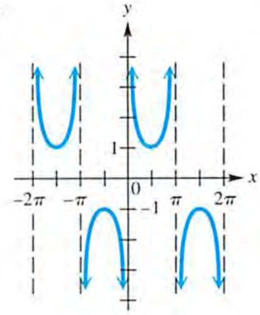
(c)



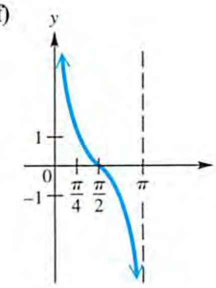
(d)



(e)

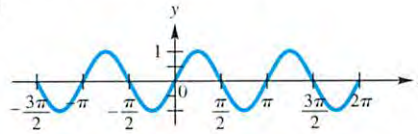


(f)

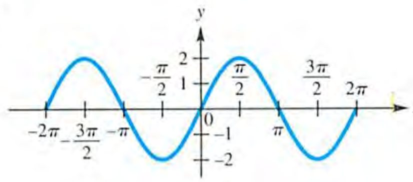


2. One of the following is the graph of $y = \sin 2x$ and the other is the graph of $y = 2 \sin x$. Identify each graph.

(a)



(b)



3. Give a short answer to each of the following.

(a) What is the domain of the cosine function?
(b) What is the range of the sine function?
(c) What is the least positive value for which the tangent function is undefined?
(d) What is the range of the secant function?

4. Consider the function defined by $y = 3 - 6\sin\left(2x + \frac{\pi}{2}\right)$.

(a) What is its period?
(b) What is the amplitude of its graph?
(c) What is its range?
(d) What is the y-intercept of its graph?
(e) What is its phase shift?

Graph each function over a two-period interval. Identify asymptotes when applicable.

5. $y = \sin(2x + \pi)$

6. $y = -\cos 2x$

7. $y = 2 + \cos x$

8. $y = -1 + 2\sin(x + \pi)$

9. $y = \tan\left(x - \frac{\pi}{2}\right)$

10. $y = -2 - \cot\left(x - \frac{\pi}{2}\right)$

11. $y = -\csc 2x$

12. $y = 3\csc \pi x$

13. ***(Modeling) Average Monthly Temperature*** The average monthly temperature (in °F) in Austin, Texas, can be modeled using the circular function defined by

$$f(x) = 17.5\sin\left[\frac{\pi}{6}(x - 4)\right] + 67.5,$$

where x is the month and $x = 1$ corresponds to January. (*Source:* Miller, A., J. Thompson, and R. Peterson, *Elements of Meteorology, Fourth Edition,* Charles E. Merrill Publishing Co., 1983.)

(a) Graph f in the window $[1, 25]$ by $[45, 90]$.
(b) Determine the amplitude, period, phase shift, and vertical translation of f.
(c) What is the average monthly temperature for the month of December?
(d) Determine the maximum and minimum average monthly temperatures and the months when they occur.
(e) What would be an approximation for the average *yearly* temperature in Austin? How is this related to the vertical translation of the sine function in the formula for f?

14. ***(Modeling) Spring*** The height of a weight attached to a spring is

$$s(t) = -4\cos 8\pi t$$

inches after t seconds.

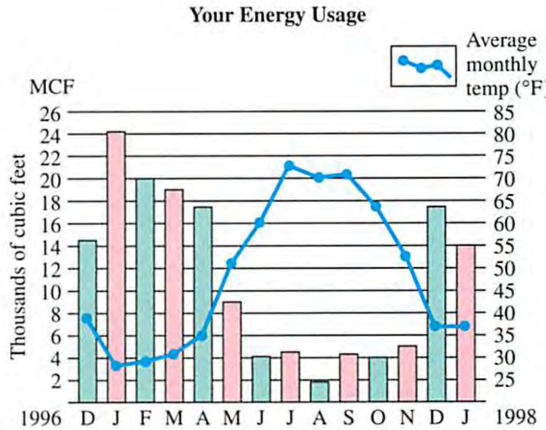
(a) Find the maximum height that the weight rises above the equilibrium position of $s(t) = 0$.
(b) When does the weight first reach its maximum height, if $t \geq 0$?
(c) What are the frequency and period?

15. Explain why the domains of the tangent and secant functions are the same, and then give a similar explanation for the cotangent and cosecant functions.

CHAPTER 4 ▶ Quantitative Reasoning

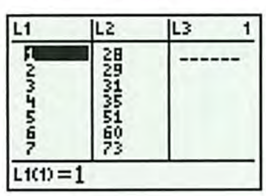
Does the fact that average monthly temperatures are periodic affect your utility bills?

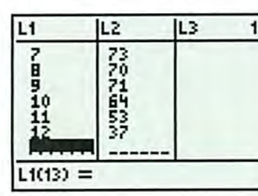
In an article entitled "I Found Sinusoids in My Gas Bill" (*Mathematics Teacher,* January 2000), Cathy G. Schloemer presents the following graph that accompanied her gas bill.



Notice that two sinusoids are suggested here: one for the behavior of the average monthly temperature and another for gas use in MCF (thousands of cubic feet).

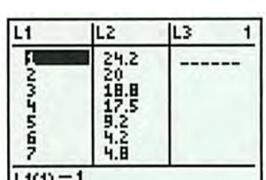
1. If January 1997 is represented by $x = 1$, the data of estimated ordered pairs (month, temperature) are given in the list shown on the two graphing calculator screens below.

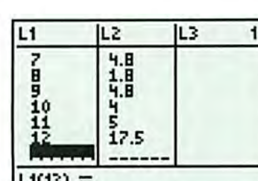




Use the sine regression feature of a graphing calculator to find a sine function that fits these data points. Then make a scatter diagram, and graph the function.

2. If January 1997 is again represented by $x = 1$, the data of estimated ordered pairs (month, gas use in MCF) are given in the list shown on the two graphing calculator screens below.





Use the sine regression feature of a graphing calculator to find a sine function that fits these data points. Then make a scatter diagram, and graph the function.

3. Answer the question posed at the top of the page, in the form of a short paragraph.