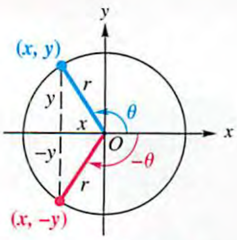
5 Trigonometric Identities

In 1831 Michael Faraday discovered that when a wire passes by a magnet, a small electric current is produced in the wire. Now we generate massive amounts of electricity by simultaneously rotating thousands of wires near large electromagnets. Because electric current alternates its direction on electrical wires, it is modeled accurately by either the sine or the cosine function.

We give many examples of applications of the trigonometric functions to electricity and other phenomena in the examples and exercises in this chapter, including a model of the wattage consumption of a toaster in Section 5.5, Example 6.

5.1 Fundamental Identities

Fundamental Identities ▪ Using the Fundamental Identities



$\sin(-\theta) = -\frac{y}{r} = -\sin\theta$

Figure 1

Recall that an **identity** is an equation that is satisfied by *every* value in the domain of its variable. (Appendix A)

Fundamental Identities

As suggested by the circle shown in Figure 1, an angle θ having the point (x, y) on its terminal side has a corresponding angle $-\theta$ with the point $(x, -y)$ on its terminal side. From the definition of sine,

$$\sin(-\theta) = \frac{-y}{r} \quad \text{and} \quad \sin\theta = \frac{y}{r}, \quad \text{(Section 1.3)}$$

so $\sin(-\theta)$ and $\sin\theta$ are negatives of each other, or

$$\mathbf{\sin(-\theta) = -\sin\theta.}$$

Figure 1 shows an angle θ in quadrant II, but the same result holds for θ in any quadrant. Also, by definition,

$$\cos(-\theta) = \frac{x}{r} \quad \text{and} \quad \cos\theta = \frac{x}{r}, \quad \text{(Section 1.3)}$$

so

$$\mathbf{\cos(-\theta) = \cos\theta.}$$

We use the identities for $\sin(-\theta)$ and $\cos(-\theta)$ to find $\tan(-\theta)$ in terms of $\tan\theta$:

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\frac{\sin\theta}{\cos\theta}, \quad \text{or} \quad \mathbf{\tan(-\theta) = -\tan\theta.}$$

Similar reasoning gives the remaining three **negative-angle** or **negative-number identities,** which, together with the reciprocal, quotient, and Pythagorean identities from **Chapter 1,** are called the **fundamental identities.**

FUNDAMENTAL IDENTITIES

Reciprocal Identities

$$\cot\theta = \frac{1}{\tan\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

Negative-Angle Identities

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$

$$\csc(-\theta) = -\csc\theta \qquad \sec(-\theta) = \sec\theta \qquad \cot(-\theta) = -\cot\theta$$

> ▶ **Note** The most commonly recognized forms of the fundamental identities are given in the preceding box. Throughout this chapter you must also recognize alternative forms of these identities. ***For example, two other forms of*** $\sin^2\theta + \cos^2\theta = 1$ ***are***
>
> $$\sin^2\theta = 1 - \cos^2\theta \quad \textit{and} \quad \cos^2\theta = 1 - \sin^2\theta.$$

Using the Fundamental Identities One way we use these identities is to find the values of other trigonometric functions from the value of a given trigonometric function. Although we could find such values using a right triangle, this is a good way to practice using the fundamental identities.

▶ EXAMPLE 1 FINDING TRIGONOMETRIC FUNCTION VALUES GIVEN ONE VALUE AND THE QUADRANT

If $\tan\theta = -\frac{5}{3}$ and θ is in quadrant II, find each function value.

(a) $\sec\theta$ **(b)** $\sin\theta$ **(c)** $\cot(-\theta)$

Solution

(a) Look for an identity that relates tangent and secant.

$$\tan^2\theta + 1 = \sec^2\theta \quad \text{Pythagorean identity}$$

$$\left(-\frac{5}{3}\right)^2 + 1 = \sec^2\theta \quad \tan\theta = -\tfrac{5}{3}$$

$$\frac{25}{9} + 1 = \sec^2\theta \quad \left(-\tfrac{5}{3}\right)^2 = -\tfrac{5}{3}\left(-\tfrac{5}{3}\right) = \tfrac{25}{9}$$

$$\frac{34}{9} = \sec^2\theta \quad \text{Combine terms.}$$

$$-\sqrt{\frac{34}{9}} = \sec\theta \quad \text{Take the negative square root. (Appendix A)}$$

Choose the correct sign.

$$-\frac{\sqrt{34}}{3} = \sec\theta \quad \text{Simplify the radical; } -\sqrt{\tfrac{34}{9}} = -\tfrac{\sqrt{34}}{\sqrt{9}} = -\tfrac{\sqrt{34}}{3}.$$

We chose the negative square root since $\sec\theta$ is negative in quadrant II.

(b)

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad \text{Quotient identity}$$

$$\cos\theta\tan\theta = \sin\theta \quad \text{Multiply each side by } \cos\theta.$$

$$\left(\frac{1}{\sec\theta}\right)\tan\theta = \sin\theta \quad \text{Reciprocal identity}$$

$$\left(-\frac{3\sqrt{34}}{34}\right)\left(-\frac{5}{3}\right) = \sin\theta \quad \frac{1}{\sec\theta} = \frac{1}{-\frac{\sqrt{34}}{3}} = -\frac{3}{\sqrt{34}} = -\frac{3}{\sqrt{34}}\cdot\frac{\sqrt{34}}{\sqrt{34}} = -\frac{3\sqrt{34}}{34};\ \tan\theta = -\frac{5}{3}$$

$$\sin\theta = \frac{5\sqrt{34}}{34} \quad \text{Multiply; rewrite.}$$

(c)
$$\cot(-\theta) = \frac{1}{\tan(-\theta)} \qquad \text{Reciprocal identity}$$
$$\cot(-\theta) = \frac{1}{-\tan\theta} \qquad \text{Negative-angle identity}$$
$$\cot(-\theta) = \frac{1}{-\left(-\frac{5}{3}\right)} = \frac{3}{5} \qquad \tan\theta = -\tfrac{5}{3}\text{; simplify the complex fraction.}$$

NOW TRY EXERCISES 7, 11, AND 25.

▶ **Caution** ***To avoid a common error, when taking the square root, be sure to choose the sign based on the quadrant of θ and the function being evaluated.***

Any trigonometric function of a number or angle can be expressed in terms of any other function.

▶ EXAMPLE 2 EXPRESSING ONE FUNCTION IN TERMS OF ANOTHER

Express $\cos x$ in terms of $\tan x$.

Solution Since $\sec x$ is related to both $\cos x$ and $\tan x$ by identities, start with $1 + \tan^2 x = \sec^2 x$.

$$\frac{1}{1+\tan^2 x} = \frac{1}{\sec^2 x} \qquad \text{Take reciprocals.}$$
$$\frac{1}{1+\tan^2 x} = \cos^2 x \qquad \text{Reciprocal identity}$$

Remember both the positive and negative roots.

$$\pm\sqrt{\frac{1}{1+\tan^2 x}} = \cos x \qquad \text{Take the square root of each side.}$$
$$\cos x = \frac{\pm 1}{\sqrt{1+\tan^2 x}} \qquad \text{Quotient rule for radicals: } \sqrt[n]{\tfrac{a}{b}} = \tfrac{\sqrt[n]{a}}{\sqrt[n]{b}}\text{; rewrite.}$$
$$\cos x = \frac{\pm\sqrt{1+\tan^2 x}}{1+\tan^2 x} \qquad \text{Rationalize the denominator.}$$

Choose the + sign or the − sign, depending on the quadrant of x.

NOW TRY EXERCISE 47.

We can use a graphing calculator to decide whether two functions are identical. See Figure 2, which supports the identity $\sin^2 x + \cos^2 x = 1$. With an identity, you should see no difference in the two graphs.

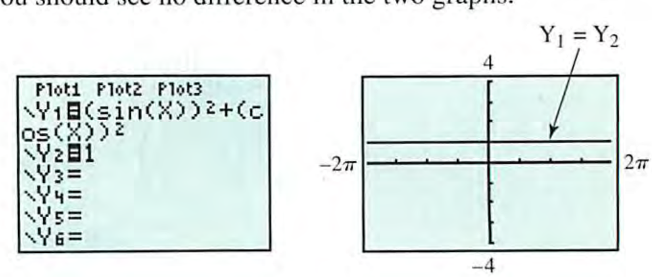


Figure 2

Each of the functions tan θ, cot θ, sec θ, and csc θ can easily be expressed in terms of sin θ and/or cos θ. We often make such substitutions in an expression to simplify it.

▶ EXAMPLE 3 REWRITING AN EXPRESSION IN TERMS OF SINE AND COSINE

Write $\tan\theta + \cot\theta$ in terms of $\sin\theta$ and $\cos\theta$, and then simplify the expression.

Solution

$$\tan\theta + \cot\theta = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} \qquad \text{Quotient identities}$$

$$= \frac{\sin\theta}{\cos\theta}\cdot\frac{\sin\theta}{\sin\theta} + \frac{\cos\theta}{\sin\theta}\cdot\frac{\cos\theta}{\cos\theta} \qquad \text{Write each fraction with the least common denominator (LCD).}$$

$$= \frac{\sin^2\theta}{\cos\theta\sin\theta} + \frac{\cos^2\theta}{\cos\theta\sin\theta} \qquad \text{Multiply.}$$

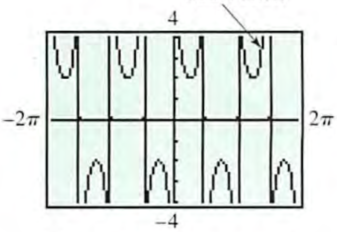
$$= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta} \qquad \text{Add fractions; } \frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}.$$

$$\tan\theta + \cot\theta = \frac{1}{\cos\theta\sin\theta} \qquad \text{Pythagorean identity}$$

NOW TRY EXERCISE 59. ◀

$y_1 = \tan x + \cot x$

$y_2 = \dfrac{1}{\cos x \sin x}$



The graph supports the result in Example 3. The graphs of y_1 and y_2 appear to be identical.

> ▶ **Caution** ***When working with trigonometric expressions and identities, be sure to write the argument of the function.*** For example, we would *not* write $\sin^2 + \cos^2 = 1$; an argument such as θ is necessary in this identity.

5.1 Exercises

Concept Check *In Exercises 1–6, use identities to fill in the blanks.*

1. If $\tan\theta = 2.6$, then $\tan(-\theta) =$ ______.

2. If $\cos\theta = -.65$, then $\cos(-\theta) =$ ______.

3. If $\tan\theta = 1.6$, then $\cot\theta =$ ______.

4. If $\cos\theta = .8$ and $\sin\theta = .6$, then $\tan(-\theta) =$ ______.

5. If $\sin\theta = \frac{2}{3}$, then $-\sin(-\theta) =$ ______.

6. If $\cos\theta = -\frac{1}{5}$, then $-\cos(-\theta) =$ ______.

Find sin θ. *See Example 1.*

7. $\cos\theta = \dfrac{3}{4}$, θ in quadrant I

8. $\cot\theta = -\dfrac{1}{3}$, θ in quadrant IV

9. $\cos(-\theta) = \dfrac{\sqrt{5}}{5}$, $\tan\theta < 0$

10. $\tan\theta = -\dfrac{\sqrt{7}}{2}$, $\sec\theta > 0$

11. $\sec\theta = \dfrac{11}{4}$, $\tan\theta < 0$

12. $\csc\theta = -\dfrac{8}{5}$

13. Why is it unnecessary to give the quadrant of θ in Exercise 12?

14. *Concept Check* What is **WRONG** with the statement of this problem?

Find $\cos(-\theta)$ *if* $\cos\theta = 3$.

RELATING CONCEPTS

For individual or collaborative investigation
(Exercises 15–20)

A function is called an ***even function*** *if* $f(-x) = f(x)$ *for all x in the domain of f. A function is called an* ***odd function*** *if* $f(-x) = -f(x)$ *for all x in the domain of f.* ***Work Exercises 15–20 in order,*** *to see the connection between the negative-angle identities and even and odd functions.*

15. Complete the statement: $\sin(-x) =$ ______.

16. Is the function defined by $f(x) = \sin x$ *even* or *odd*?

17. Complete the statement: $\cos(-x) =$ ______.

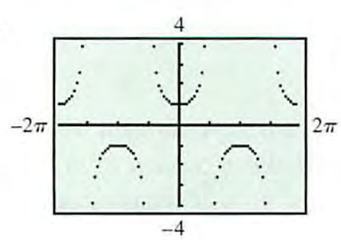
18. Is the function defined by $f(x) = \cos x$ *even* or *odd*?

19. Complete the statement: $\tan(-x) =$ ______.

20. Is the function defined by $f(x) = \tan x$ *even* or *odd*?

Concept Check *For each graph of a circular function* $y = f(x)$ *in dot mode, determine whether* $f(-x) = f(x)$ *or* $f(-x) = -f(x)$ *is true.*

21.



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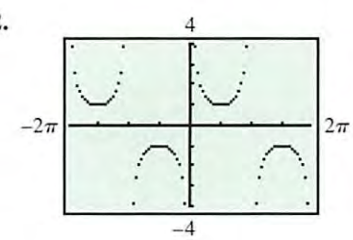

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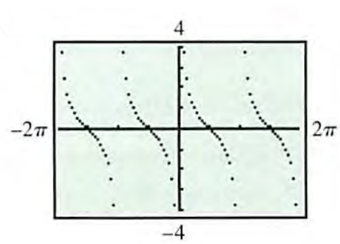

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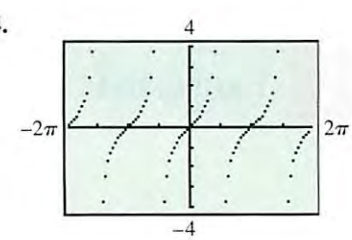


Find the remaining five trigonometric functions of θ. *See Example 1.*

25. $\sin\theta = \frac{2}{3}$, θ in quadrant II

26. $\cos\theta = \frac{1}{5}$, θ in quadrant I

27. $\tan\theta = -\frac{1}{4}$, θ in quadrant IV

28. $\csc\theta = -\frac{5}{2}$, θ in quadrant III

29. $\cot\theta = \frac{4}{3}$, $\sin\theta > 0$

30. $\sin\theta = -\frac{4}{5}$, $\cos\theta < 0$

31. $\sec\theta = \frac{4}{3}$, $\sin\theta < 0$

32. $\cos\theta = -\frac{1}{4}$, $\sin\theta > 0$

Concept Check *For each expression in Column I, choose the expression from Column II that completes an identity.*

I	II
33. $\frac{\cos x}{\sin x} =$ ______	**A.** $\sin^2 x + \cos^2 x$
34. $\tan x =$ ______	**B.** $\cot x$
35. $\cos(-x) =$ ______	**C.** $\sec^2 x$
36. $\tan^2 x + 1 =$ ______	**D.** $\frac{\sin x}{\cos x}$
37. $1 =$ ______	**E.** $\cos x$

Concept Check *For each expression in Column I, choose the expression from Column II that completes an identity. You may have to rewrite one or both expressions.*

I	II
38. $-\tan x \cos x =$ ______	**A.** $\frac{\sin^2 x}{\cos^2 x}$
39. $\sec^2 x - 1 =$ ______	**B.** $\frac{1}{\sec^2 x}$
40. $\frac{\sec x}{\csc x} =$ ______	**C.** $\sin(-x)$
41. $1 + \sin^2 x =$ ______	**D.** $\csc^2 x - \cot^2 x + \sin^2 x$
42. $\cos^2 x =$ ______	**E.** $\tan x$

43. A student writes "$1 + \cot^2 = \csc^2$." Comment on this student's work.

44. Another student makes the following claim: "Since $\sin^2 \theta + \cos^2 \theta = 1$, I should be able to also say that $\sin \theta + \cos \theta = 1$ if I take the square root of both sides." Comment on this student's statement.

45. **Concept Check** Suppose that $\cos \theta = \frac{x}{x+1}$. Find an expression in x for $\sin \theta$.

46. **Concept Check** Suppose that $\sec \theta = \frac{p+4}{p}$. Find an expression in p for $\tan \theta$.

Write the first trigonometric function in terms of the second trigonometric function. See Example 2.

47. $\sin x$; $\cos x$
48. $\cot x$; $\sin x$
49. $\tan x$; $\sec x$
50. $\cot x$; $\csc x$
51. $\csc x$; $\cos x$
52. $\sec x$; $\sin x$

Write each expression in terms of sine and cosine, and simplify so that no quotients appear in the final expression. See Example 3.

53. $\cot \theta \sin \theta$
54. $\sec \theta \cot \theta \sin \theta$
55. $\cos \theta \csc \theta$
56. $\cot^2 \theta (1 + \tan^2 \theta)$
57. $\sin^2 \theta (\csc^2 \theta - 1)$
58. $(\sec \theta - 1)(\sec \theta + 1)$
59. $(1 - \cos \theta)(1 + \sec \theta)$
60. $\frac{\cos \theta + \sin \theta}{\sin \theta}$
61. $\frac{\cos^2 \theta - \sin^2 \theta}{\sin \theta \cos \theta}$
62. $\frac{1 - \sin^2 \theta}{1 + \cot^2 \theta}$
63. $\sec \theta - \cos \theta$
64. $(\sec \theta + \csc \theta)(\cos \theta - \sin \theta)$
65. $\sin \theta (\csc \theta - \sin \theta)$
66. $\frac{1 + \tan^2 \theta}{1 + \cot^2 \theta}$

67. $\sin^2\theta + \tan^2\theta + \cos^2\theta$

68. $\dfrac{\tan(-\theta)}{\sec\theta}$

69. Let $\cos x = \frac{1}{5}$. Find all possible values of $\frac{\sec x - \tan x}{\sin x}$.

70. Let $\csc x = -3$. Find all possible values of $\frac{\sin x + \cos x}{\sec x}$.

RELATING CONCEPTS

***For individual or collaborative investigation* (Exercises 71–76)**

In ***Chapter 4*** *we graphed functions defined by*

$$y = c + a \cdot f[b(x - d)]$$

with the assumption that $b > 0$. *To see what happens when* $b < 0$, ***work Exercises 71–76 in order.***

71. Use a negative-angle identity to write $y = \sin(-2x)$ as a function of $2x$.

72. How does your answer to Exercise 71 relate to $y = \sin(2x)$?

73. Use a negative-angle identity to write $y = \cos(-4x)$ as a function of $4x$.

74. How does your answer to Exercise 73 relate to $y = \cos(4x)$?

75. Use your results from Exercises 71–74 to rewrite the following with a positive value of b.

(a) $y = \sin(-4x)$ **(b)** $y = \cos(-2x)$ **(c)** $y = -5\sin(-3x)$

76. Write a short response to this statement, often used by one of the authors of this text in trigonometry classes: *Students who tend to ignore negative signs should enjoy graphing functions involving the cosine and the secant.*

Use a graphing calculator to make a conjecture as to whether each equation is an identity. (Hint: In Exercises 81 and 82, graph as a function of x for a few different values of y (in radians).)

77. $\cos 2x = 1 - 2\sin^2 x$

78. $2\sin s = \sin 2s$

79. $\sin x = \sqrt{1 - \cos^2 x}$

80. $\cos 2x = \cos^2 x - \sin^2 x$

81. $\cos(x - y) = \cos x - \cos y$

82. $\sin(x + y) = \sin x + \sin y$

5.2 Verifying Trigonometric Identities

Verifying Identities by Working with One Side ▪ Verifying Identities by Working with Both Sides

Recall that an identity is an equation that is satisfied for all meaningful replacements of the variable. One of the skills required for more advanced work in mathematics, especially in calculus, is the ability to use identities to write expressions in alternative forms. We develop this skill by using the fundamental identities to verify that a trigonometric equation is an identity (for those values of the variable for which it is defined). Here are some hints to help you get started.

▼ LOOKING AHEAD TO CALCULUS

Trigonometric identities are used in calculus to simplify trigonometric expressions, determine derivatives of trigonometric functions, and change the form of some integrals.

HINTS FOR VERIFYING IDENTITIES

1. ***Learn the fundamental identities given in Section 5.1.*** Whenever you see either side of a fundamental identity, the other side should come to mind. ***Also, be aware of equivalent forms of the fundamental identities.*** For example, $\sin^2\theta = 1 - \cos^2\theta$ is an alternative form of the identity $\sin^2\theta + \cos^2\theta = 1$.
2. ***Try to rewrite the more complicated side*** of the equation so that it is identical to the simpler side.
3. ***It is sometimes helpful to express all trigonometric functions in the equation in terms of sine and cosine*** and then simplify the result.
4. ***Usually, any factoring or indicated algebraic operations should be performed.*** For example, the expression

   $$\sin^2 x + 2\sin x + 1 \quad \text{can be factored as} \quad (\sin x + 1)^2.$$

   The sum or difference of two trigonometric expressions, such as $\frac{1}{\sin\theta} + \frac{1}{\cos\theta}$, can be added or subtracted in the same way as any other rational expression.

   $$\frac{1}{\sin\theta} + \frac{1}{\cos\theta} = \frac{\cos\theta}{\sin\theta\cos\theta} + \frac{\sin\theta}{\sin\theta\cos\theta} \qquad \text{Write with the LCD.}$$

   $$= \frac{\cos\theta + \sin\theta}{\sin\theta\cos\theta} \qquad \frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

5. ***As you select substitutions, keep in mind the side you are not changing, because it represents your goal.*** For example, to verify the identity

   $$\tan^2 x + 1 = \frac{1}{\cos^2 x},$$

   try to think of an identity that relates $\tan x$ to $\cos x$. In this case, since $\sec x = \frac{1}{\cos x}$ and $\sec^2 x = \tan^2 x + 1$, the secant function is the best link between the two sides.
6. If an expression contains $1 + \sin x$, ***multiplying both numerator and denominator*** by $1 - \sin x$ would give $1 - \sin^2 x$, which could be replaced with $\cos^2 x$. Similar results for $1 - \sin x$, $1 + \cos x$, and $1 - \cos x$ may be useful.

▶ **Caution** ***Verifying identities is not the same as solving equations.*** Techniques used in solving equations, such as adding the same terms to both sides, or multiplying both sides by the same term, should not be used when working with identities since you are starting with a statement (to be verified) that may not be true.

Verifying Identities by Working with One Side To avoid the temptation to use algebraic properties of equations to verify identities, ***one strategy is to work with only one side and rewrite it to match the other side,*** as shown in Examples 1–4.

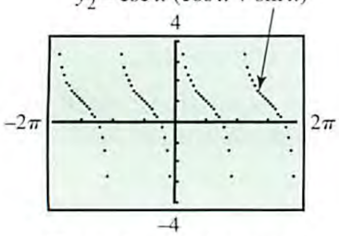
▶ EXAMPLE 1 VERIFYING AN IDENTITY (WORKING WITH ONE SIDE)

Verify that the following equation is an identity.

$$\cot \theta + 1 = \csc \theta(\cos \theta + \sin \theta)$$

Solution We use the fundamental identities from **Section 5.1** to rewrite one side of the equation so that it is identical to the other side. Since the right side is more complicated, we work with it, using the third hint to change all functions to sine or cosine.

For $\theta = x$,
$y_1 = \cot x + 1$
$y_2 = \csc x\,(\cos x + \sin x)$



The graphs coincide, supporting the conclusion in Example 1.

Steps	Reasons
$\underbrace{\csc \theta(\cos \theta + \sin \theta)}_{\text{Right side of given equation}} = \frac{1}{\sin \theta}(\cos \theta + \sin \theta)$	$\csc \theta = \frac{1}{\sin \theta}$
$= \frac{\cos \theta}{\sin \theta} + \frac{\sin \theta}{\sin \theta}$	Distributive property; $a(b + c) = ab + ac$
$= \underbrace{\cot \theta + 1}_{\text{Left side of given equation}}$	$\frac{\cos \theta}{\sin \theta} = \cot \theta$; $\frac{\sin \theta}{\sin \theta} = 1$

The given equation is an identity. The right side is identical to the left side.

NOW TRY EXERCISE 35. ◀

▶ EXAMPLE 2 VERIFYING AN IDENTITY (WORKING WITH ONE SIDE)

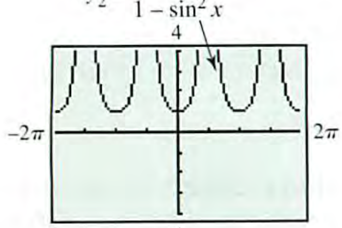
Verify that the following equation is an identity.

$$\tan^2 x(1 + \cot^2 x) = \frac{1}{1 - \sin^2 x}$$

Solution We work with the more complicated left side, as suggested in the second hint. Again, we use the fundamental identities from **Section 5.1.**

$$\begin{aligned}
\tan^2 x(1 + \cot^2 x) &= \tan^2 x + \tan^2 x \cot^2 x && \text{Distributive property} \\
&= \tan^2 x + \tan^2 x \cdot \frac{1}{\tan^2 x} && \cot^2 x = \frac{1}{\tan^2 x} \\
&= \tan^2 x + 1 && \tan^2 x \cdot \frac{1}{\tan^2 x} = 1 \\
&= \sec^2 x && \tan^2 x + 1 = \sec^2 x \\
&= \frac{1}{\cos^2 x} && \sec^2 x = \frac{1}{\cos^2 x} \\
&= \frac{1}{1 - \sin^2 x} && \cos^2 x = 1 - \sin^2 x
\end{aligned}$$

$y_1 = \tan^2 x(1 + \cot^2 x)$
$y_2 = \frac{1}{1 - \sin^2 x}$



The screen supports the conclusion in Example 2.

Since the left side is identical to the right side, the given equation is an identity.

NOW TRY EXERCISE 39. ◀

▶ EXAMPLE 3 VERIFYING AN IDENTITY (WORKING WITH ONE SIDE)

Verify that the following equation is an identity.

$$\frac{\tan t - \cot t}{\sin t \cos t} = \sec^2 t - \csc^2 t$$

Solution We transform the more complicated left side to match the right side.

$$\begin{aligned}
\frac{\tan t - \cot t}{\sin t \cos t} &= \frac{\tan t}{\sin t \cos t} - \frac{\cot t}{\sin t \cos t} && \frac{a-b}{c} = \frac{a}{c} - \frac{b}{c} \\
&= \tan t \cdot \frac{1}{\sin t \cos t} - \cot t \cdot \frac{1}{\sin t \cos t} && \frac{a}{b} = a \cdot \frac{1}{b} \\
&= \frac{\sin t}{\cos t} \cdot \frac{1}{\sin t \cos t} - \frac{\cos t}{\sin t} \cdot \frac{1}{\sin t \cos t} && \tan t = \frac{\sin t}{\cos t};\ \cot t = \frac{\cos t}{\sin t} \\
&= \frac{1}{\cos^2 t} - \frac{1}{\sin^2 t} && \text{Multiply.} \\
&= \sec^2 t - \csc^2 t && \frac{1}{\cos^2 t} = \sec^2 t;\ \frac{1}{\sin^2 t} = \csc^2 t
\end{aligned}$$

The third hint about writing all trigonometric functions in terms of sine and cosine was used in the third line of the solution.

NOW TRY EXERCISE 43. ◀

▶ EXAMPLE 4 VERIFYING AN IDENTITY (WORKING WITH ONE SIDE)

Verify that the following equation is an identity.

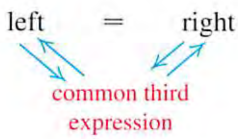
$$\frac{\cos x}{1 - \sin x} = \frac{1 + \sin x}{\cos x}$$

Solution We work on the right side, using the last hint in the list given earlier to multiply numerator and denominator on the right by $1 - \sin x$.

$$\begin{aligned}
\frac{1 + \sin x}{\cos x} &= \frac{(1 + \sin x)(1 - \sin x)}{\cos x(1 - \sin x)} && \text{Multiply by 1 in the form } \frac{1 - \sin x}{1 - \sin x}. \\
&= \frac{1 - \sin^2 x}{\cos x(1 - \sin x)} && (x + y)(x - y) = x^2 - y^2 \\
&= \frac{\cos^2 x}{\cos x(1 - \sin x)} && 1 - \sin^2 x = \cos^2 x \\
&= \frac{\cos x}{1 - \sin x} && \text{Lowest terms}
\end{aligned}$$

NOW TRY EXERCISE 49. ◀

Verifying Identities by Working with Both Sides If both sides of an identity appear to be equally complex, the identity can be verified by working independently on the left side and on the right side, until each side is changed into some common third result. ***Each step, on each side, must be reversible.*** With all



steps reversible, the procedure is as shown in the margin. The left side leads to a common third expression, which leads back to the right side. This procedure is just a shortcut for the procedure used in Examples 1–4: one side is changed into the other side, but by going through an intermediate step.

▶ EXAMPLE 5 VERIFYING AN IDENTITY (WORKING WITH BOTH SIDES)

Verify that the following equation is an identity.

$$\frac{\sec\alpha + \tan\alpha}{\sec\alpha - \tan\alpha} = \frac{1 + 2\sin\alpha + \sin^2\alpha}{\cos^2\alpha}$$

Solution Both sides appear equally complex, so we verify the identity by changing each side into a common third expression. We work first on the left, multiplying numerator and denominator by $\cos\alpha$.

$$\frac{\sec\alpha + \tan\alpha}{\sec\alpha - \tan\alpha} = \frac{(\sec\alpha + \tan\alpha)\cos\alpha}{(\sec\alpha - \tan\alpha)\cos\alpha} \qquad \text{Multiply by 1 in the form } \frac{\cos\alpha}{\cos\alpha}.$$

$$= \frac{\sec\alpha\cos\alpha + \tan\alpha\cos\alpha}{\sec\alpha\cos\alpha - \tan\alpha\cos\alpha} \qquad \text{Distributive property}$$

$$= \frac{1 + \tan\alpha\cos\alpha}{1 - \tan\alpha\cos\alpha} \qquad \sec\alpha\cos\alpha = 1$$

$$= \frac{1 + \frac{\sin\alpha}{\cos\alpha}\cdot\cos\alpha}{1 - \frac{\sin\alpha}{\cos\alpha}\cdot\cos\alpha} \qquad \tan\alpha = \frac{\sin\alpha}{\cos\alpha}$$

$$= \frac{1 + \sin\alpha}{1 - \sin\alpha} \qquad \text{Simplify.}$$

On the right side of the original equation, begin by factoring.

$$\frac{1 + 2\sin\alpha + \sin^2\alpha}{\cos^2\alpha} = \frac{(1 + \sin\alpha)^2}{\cos^2\alpha} \qquad x^2 + 2xy + y^2 = (x + y)^2$$

$$= \frac{(1 + \sin\alpha)^2}{1 - \sin^2\alpha} \qquad \cos^2\alpha = 1 - \sin^2\alpha$$

$$= \frac{(1 + \sin\alpha)^2}{(1 + \sin\alpha)(1 - \sin\alpha)} \qquad \text{Factor the denominator; } x^2 - y^2 = (x + y)(x - y).$$

$$= \frac{1 + \sin\alpha}{1 - \sin\alpha} \qquad \text{Lowest terms}$$

We have shown that

$$\underbrace{\frac{\sec\alpha + \tan\alpha}{\sec\alpha - \tan\alpha}}_{\text{Left side of given equation}} = \underbrace{\frac{1 + \sin\alpha}{1 - \sin\alpha}}_{\text{Common third expression}} = \underbrace{\frac{1 + 2\sin\alpha + \sin^2\alpha}{\cos^2\alpha}}_{\text{Right side of given equation}},$$

verifying that the given equation is an identity.

NOW TRY EXERCISE 65. ◀

▶ **Caution** Use the method of Example 5 *only* if the steps are reversible.

There are usually several ways to verify a given identity. For instance, another way to begin verifying the identity in Example 5 is to work on the left as follows.

$$\frac{\sec\alpha + \tan\alpha}{\sec\alpha - \tan\alpha} = \frac{\frac{1}{\cos\alpha} + \frac{\sin\alpha}{\cos\alpha}}{\frac{1}{\cos\alpha} - \frac{\sin\alpha}{\cos\alpha}} \quad \text{Fundamental identities (Section 5.1)}$$

$$= \frac{\frac{1 + \sin\alpha}{\cos\alpha}}{\frac{1 - \sin\alpha}{\cos\alpha}} \quad \text{Add and subtract fractions.}$$

Multiply by $\frac{\cos\alpha}{\cos\alpha}$ here.

$$= \frac{1 + \sin\alpha}{1 - \sin\alpha} \quad \text{Simplify the complex fraction.}$$

Compare this with the result shown in Example 5 for the right side to see that the two sides indeed agree.

▶ EXAMPLE 6 APPLYING A PYTHAGOREAN IDENTITY TO RADIOS

Tuners in radios select a radio station by adjusting the frequency. A tuner may contain an inductor L and a capacitor C, as illustrated in Figure 3. The energy stored in the inductor at time t is given by

$$L(t) = k\sin^2(2\pi Ft)$$

and the energy stored in the capacitor is given by

$$C(t) = k\cos^2(2\pi Ft),$$

where F is the frequency of the radio station and k is a constant. The total energy E in the circuit is given by

$$E(t) = L(t) + C(t).$$

Show that E is a constant function. (*Source:* Weidner, R. and R. Sells, *Elementary Classical Physics,* Vol. 2, Allyn & Bacon, 1973.)

An Inductor and a Capacitor

Figure 3

Solution

$$\begin{aligned} E(t) &= L(t) + C(t) && \text{Given equation} \\ &= k\sin^2(2\pi Ft) + k\cos^2(2\pi Ft) && \text{Substitute.} \\ &= k[\sin^2(2\pi Ft) + \cos^2(2\pi Ft)] && \text{Factor out } k. \\ &= k(1) && \sin^2\theta + \cos^2\theta = 1 \text{ (Here } \theta = 2\pi Ft.) \\ &= k \end{aligned}$$

Since k is a constant, $E(t)$ is a constant function.

NOW TRY EXERCISE 95. ◀

5.2 Exercises

Perform each indicated operation and simplify the result.

1. $\cot \theta + \frac{1}{\cot \theta}$

2. $\frac{\sec x}{\csc x} + \frac{\csc x}{\sec x}$

3. $\tan s(\cot s + \csc s)$

4. $\cos \beta(\sec \beta + \csc \beta)$

5. $\frac{1}{\csc^2 \theta} + \frac{1}{\sec^2 \theta}$

6. $\frac{1}{\sin \alpha - 1} - \frac{1}{\sin \alpha + 1}$

7. $\frac{\cos x}{\sec x} + \frac{\sin x}{\csc x}$

8. $\frac{\cos \theta}{\sin \theta} + \frac{\sin \theta}{1 + \cos \theta}$

9. $(1 + \sin t)^2 + \cos^2 t$

10. $(1 + \tan s)^2 - 2 \tan s$

11. $\frac{1}{1 + \cos x} - \frac{1}{1 - \cos x}$

12. $(\sin \alpha - \cos \alpha)^2$

Factor each trigonometric expression.

13. $\sin^2 \theta - 1$

14. $\sec^2 \theta - 1$

15. $(\sin x + 1)^2 - (\sin x - 1)^2$

16. $(\tan x + \cot x)^2 - (\tan x - \cot x)^2$

17. $2 \sin^2 x + 3 \sin x + 1$

18. $4 \tan^2 \beta + \tan \beta - 3$

19. $\cos^4 x + 2 \cos^2 x + 1$

20. $\cot^4 x + 3 \cot^2 x + 2$

21. $\sin^3 x - \cos^3 x$

22. $\sin^3 \alpha + \cos^3 \alpha$

Each expression simplifies to a constant, a single function, or a power of a function. Use fundamental identities to simplify each expression.

23. $\tan \theta \cos \theta$

24. $\cot \alpha \sin \alpha$

25. $\sec r \cos r$

26. $\cot t \tan t$

27. $\frac{\sin \beta \tan \beta}{\cos \beta}$

28. $\frac{\csc \theta \sec \theta}{\cot \theta}$

29. $\sec^2 x - 1$

30. $\csc^2 t - 1$

31. $\frac{\sin^2 x}{\cos^2 x} + \sin x \csc x$

32. $\frac{1}{\tan^2 \alpha} + \cot \alpha \tan \alpha$

33. $1 - \frac{1}{\csc^2 x}$

34. $1 - \frac{1}{\sec^2 x}$

In Exercises 35–78, verify that each trigonometric equation is an identity. See Examples 1–5.

35. $\frac{\cot \theta}{\csc \theta} = \cos \theta$

36. $\frac{\tan \alpha}{\sec \alpha} = \sin \alpha$

37. $\frac{1 - \sin^2 \beta}{\cos \beta} = \cos \beta$

38. $\frac{\tan^2 \alpha + 1}{\sec \alpha} = \sec \alpha$

39. $\cos^2 \theta(\tan^2 \theta + 1) = 1$

40. $\sin^2 \beta(1 + \cot^2 \beta) = 1$

41. $\cot s + \tan s = \sec s \csc s$

42. $\sin^2 \alpha + \tan^2 \alpha + \cos^2 \alpha = \sec^2 \alpha$

43. $\frac{\cos \alpha}{\sec \alpha} + \frac{\sin \alpha}{\csc \alpha} = \sec^2 \alpha - \tan^2 \alpha$

44. $\frac{\sin^2 \theta}{\cos \theta} = \sec \theta - \cos \theta$

45. $\sin^4 \theta - \cos^4 \theta = 2 \sin^2 \theta - 1$

46. $\frac{\cos \theta}{\sin \theta \cot \theta} = 1$

47. $\frac{1 - \cos x}{1 + \cos x} = (\cot x - \csc x)^2$

48. $\sin^2 \theta(1 + \cot^2 \theta) - 1 = 0$

49. $\dfrac{\cos\theta + 1}{\tan^2\theta} = \dfrac{\cos\theta}{\sec\theta - 1}$

50. $\dfrac{(\sec\theta - \tan\theta)^2 + 1}{\sec\theta\csc\theta - \tan\theta\csc\theta} = 2\tan\theta$

51. $\dfrac{1}{1 - \sin\theta} + \dfrac{1}{1 + \sin\theta} = 2\sec^2\theta$

52. $\dfrac{1}{\sec\alpha - \tan\alpha} = \sec\alpha + \tan\alpha$

53. $\dfrac{\cot\alpha + 1}{\cot\alpha - 1} = \dfrac{1 + \tan\alpha}{1 - \tan\alpha}$

54. $\dfrac{\csc\theta + \cot\theta}{\tan\theta + \sin\theta} = \cot\theta\csc\theta$

55. $\sec^4 x - \sec^2 x = \tan^4 x + \tan^2 x$

56. $(\sec\alpha - \tan\alpha)^2 = \dfrac{1 - \sin\alpha}{1 + \sin\alpha}$

57. $\dfrac{\sec^4 s - \tan^4 s}{\sec^2 s + \tan^2 s} = \sec^2 s - \tan^2 s$

58. $\dfrac{\cot^2 t - 1}{1 + \cot^2 t} = 1 - 2\sin^2 t$

59. $\dfrac{\tan^2 t - 1}{\sec^2 t} = \dfrac{\tan t - \cot t}{\tan t + \cot t}$

60. $\dfrac{\sin^4\alpha - \cos^4\alpha}{\sin^2\alpha - \cos^2\alpha} = 1$

61. $(1 - \cos^2\alpha)(1 + \cos^2\alpha) = 2\sin^2\alpha - \sin^4\alpha$

62. $\tan^2\alpha\sin^2\alpha = \tan^2\alpha + \cos^2\alpha - 1$

63. $\sin^2\alpha\sec^2\alpha + \sin^2\alpha\csc^2\alpha = \sec^2\alpha$

64. $\dfrac{-1}{\tan\alpha - \sec\alpha} + \dfrac{-1}{\tan\alpha + \sec\alpha} = 2\tan\alpha$

65. $\dfrac{\tan s}{1 + \cos s} + \dfrac{\sin s}{1 - \cos s} = \cot s + \sec s\csc s$

66. $\dfrac{1 - \cos x}{1 + \cos x} = \csc^2 x - 2\csc x\cot x + \cot^2 x$

67. $\dfrac{1 - \sin\theta}{1 + \sin\theta} = \sec^2\theta - 2\sec\theta\tan\theta + \tan^2\theta$

68. $\sin\theta + \cos\theta = \dfrac{\sin\theta}{1 - \dfrac{\cos\theta}{\sin\theta}} + \dfrac{\cos\theta}{1 - \dfrac{\sin\theta}{\cos\theta}}$

69. $\dfrac{\sin\theta}{1 - \cos\theta} - \dfrac{\sin\theta\cos\theta}{1 + \cos\theta} = \csc\theta(1 + \cos^2\theta)$

70. $(1 + \sin x + \cos x)^2 = 2(1 + \sin x)(1 + \cos x)$

71. $\dfrac{1 + \cos x}{1 - \cos x} - \dfrac{1 - \cos x}{1 + \cos x} = 4\cot x\csc x$

72. $(\sec\alpha + \csc\alpha)(\cos\alpha - \sin\alpha) = \cot\alpha - \tan\alpha$

73. $\dfrac{1 + \sin\theta}{1 - \sin\theta} - \dfrac{1 - \sin\theta}{1 + \sin\theta} = 4\tan\theta\sec\theta$

74. $\dfrac{1 - \cos\theta}{1 + \cos\theta} = 2\csc^2\theta - 2\csc\theta\cot\theta - 1$

75. $(2\sin x + \cos x)^2 + (2\cos x - \sin x)^2 = 5$

76. $\sin^2 x(1 + \cot x) + \cos^2 x(1 - \tan x) + \cot^2 x = \csc^2 x$

77. $\sec x - \cos x + \csc x - \sin x - \sin x\tan x = \cos x\cot x$

78. $\sin^3\theta + \cos^3\theta = (\cos\theta + \sin\theta)(1 - \cos\theta\sin\theta)$

Graph each expression and make a conjecture, predicting what might be an identity. Then verify your conjecture algebraically.

79. $(\sec \theta + \tan \theta)(1 - \sin \theta)$

80. $(\csc \theta + \cot \theta)(\sec \theta - 1)$

81. $\dfrac{\cos \theta + 1}{\sin \theta + \tan \theta}$

82. $\tan \theta \sin \theta + \cos \theta$

Graph the expressions on each side of the equals symbol to determine whether the equation might be an identity. (Note: Use a domain whose length is at least 2π.) If the equation looks like an identity, verify it algebraically. See Example 1.

83. $\dfrac{2 + 5 \cos x}{\sin x} = 2 \csc x + 5 \cot x$

84. $1 + \cot^2 x = \dfrac{\sec^2 x}{\sec^2 x - 1}$

85. $\dfrac{\tan x - \cot x}{\tan x + \cot x} = 2 \sin^2 x$

86. $\dfrac{1}{1 + \sin x} + \dfrac{1}{1 - \sin x} = \sec^2 x$

By substituting a number for s or t, show that the equation is not an identity.

87. $\sin(\csc s) = 1$

88. $\sqrt{\cos^2 s} = \cos s$

89. $\csc t = \sqrt{1 + \cot^2 t}$

90. $\cos t = \sqrt{1 - \sin^2 t}$

91. ***Concept Check*** When is $\sin x = \sqrt{1 - \cos^2 x}$ a true statement?

92. ***Concept Check*** When is $\cos x = \sqrt{1 - \sin^2 x}$ a true statement?

(Modeling) *Work each problem.*

93. ***Intensity of a Lamp*** According to Lambert's law, the intensity of light from a single source on a flat surface at point P is given by

$$I = k \cos^2 \theta,$$

where k is a constant. (*Source:* Winter, C., *Solar Power Plants,* Springer-Verlag, 1991.)

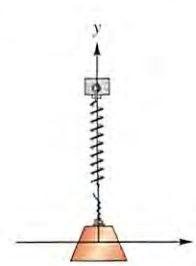
(a) Write I in terms of the sine function.
(b) Why does the maximum value of I occur when $\theta = 0$?

94. ***Oscillating Spring*** The distance or displacement y of a weight attached to an oscillating spring from its natural position is modeled by

$$y = 4 \cos(2\pi t),$$

where t is time in seconds. Potential energy is the energy of position and is given by

$$P = ky^2,$$



where k is a constant. The weight has the greatest potential energy when the spring is stretched the most. (*Source:* Weidner, R. and R. Sells, *Elementary Classical Physics,* Vol. 1, Allyn & Bacon, 1973.)

(a) Write an expression for P that involves the cosine function.
(b) Use a fundamental identity to write P in terms of $\sin(2\pi t)$.

95. ***Radio Tuners*** Refer to Example 6. Let the energy stored in the inductor be given by

$$L(t) = 3\cos^2(6{,}000{,}000t)$$

and the energy in the capacitor be given by

$$C(t) = 3\sin^2(6{,}000{,}000t),$$

where t is time in seconds. The total energy E in the circuit is given by $E(t) = L(t) + C(t)$.

(a) Graph L, C, and E in the window $[0, 10^{-6}]$ by $[-1, 4]$, with Xscl $= 10^{-7}$ and Yscl $= 1$. Interpret the graph.

(b) Make a table of values for L, C, and E starting at $t = 0$, incrementing by 10^{-7}. Interpret your results.

(c) Use a fundamental identity to derive a simplified expression for $E(t)$.

5.3 Sum and Difference Identities for Cosine

Difference Identity for Cosine ▪ Sum Identity for Cosine ▪ Cofunction Identities ▪ Applying the Sum and Difference Identities

Difference Identity for Cosine Several examples presented earlier should have convinced you by now that $\cos(A - B)$ *does not equal* $\cos A - \cos B$. For example, if $A = \frac{\pi}{2}$ and $B = 0$, then

$$\cos(A - B) = \cos\left(\frac{\pi}{2} - 0\right) = \cos\frac{\pi}{2} = 0,$$

while $$\cos A - \cos B = \cos\frac{\pi}{2} - \cos 0 = 0 - 1 = -1.$$

We can now derive a formula for $\cos(A - B)$. We start by locating angles A and B in standard position on a unit circle, with $B < A$. Let S and Q be the points where the terminal sides of angles A and B, respectively, intersect the circle. Let P be the point $(1, 0)$, and locate point R on the unit circle so that angle POR equals the difference $A - B$. See Figure 4.

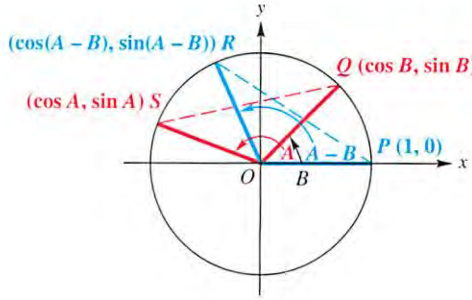


Figure 4

Point Q is on the unit circle, thus the x-coordinate of Q is the cosine of angle B, while the y-coordinate of Q is the sine of angle B.

Q has coordinates $(\cos B, \sin B)$.

In the same way,

S has coordinates $(\cos A, \sin A)$,

and R has coordinates $(\cos(A - B), \sin(A - B))$.

Angle SOQ also equals $A - B$. Since the central angles SOQ and POR are equal, chords PR and SQ are equal. By the distance formula, since $PR = SQ$,

$$\sqrt{[\cos(A - B) - 1]^2 + [\sin(A - B) - 0]^2}$$
$$= \sqrt{(\cos A - \cos B)^2 + (\sin A - \sin B)^2}.$$ (Appendix B)

Squaring both sides and clearing parentheses gives

$$\cos^2(A - B) - 2\cos(A - B) + 1 + \sin^2(A - B)$$
$$= \cos^2 A - 2\cos A \cos B + \cos^2 B + \sin^2 A - 2\sin A \sin B + \sin^2 B.$$

Since $\sin^2 x + \cos^2 x = 1$ for any value of x, we can rewrite the equation as

$$2 - 2\cos(A - B) = 2 - 2\cos A \cos B - 2\sin A \sin B$$
$$\mathbf{\cos(A - B) = \cos A \cos B + \sin A \sin B.}$$ Subtract 2; divide by -2.

This is the identity for $\cos(A - B)$. Although Figure 4 shows angles A and B in the second and first quadrants, respectively, this result is the same for any values of these angles.

Sum Identity for Cosine To find a similar expression for $\cos(A + B)$, rewrite $A + B$ as $A - (-B)$ and use the identity for $\cos(A - B)$.

$$\cos(A + B) = \cos[A - (-B)]$$
$$= \cos A \cos(-B) + \sin A \sin(-B)$$ Cosine difference identity
$$= \cos A \cos B + \sin A(-\sin B)$$ Negative-angle identities (Section 5.1)
$$\mathbf{\cos(A + B) = \cos A \cos B - \sin A \sin B}$$

COSINE OF A SUM OR DIFFERENCE

$$\mathbf{\cos(A + B) = \cos A \cos B - \sin A \sin B}$$
$$\mathbf{\cos(A - B) = \cos A \cos B + \sin A \sin B}$$

These identities are important in calculus and useful in certain applications. For example, the method shown in Example 1 can be applied to get an exact value for $\cos 15°$, as well as to practice using the sum and difference identities.

▶ EXAMPLE 1 FINDING EXACT COSINE FUNCTION VALUES

Find the *exact* value of each expression.

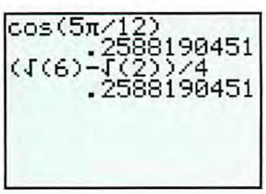
(a) $\cos 15^\circ$ **(b)** $\cos \frac{5\pi}{12}$ **(c)** $\cos 87^\circ \cos 93^\circ - \sin 87^\circ \sin 93^\circ$

Solution

(a) To find $\cos 15^\circ$, we write 15° as the sum or difference of two angles with known function values, such as 45° and 30°, since $15^\circ = 45^\circ - 30^\circ$. (We could also use $60^\circ - 45^\circ$.) Then we use the cosine difference identity.

$$\cos 15^\circ = \cos(45^\circ - 30^\circ) \qquad 15^\circ = 45^\circ - 30^\circ$$
$$= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ \qquad \text{Cosine difference identity}$$
$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} \qquad \text{Substitute known values. (Section 2.1)}$$
$$= \frac{\sqrt{6} + \sqrt{2}}{4} \qquad \text{Multiply; add fractions.}$$

(b)
$$\cos \frac{5\pi}{12} = \cos\left(\frac{\pi}{6} + \frac{\pi}{4}\right) \qquad \frac{\pi}{6} = \frac{2\pi}{12};\ \frac{\pi}{4} = \frac{3\pi}{12}$$
$$= \cos \frac{\pi}{6} \cos \frac{\pi}{4} - \sin \frac{\pi}{6} \sin \frac{\pi}{4} \qquad \text{Cosine sum identity}$$
$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2} \qquad \text{Substitute known values. (Section 3.1)}$$
$$= \frac{\sqrt{6} - \sqrt{2}}{4} \qquad \text{Multiply; subtract fractions.}$$



The screen supports the solution in Example 1(b) by showing that $\cos \frac{5\pi}{12} = \frac{\sqrt{6} - \sqrt{2}}{4}$.

(c)
$$\cos 87^\circ \cos 93^\circ - \sin 87^\circ \sin 93^\circ = \cos(87^\circ + 93^\circ) \qquad \text{Cosine sum identity}$$
$$= \cos 180^\circ \qquad \text{Add.}$$
$$= -1 \qquad \text{(Section 1.3)}$$

NOW TRY EXERCISES 7, 9, AND 11. ◀

Cofunction Identities We can use the identity for the cosine of the difference of two angles and the fundamental identities to derive the *cofunction identities,* presented in **Section 2.1** for values of θ in the interval $[0^\circ, 90^\circ]$.

COFUNCTION IDENTITIES

$\cos(90^\circ - \theta) = \sin \theta$	$\cot(90^\circ - \theta) = \tan \theta$
$\sin(90^\circ - \theta) = \cos \theta$	$\sec(90^\circ - \theta) = \csc \theta$
$\tan(90^\circ - \theta) = \cot \theta$	$\csc(90^\circ - \theta) = \sec \theta$

Similar identities can be obtained for a real number domain by replacing 90° with $\frac{\pi}{2}$.

Substituting 90° for A and θ for B in the identity for $\cos(A - B)$ gives

$$\begin{aligned}\cos(90° - \theta) &= \cos 90° \cos\theta + \sin 90° \sin\theta \\ &= 0 \cdot \cos\theta + 1 \cdot \sin\theta \\ &= \sin\theta.\end{aligned}$$

This result is true for *any* value of θ since the identity for $\cos(A - B)$ is true for any values of A and B.

▶ EXAMPLE 2 USING COFUNCTION IDENTITIES TO FIND θ

Find an angle θ that satisfies each of the following.

(a) $\cot\theta = \tan 25°$ **(b)** $\sin\theta = \cos(-30°)$ **(c)** $\csc\frac{3\pi}{4} = \sec\theta$

Solution

(a) Since tangent and cotangent are cofunctions, $\tan(90° - \theta) = \cot\theta$.

$$\begin{aligned}\cot\theta &= \tan 25° && \\ \tan(90° - \theta) &= \tan 25° && \text{Cofunction identity} \\ 90° - \theta &= 25° && \text{Set angle measures equal.} \\ \theta &= 65° && \text{Solve for } \theta.\end{aligned}$$

(b)

$$\begin{aligned}\sin\theta &= \cos(-30°) && \\ \cos(90° - \theta) &= \cos(-30°) && \text{Cofunction identity} \\ 90° - \theta &= -30° && \\ \theta &= 120° &&\end{aligned}$$

(c)

$$\begin{aligned}\csc\frac{3\pi}{4} &= \sec\theta && \\ \sec\left(\frac{\pi}{2} - \frac{3\pi}{4}\right) &= \sec\theta && \text{Cofunction identity} \\ \sec\left(-\frac{\pi}{4}\right) &= \sec\theta && \text{Combine terms.} \\ -\frac{\pi}{4} &= \theta &&\end{aligned}$$

NOW TRY EXERCISES 33 AND 37. ◀

▶ **Note** Because trigonometric (circular) functions are periodic, the solutions in Example 2 are not unique. We give only one of infinitely many possibilities.

Applying the Sum and Difference Identities If one of the angles A or B in the identities for $\cos(A + B)$ and $\cos(A - B)$ is a quadrantal angle, then the identity allows us to write the expression in terms of a single function of A or B.

▶ EXAMPLE 3 **REDUCING cos(A − B) TO A FUNCTION OF A SINGLE VARIABLE**

Write $\cos(180° - \theta)$ as a trigonometric function of θ alone.

Solution $\cos(180° - \theta) = \cos 180° \cos \theta + \sin 180° \sin \theta$ Cosine difference identity

$= (-1)\cos\theta + (0)\sin\theta$ (Section 1.3)

$= -\cos\theta$

NOW TRY EXERCISE 39. ◀

CONNECTIONS (This discussion applies to functions of both angles and real numbers.)

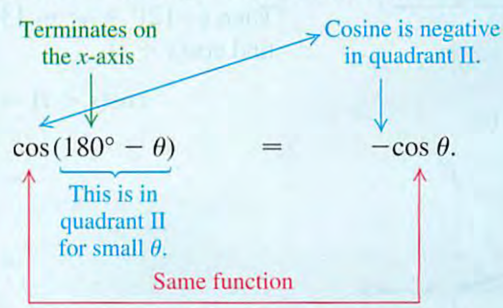
The result of Example 3 can be written as an identity:

$$\mathbf{\cos(180° - \theta) = -\cos\theta.}$$

This is an example of a **reduction formula,** which is an identity that *reduces* a function of a quadrantal angle plus or minus θ to a function of θ alone. Another example of a reduction formula is $\cos(270° + \theta) = \sin\theta$. (Verify this using the formula for the cosine of the sum.)

Here is an interesting method for quickly determining a reduction formula for a trigonometric function f of the form $f(Q \pm \theta)$, where Q is a quadrantal angle. ***There are two cases to consider, and in each case, think of θ as a small positive angle*** so that you can determine the quadrant in which $Q \pm \theta$ will lie.

Case 1: **Suppose that Q is a quadrantal angle whose terminal side lies along the x-axis.** Determine the quadrant in which $Q \pm \theta$ will lie for a small positive angle θ. If the given function f is positive in that quadrant, use a + sign on the reduced form; if f is negative in that quadrant, use a − sign. The reduced form will have that sign, f as the function, and θ as the argument. For example:



Case 2: **Suppose that Q is a quadrantal angle whose terminal side lies along the y-axis.** Determine the quadrant in which $Q \pm \theta$ will lie for a small positive angle θ. If the given function f is positive in that quadrant, use a + sign on the reduced form; if f is negative in that quadrant, use a − sign. The reduced form will have that sign, the ***cofunction of f*** as the function, and θ as the argument.

(continued)

For example:

Terminates on the y-axis → Cosine is positive in quadrant IV.

$\cos(270° + \theta) \quad = \quad +\sin\theta$, (or $\sin\theta$, as it is usually written).

This is in quadrant IV for small θ.

Cofunctions

FOR DISCUSSION OR WRITING

Use these ideas to write reduction formulas for the following. (Those involving sine and tangent can be verified after studying the identities in **Section 5.4.**)

1. $\cos(90° + \theta)$ **2.** $\cos(270° - \theta)$ **3.** $\cos(180° + \theta)$

4. $\cos(180° - \theta)$ **5.** $\sin(180° + \theta)$ **6.** $\tan(270° - \theta)$

▶ EXAMPLE 4 FINDING $\cos(s + t)$ GIVEN INFORMATION ABOUT s AND t

Suppose that $\sin s = \frac{3}{5}$, $\cos t = -\frac{12}{13}$, and both s and t are in quadrant II. Find $\cos(s + t)$.

Solution By the cosine sum identity, $\cos(s + t) = \cos s \cos t - \sin s \sin t$. The values of $\sin s$ and $\cos t$ are given, so we can find $\cos(s + t)$ if we know the values of $\cos s$ and $\sin t$. To find $\cos s$ and $\sin t$, we sketch two angles in the second quadrant, one with $\sin s = \frac{3}{5}$ and the other with $\cos t = -\frac{12}{13}$. See Figure 5.

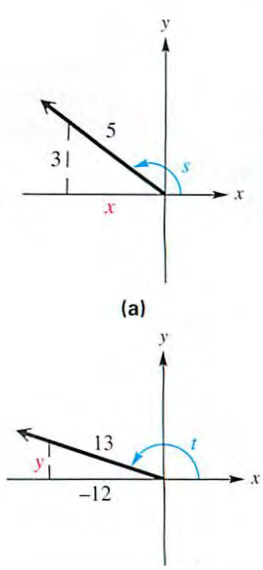
In Figure 5(a), since $\sin s = \frac{3}{5} = \frac{y}{r}$, we let $y = 3$ and $r = 5$. Substituting in the Pythagorean theorem, we get $x^2 + 3^2 = 5^2$ and solve to find $x = -4$. Thus, $\cos s = -\frac{4}{5}$. In Figure 5(b), $\cos t = -\frac{12}{13} = \frac{x}{r}$, so we let $x = -12$ and $r = 13$. Then $(-12)^2 + y^2 = 13^2$; we solve to get $y = 5$. Thus, $\sin t = \frac{5}{13}$. Now we can find $\cos(s + t)$.

$$\cos(s + t) = \cos s \cos t - \sin s \sin t \qquad \text{Cosine sum identity}$$

$$= -\frac{4}{5}\left(-\frac{12}{13}\right) - \frac{3}{5} \cdot \frac{5}{13} \qquad \text{Substitute.}$$

$$= \frac{48}{65} - \frac{15}{65} = \frac{33}{65}$$

NOW TRY EXERCISE 47. ◀



(a)

(b)

Figure 5

▶ **Note** In Example 4, the values of $\cos s$ and $\sin t$ could also be found by using the Pythagorean identities. The problem could then be solved using the identity for $\cos(s + t)$ as shown in the example.

▶ EXAMPLE 5 APPLYING THE COSINE DIFFERENCE IDENTITY TO VOLTAGE

Common household electric current is called **alternating current** because the current alternates direction within the wires. The voltage V in a typical 115-volt outlet can be expressed by the function

$$V(t) = 163 \sin \omega t,$$

where ω is the angular speed (in radians per second) of the rotating generator at the electrical plant and t is time measured in seconds. (*Source:* Bell, D., *Fundamentals of Electric Circuits,* Fourth Edition, Prentice-Hall, 1988.)

(a) It is essential for electric generators to rotate at precisely 60 cycles per sec so household appliances and computers will function properly. Determine ω for these electric generators.

(b) Graph V in the window $[0, .05]$ by $[-200, 200]$.

(c) Determine a value of ϕ so that the graph of $V(t) = 163 \cos(\omega t - \phi)$ is the same as the graph of $V(t) = 163 \sin \omega t$.

Solution

(a) Each cycle is 2π radians at 60 cycles per sec, so the angular speed is $\omega = 60(2\pi) = 120\pi$ radians per sec.

(b) $V(t) = 163 \sin \omega t = 163 \sin 120\pi t$. Because the amplitude of the function is 163 (from **Section 4.1**), $[-200, 200]$ is an appropriate interval for the range, as shown in Figure 6.

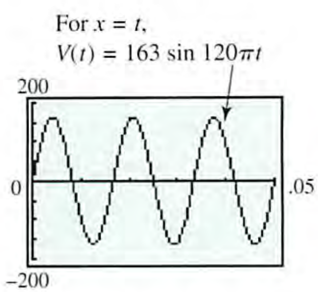


Figure 6

(c) Using the negative-angle identity for cosine and a cofunction identity,

$$\cos\left(x - \frac{\pi}{2}\right) = \cos\left[-\left(\frac{\pi}{2} - x\right)\right] = \cos\left(\frac{\pi}{2} - x\right) = \sin x.$$

Therefore, if $\phi = \frac{\pi}{2}$, then

$$V(t) = 163 \cos(\omega t - \phi) = 163 \cos\left(\omega t - \frac{\pi}{2}\right) = 163 \sin \omega t.$$

NOW TRY EXERCISE 71. ◀

5.3 Exercises

Concept Check *Match each expression in Column I with the correct expression in Column II to form an identity.*

I	II
1. $\cos(x + y) =$ ______	**A.** $\cos x \cos y + \sin x \sin y$
2. $\cos(x - y) =$ ______	**B.** $\cos x$
3. $\cos(90° - x) =$ ______	**C.** $\cos x + \sin x$
4. $\sin(90° - x) =$ ______	**D.** $\cos x - \sin x$
	E. $\sin x$
	F. $\cos x \cos y - \sin x \sin y$

Use identities to find each exact value. (Do not use a calculator.) See Example 1.

5. $\cos 75°$

6. $\cos(-15°)$

7. $\cos 105°$ (*Hint:* $105° = 60° + 45°$)

8. $\cos(-105°)$ (*Hint:* $-105° = -60° + (-45°)$)

9. $\cos \frac{7\pi}{12}$

10. $\cos\left(-\frac{\pi}{12}\right)$

11. $\cos 40° \cos 50° - \sin 40° \sin 50°$

12. $\cos \frac{7\pi}{9} \cos \frac{2\pi}{9} - \sin \frac{7\pi}{9} \sin \frac{2\pi}{9}$

Use a graphing or scientific calculator to support your answer for each of the following. See Example 1.

13. Exercise 11

14. Exercise 12

Write each function value in terms of the cofunction of a complementary angle. See Example 2.

15. $\tan 87°$

16. $\sin 15°$

17. $\cos \frac{\pi}{12}$

18. $\sin \frac{2\pi}{5}$

19. $\csc(-14° 24')$

20. $\sin 142° 14'$

21. $\sin \frac{5\pi}{8}$

22. $\cot \frac{9\pi}{10}$

23. $\sec 146° 42'$

24. $\tan 174° 3'$

25. $\cot 176.9814°$

26. $\sin 98.0142°$

Use identities to fill in each blank with the appropriate trigonometric function name. See Example 2.

27. $\cot \frac{\pi}{3} =$ _____ $\frac{\pi}{6}$

28. $\sin \frac{2\pi}{3} =$ _____ $\left(-\frac{\pi}{6}\right)$

29. _____ $33° = \sin 57°$

30. _____ $72° = \cot 18°$

31. $\cos 70° = \frac{1}{\text{\_\_\_\_\_ } 20°}$

32. $\tan 24° = \frac{1}{\text{\_\_\_\_\_ } 66°}$

Find an angle θ that makes each statement true. See Example 2.

33. $\tan \theta = \cot(45° + 2\theta)$

34. $\sin \theta = \cos(2\theta - 10°)$

35. $\sec \theta = \csc\left(\frac{\theta}{2} + 20°\right)$

36. $\cos \theta = \sin(3\theta + 10°)$

37. $\sin(3\theta - 15°) = \cos(\theta + 25°)$

38. $\cot(\theta - 10°) = \tan(2\theta + 20°)$

Use identities to write each expression as a function of θ. See Example 3.

39. $\cos(0° - \theta)$

40. $\cos(90° - \theta)$

41. $\cos(180° - \theta)$

42. $\cos(270° - \theta)$

43. $\cos(0° + \theta)$

44. $\cos(90° + \theta)$

45. $\cos(180° + \theta)$

46. $\cos(270° + \theta)$

Find $\cos(s + t)$ *and* $\cos(s - t)$*. See Example 4.*

47. $\cos s = -\frac{1}{5}$ and $\sin t = \frac{3}{5}$, s and t in quadrant II

48. $\sin s = \frac{2}{3}$ and $\sin t = -\frac{1}{3}$, s in quadrant II and t in quadrant IV

49. $\sin s = \frac{3}{5}$ and $\sin t = -\frac{12}{13}$, s in quadrant I and t in quadrant III

50. $\cos s = -\frac{8}{17}$ and $\cos t = -\frac{3}{5}$, s and t in quadrant III

51. $\sin s = \frac{\sqrt{5}}{7}$ and $\sin t = \frac{\sqrt{6}}{8}$, s and t in quadrant I

52. $\cos s = \frac{\sqrt{2}}{4}$ and $\sin t = -\frac{\sqrt{5}}{6}$, s and t in quadrant IV

Concept Check *Tell whether each statement is* true *or* false.

53. $\cos 42° = \cos(30° + 12°)$

54. $\cos(-24°) = \cos 16° - \cos 40°$

55. $\cos 74° = \cos 60° \cos 14° + \sin 60° \sin 14°$

56. $\cos 140° = \cos 60° \cos 80° - \sin 60° \sin 80°$

57. $\cos \frac{\pi}{3} = \cos \frac{\pi}{12} \cos \frac{\pi}{4} - \sin \frac{\pi}{12} \sin \frac{\pi}{4}$

58. $\cos \frac{2\pi}{3} = \cos \frac{11\pi}{12} \cos \frac{\pi}{4} + \sin \frac{11\pi}{12} \sin \frac{\pi}{4}$

59. $\cos 70° \cos 20° - \sin 70° \sin 20° = 0$

60. $\cos 85° \cos 40° + \sin 85° \sin 40° = \frac{\sqrt{2}}{2}$

61. $\tan\left(\theta - \frac{\pi}{2}\right) = \cot \theta$

62. $\sin\left(\theta - \frac{\pi}{2}\right) = \cos \theta$

Verify that each equation is an identity.

63. $\cos\left(\frac{\pi}{2} + x\right) = -\sin x$

64. $\sec(\pi - x) = -\sec x$

65. $\cos 2x = \cos^2 x - \sin^2 x$ (*Hint:* $\cos 2x = \cos(x + x)$.)

66. $1 + \cos 2x - \cos^2 x = \cos^2 x$ (*Hint:* Use the result from Exercise 65.)

RELATING CONCEPTS

***For individual or collaborative investigation* (Exercises 67–70)**

The identities for $\cos(A + B)$ *and* $\cos(A - B)$ *can be used to find exact values of expressions like* $\cos 195°$ *and* $\cos 255°$, *where the angle is not in the first quadrant.* ***Work Exercises 67–70 in order,*** *to see how this is done.*

67. By writing 195° as 180° + 15°, use the identity for $\cos(A + B)$ to express $\cos 195°$ as $-\cos 15°$.

68. Use the identity for $\cos(A - B)$ to find $-\cos 15°$.

69. By the results of Exercises 67 and 68, $\cos 195° =$ ______.

70. Find each exact value using the method shown in Exercises 67–69.

(a) $\cos 255°$ **(b)** $\cos \frac{11\pi}{12}$

(Modeling) Solve each problem.

71. ***Electric Current*** Refer to Example 5.

(a) How many times does the current oscillate in .05 sec?

(b) What are the maximum and minimum voltages in this outlet? Is the voltage always equal to 115 volts?

72. ***Sound Waves*** Sound is a result of waves applying pressure to a person's eardrum. For a pure sound wave radiating outward in a spherical shape, the trigonometric function defined by

$$P = \frac{a}{r} \cos\left(\frac{2\pi r}{\lambda} - ct\right)$$

can be used to model the sound pressure at a radius of r feet from the source, where t is time in seconds, λ is length of the sound wave in feet, c is speed of sound in feet per second, and a is maximum sound pressure at the source measured in pounds per square foot. (*Source:* Beranek, L., *Noise and Vibration Control,* Institute of Noise Control Engineering, Washington, D.C., 1988.) Let $\lambda = 4.9$ ft and $c = 1026$ ft per sec.

(a) Let $a = .4$ lb per ft^2. Graph the sound pressure at distance $r = 10$ ft from its source in the window [0, .05] by [−.05, .05]. Describe P at this distance.

(b) Now let $a = 3$ and $t = 10$. Graph the sound pressure in the window [0, 20] by [−2, 2]. What happens to pressure P as radius r increases?

(c) Suppose a person stands at a radius r so that $r = n\lambda$, where n is a positive integer. Use the difference identity for cosine to simplify P in this situation.

5.4 Sum and Difference Identities for Sine and Tangent

Sum and Difference Identities for Sine ▪ Sum and Difference Identities for Tangent ▪ Applying the Sum and Difference Identities

Sum and Difference Identities for Sine

We can use the cosine sum and difference identities to derive similar identities for sine and tangent. Since $\sin \theta = \cos(90° - \theta)$, we replace θ with $A + B$ to get

$$\sin(A + B) = \cos[90° - (A + B)] \quad \text{Cofunction identity (Section 5.3)}$$
$$= \cos[(90° - A) - B]$$
$$= \cos(90° - A)\cos B + \sin(90° - A)\sin B \quad \text{Cosine difference identity (Section 5.3)}$$
$$\mathbf{\sin(A + B) = \sin A \cos B + \cos A \sin B.} \quad \text{Cofunction identities}$$

Now we write $\sin(A - B)$ as $\sin[A + (-B)]$ and use the identity for $\sin(A + B)$.

$$\sin(A - B) = \sin[A + (-B)]$$
$$= \sin A \cos(-B) + \cos A \sin(-B) \quad \text{Sine sum identity}$$
$$\mathbf{\sin(A - B) = \sin A \cos B - \cos A \sin B} \quad \text{Negative-angle identities (Section 5.1)}$$

SINE OF A SUM OR DIFFERENCE

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$
$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

Sum and Difference Identities for Tangent

To derive the identity for $\tan(A + B)$, we start with

$$\tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)}$$ Fundamental identity (Section 5.1)

$$= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}.$$ Sum identities

We express this result in terms of the tangent function by multiplying both numerator and denominator by $\frac{1}{\cos A \cos B}$.

$$\tan(A + B) = \frac{\dfrac{\sin A \cos B + \cos A \sin B}{1}}{\dfrac{\cos A \cos B - \sin A \sin B}{1}} \cdot \frac{\dfrac{1}{\cos A \cos B}}{\dfrac{1}{\cos A \cos B}}$$ Simplify the complex fraction.

$$= \frac{\dfrac{\sin A \cos B}{\cos A \cos B} + \dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} - \dfrac{\sin A \sin B}{\cos A \cos B}}$$ Multiply numerators; multiply denominators.

$$= \frac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A}{\cos A} \cdot \dfrac{\sin B}{\cos B}}$$ Simplify.

Since $\frac{\sin \theta}{\cos \theta} = \tan \theta$, we have

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Replacing B with $-B$ and using the fact that $\tan(-B) = -\tan B$ gives the identity for the tangent of the difference of two angles.

TANGENT OF A SUM OR DIFFERENCE

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \qquad \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Applying the Sum and Difference Identities

▶ EXAMPLE 1 FINDING EXACT SINE AND TANGENT FUNCTION VALUES

Find the *exact* value of each expression.

(a) $\sin 75°$ **(b)** $\tan \frac{7\pi}{12}$ **(c)** $\sin 40° \cos 160° - \cos 40° \sin 160°$

Solution

(a)
$$\sin 75° = \sin(45° + 30°) \quad 75° = 45° + 30°$$
$$= \sin 45° \cos 30° + \cos 45° \sin 30° \quad \text{Sine sum identity}$$
$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} \quad \text{Substitute known values. (Section 2.1)}$$
$$= \frac{\sqrt{6} + \sqrt{2}}{4} \quad \text{Multiply; add fractions.}$$

(b)
$$\tan \frac{7\pi}{12} = \tan\left(\frac{\pi}{3} + \frac{\pi}{4}\right) \quad \frac{\pi}{3} = \frac{4\pi}{12};\ \frac{\pi}{4} = \frac{3\pi}{12}$$
$$= \frac{\tan \frac{\pi}{3} + \tan \frac{\pi}{4}}{1 - \tan \frac{\pi}{3} \tan \frac{\pi}{4}} \quad \text{Tangent sum identity}$$
$$= \frac{\sqrt{3} + 1}{1 - \sqrt{3} \cdot 1} \quad \text{Substitute known values. (Section 3.1)}$$
$$= \frac{\sqrt{3} + 1}{1 - \sqrt{3}} \cdot \frac{1 + \sqrt{3}}{1 + \sqrt{3}} \quad \text{Rationalize the denominator.}$$
$$= \frac{\sqrt{3} + 3 + 1 + \sqrt{3}}{1 - 3} \quad \text{Multiply.}$$
$$= \frac{4 + 2\sqrt{3}}{-2} \quad \text{Combine terms.}$$
$$= \frac{2(2 + \sqrt{3})}{2(-1)} \quad \text{Factor out 2.}$$

Factor first. Then divide out the common factor.

$$= -2 - \sqrt{3} \quad \text{Lowest terms}$$

(c)
$$\sin 40° \cos 160° - \cos 40° \sin 160° = \sin(40° - 160°) \quad \text{Sine difference identity}$$
$$= \sin(-120°) \quad \text{Subtract.}$$
$$= -\sin 120° \quad \text{Negative-angle identity}$$
$$= -\frac{\sqrt{3}}{2} \quad \text{(Section 2.2)}$$

NOW TRY EXERCISES 9, 11, AND 15. ◀

▶ EXAMPLE 2 WRITING FUNCTIONS AS EXPRESSIONS INVOLVING FUNCTIONS OF θ

Write each function as an expression involving functions of θ.

(a) $\sin(30° + \theta)$ **(b)** $\tan(45° - \theta)$ **(c)** $\sin(180° + \theta)$

Solution

(a) Using the identity for $\sin(A + B)$,

$$\sin(30° + \theta) = \sin 30° \cos\theta + \cos 30° \sin\theta$$
$$= \frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta.$$

(b) $$\tan(45° - \theta) = \frac{\tan 45° - \tan\theta}{1 + \tan 45° \tan\theta} \quad \text{Tangent difference identity}$$
$$= \frac{1 - \tan\theta}{1 + \tan\theta}$$

(c) $$\sin(180° + \theta) = \sin 180° \cos\theta + \cos 180° \sin\theta \quad \text{Sine sum identity}$$
$$= 0 \cdot \cos\theta + (-1)\sin\theta \quad \text{(Section 1.3)}$$
$$= -\sin\theta$$

NOW TRY EXERCISES 29 AND 33. ◀

▶ EXAMPLE 3 FINDING FUNCTION VALUES AND THE QUADRANT OF A + B

Suppose that A and B are angles in standard position, with $\sin A = \frac{4}{5}$, $\frac{\pi}{2} < A < \pi$, and $\cos B = -\frac{5}{13}$, $\pi < B < \frac{3\pi}{2}$. Find each of the following.

(a) $\sin(A + B)$ **(b)** $\tan(A + B)$ **(c)** the quadrant of $A + B$

Solution

(a) The identity for $\sin(A + B)$ requires $\sin A$, $\cos A$, $\sin B$, and $\cos B$. We are given values of $\sin A$ and $\cos B$. We must find values of $\cos A$ and $\sin B$.

$$\sin^2 A + \cos^2 A = 1 \quad \text{Fundamental identity (Section 5.1)}$$
$$\frac{16}{25} + \cos^2 A = 1 \quad \sin A = \tfrac{4}{5}$$
$$\cos^2 A = \frac{9}{25} \quad \text{Subtract } \tfrac{16}{25}.$$
$$\cos A = -\frac{3}{5} \quad \text{Take square roots (Appendix A); since } A \text{ is in quadrant II, } \cos A < 0.$$

In the same way, $\sin B = -\frac{12}{13}$. Now use the formula for $\sin(A + B)$.

$$\sin(A + B) = \frac{4}{5}\left(-\frac{5}{13}\right) + \left(-\frac{3}{5}\right)\left(-\frac{12}{13}\right)$$
$$= -\frac{20}{65} + \frac{36}{65} = \frac{16}{65}$$

(b) To find $\tan(A + B)$, first use the values of sine and cosine from part (a), $\sin A = \frac{4}{5}$, $\cos A = -\frac{3}{5}$, $\sin B = -\frac{12}{13}$, and $\cos B = -\frac{5}{13}$, to get $\tan A = -\frac{4}{3}$ and $\tan B = \frac{12}{5}$.

$$\tan(A + B) = \frac{-\frac{4}{3} + \frac{12}{5}}{1 - \left(-\frac{4}{3}\right)\left(\frac{12}{5}\right)} = \frac{\frac{16}{15}}{1 + \frac{48}{15}} = \frac{\frac{16}{15}}{\frac{63}{15}} = \frac{16}{15} \div \frac{63}{15} = \frac{16}{15} \cdot \frac{15}{63} = \frac{16}{63}$$

(c) From parts (a) and (b),

$$\sin(A + B) = \frac{16}{65} \quad \text{and} \quad \tan(A + B) = \frac{16}{63},$$

both positive. Therefore, $A + B$ must be in quadrant I, since it is the only quadrant in which both sine and tangent are positive.

NOW TRY EXERCISE 41.

▶ EXAMPLE 4 VERIFYING AN IDENTITY USING SUM AND DIFFERENCE IDENTITIES

Verify that the equation is an identity.

$$\sin\left(\frac{\pi}{6} + \theta\right) + \cos\left(\frac{\pi}{3} + \theta\right) = \cos\theta$$

Solution Work on the left side, using the sum identities for $\sin(A + B)$ and $\cos(A + B)$.

$$\sin\left(\frac{\pi}{6} + \theta\right) + \cos\left(\frac{\pi}{3} + \theta\right)$$

$$= \left(\sin\frac{\pi}{6}\cos\theta + \cos\frac{\pi}{6}\sin\theta\right) \quad \text{Sine sum identity}$$

$$+ \left(\cos\frac{\pi}{3}\cos\theta - \sin\frac{\pi}{3}\sin\theta\right) \quad \text{Cosine sum identity (Section 5.3)}$$

$$= \left(\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta\right) \quad \sin\tfrac{\pi}{6} = \tfrac{1}{2};\ \cos\tfrac{\pi}{6} = \tfrac{\sqrt{3}}{2}$$

$$+ \left(\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta\right) \quad \cos\tfrac{\pi}{3} = \tfrac{1}{2};\ \sin\tfrac{\pi}{3} = \tfrac{\sqrt{3}}{2}$$

$$= \frac{1}{2}\cos\theta + \frac{1}{2}\cos\theta \quad \text{Simplify.}$$

$$= \cos\theta$$

NOW TRY EXERCISE 59.

5.4 Exercises

Concept Check *Match each expression in Column I with its value in Column II. See Example 1.*

I	II
1. $\sin 15°$	**A.** $\frac{\sqrt{6}+\sqrt{2}}{4}$
2. $\sin 105°$	**B.** $\frac{-\sqrt{6}-\sqrt{2}}{4}$
3. $\tan 15°$	**C.** $\frac{\sqrt{6}-\sqrt{2}}{4}$
4. $\tan 105°$	**D.** $2+\sqrt{3}$
5. $\sin(-105°)$	**E.** $2-\sqrt{3}$
6. $\tan(-105°)$	**F.** $-2-\sqrt{3}$

7. Compare the formulas for $\sin(A - B)$ and $\sin(A + B)$. How do they differ? How are they alike?

8. Compare the formulas for $\tan(A - B)$ and $\tan(A + B)$. How do they differ? How are they alike?

Use identities to find each exact value. See Example 1.

9. $\sin \frac{5\pi}{12}$ **10.** $\tan \frac{5\pi}{12}$ **11.** $\tan \frac{\pi}{12}$

12. $\sin \frac{\pi}{12}$ **13.** $\sin\left(-\frac{7\pi}{12}\right)$ **14.** $\tan\left(-\frac{7\pi}{12}\right)$

15. $\sin 76° \cos 31° - \cos 76° \sin 31°$

16. $\sin 40° \cos 50° + \cos 40° \sin 50°$

17. $\frac{\tan 80° + \tan 55°}{1 - \tan 80° \tan 55°}$

18. $\frac{\tan 80° - \tan(-55°)}{1 + \tan 80° \tan(-55°)}$

19. $\frac{\tan 100° + \tan 80°}{1 - \tan 100° \tan 80°}$

20. $\sin 100° \cos 10° - \cos 100° \sin 10°$

21. $\sin \frac{\pi}{5} \cos \frac{3\pi}{10} + \cos \frac{\pi}{5} \sin \frac{3\pi}{10}$

22. $\frac{\tan \frac{5\pi}{12} + \tan \frac{\pi}{4}}{1 - \tan \frac{5\pi}{12} \tan \frac{\pi}{4}}$

Use identities to write each expression as a single function of x or θ. See Example 2.

23. $\cos(30° + \theta)$ **24.** $\cos(45° - \theta)$ **25.** $\cos(60° + \theta)$

26. $\cos(\theta - 30°)$ **27.** $\cos\left(\frac{3\pi}{4} - x\right)$ **28.** $\sin(45° + \theta)$

29. $\tan(\theta + 30°)$ **30.** $\tan\left(\frac{\pi}{4} + x\right)$ **31.** $\sin\left(\frac{\pi}{4} + x\right)$

32. $\sin(180° - \theta)$ **33.** $\sin(270° - \theta)$ **34.** $\tan(180° + \theta)$

35. $\tan(360° - \theta)$ **36.** $\sin(\pi + \theta)$ **37.** $\tan(\pi - \theta)$

38. Why is it not possible to use the method of Example 2 to find a formula for $\tan(270° - \theta)$?

39. Why is it that standard trigonometry texts usually do not develop formulas for the cotangent, secant, and cosecant of the sum and difference of two numbers or angles?

40. Show that if A, B, and C are the angles of a triangle, then $\sin(A + B + C) = 0$.

*Use the given information to find **(a)** $\sin(s + t)$, **(b)** $\tan(s + t)$, and **(c)** the quadrant of $s + t$. See Example 3.*

41. $\cos s = \frac{3}{5}$ and $\sin t = \frac{5}{13}$, s and t in quadrant I

42. $\cos s = -\frac{1}{5}$ and $\sin t = \frac{3}{5}$, s and t in quadrant II

43. $\sin s = \frac{2}{3}$ and $\sin t = -\frac{1}{3}$, s in quadrant II and t in quadrant IV

44. $\sin s = \frac{3}{5}$ and $\sin t = -\frac{12}{13}$, s in quadrant I and t in quadrant III

45. $\cos s = -\frac{8}{17}$ and $\cos t = -\frac{3}{5}$, s and t in quadrant III

46. $\cos s = -\frac{15}{17}$ and $\sin t = \frac{4}{5}$, s in quadrant II and t in quadrant I

*Find each exact value. Use the technique developed in **Section 5.3,** Exercises 67–70.*

47. $\sin 165°$ **48.** $\tan 165°$ **49.** $\sin 255°$

50. $\tan 285°$ **51.** $\tan \frac{11\pi}{12}$ **52.** $\sin\left(-\frac{13\pi}{12}\right)$

Graph each expression and use the graph to make a conjecture, predicting what might be an identity. Then verify your conjecture algebraically.

53. $\sin\left(\frac{\pi}{2} + \theta\right)$ **54.** $\sin\left(\frac{3\pi}{2} + \theta\right)$ **55.** $\tan\left(\frac{\pi}{2} + \theta\right)$ **56.** $\frac{1 + \tan\theta}{1 - \tan\theta}$

Verify that each equation is an identity. See Example 4.

57. $\sin 2x = 2 \sin x \cos x$ (*Hint:* $\sin 2x = \sin(x + x)$)

58. $\sin(x + y) + \sin(x - y) = 2 \sin x \cos y$

59. $\sin(210° + x) - \cos(120° + x) = 0$

60. $\tan(x - y) - \tan(y - x) = \frac{2(\tan x - \tan y)}{1 + \tan x \tan y}$

61. $\frac{\cos(\alpha - \beta)}{\cos\alpha \sin\beta} = \tan\alpha + \cot\beta$

62. $\frac{\sin(s + t)}{\cos s \cos t} = \tan s + \tan t$

63. $\frac{\sin(x - y)}{\sin(x + y)} = \frac{\tan x - \tan y}{\tan x + \tan y}$

64. $\frac{\sin(x + y)}{\cos(x - y)} = \frac{\cot x + \cot y}{1 + \cot x \cot y}$

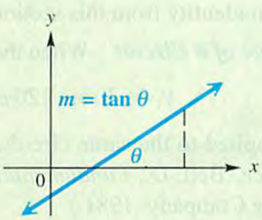
65. $\frac{\sin(s - t)}{\sin t} + \frac{\cos(s - t)}{\cos t} = \frac{\sin s}{\sin t \cos t}$

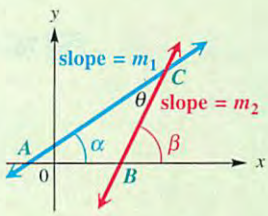
66. $\frac{\tan(\alpha + \beta) - \tan\beta}{1 + \tan(\alpha + \beta)\tan\beta} = \tan\alpha$

RELATING CONCEPTS

***For individual or collaborative investigation* (Exercises 67–72)**

Refer to the figure on the left below. By the definition of tan θ, $m = \tan \theta$, *where* m *is the slope and* θ *is the angle of inclination of the line. The following exercises, which depend on properties of triangles, refer to triangle* ABC *in the figure on the right below.* ***Work Exercises 67–72 in order.*** *Assume that all angles are measured in degrees.*





67. In terms of β, what is the measure of angle ABC?

68. Use the fact that the sum of the angles in a triangle is 180° to express θ in terms of α and β.

69. Apply the formula for $\tan(A - B)$ to obtain an expression for $\tan \theta$ in terms of $\tan \alpha$ and $\tan \beta$.

70. Replace $\tan \alpha$ with m_1 and $\tan \beta$ with m_2 to obtain $\tan \theta = \dfrac{m_2 - m_1}{1 + m_1 m_2}$.

In Exercises 71 and 72, use the result from Exercise 70 to find the acute angle between each pair of lines. (Note that the tangent of the angle will be positive.) Use a calculator and round to the nearest tenth of a degree.

71. $x + y = 9, \quad 2x + y = -1$

72. $5x - 2y + 4 = 0, \quad 3x + 5y = 6$

(Modeling) *Solve each problem.*

73. ***Back Stress*** If a person bends at the waist with a straight back making an angle of θ degrees with the horizontal, then the force F exerted on the back muscles can be modeled by the equation

$$F = \frac{.6W \sin(\theta + 90°)}{\sin 12°},$$

where W is the weight of the person. (*Source:* Metcalf, H., *Topics in Classical Biophysics*, Prentice-Hall, 1980.)

(a) Calculate F when $W = 170$ lb and $\theta = 30°$.

(b) Use an identity to show that F is approximately equal to $2.9W \cos \theta$.

(c) For what value of θ is F maximum?

74. ***Back Stress*** Refer to Exercise 73.

(a) Suppose a 200-lb person bends at the waist so that $\theta = 45°$. Estimate the force exerted on the person's back muscles.

(b) Approximate graphically the value of θ that results in the back muscles exerting a force of 400 lb.

75. ***Voltage*** A coil of wire rotating in a magnetic field induces a voltage

$$E = 20 \sin\left(\frac{\pi t}{4} - \frac{\pi}{2}\right).$$

Use an identity from this section to express this in terms of $\cos \frac{\pi t}{4}$.

76. ***Voltage of a Circuit*** When the two voltages

$$V_1 = 30 \sin 120\pi t \qquad \text{and} \qquad V_2 = 40 \cos 120\pi t$$

are applied to the same circuit, the resulting voltage V will be equal to their sum. (*Source:* Bell, D., *Fundamentals of Electric Circuits,* Second Edition, Reston Publishing Company, 1981.)

(a) Graph the sum in the window $[0, .05]$ by $[-60, 60]$.

(b) Use the graph to estimate values for a and ϕ so that $V = a \sin(120\pi t + \phi)$.

(c) Use identities to verify that your expression for V is valid.

CHAPTER 5 ▶ Quiz (Sections 5.1–5.4)

1. If $\sin \theta = -\frac{7}{25}$ and θ is in quadrant IV, find the remaining five trigonometric function values of θ.

2. Express $\cot^2 x + \csc^2 x$ in terms of $\sin x$ and $\cos x$, and simplify.

3. Find the exact value of $\sin\left(-\frac{7\pi}{12}\right)$.

4. Express $\cos(180° - \theta)$ as a function of θ alone.

5. If $\cos A = \frac{3}{5}$, $\sin B = -\frac{5}{13}$, $0 < A < \frac{\pi}{2}$, and $\pi < B < \frac{3\pi}{2}$, find each of the following.

(a) $\cos(A + B)$ **(b)** $\sin(A + B)$ **(c)** the quadrant of $A + B$

6. Express $\tan\left(\frac{3\pi}{4} + x\right)$ as a function of x alone.

Verify each identity.

7. $\dfrac{1 + \sin \theta}{\cot^2 \theta} = \dfrac{\sin \theta}{\csc \theta - 1}$

8. $\dfrac{\sin^2 \theta - \cos^2 \theta}{\sin^4 \theta - \cos^4 \theta} = 1$

9. $\sin\left(\dfrac{\pi}{3} + \theta\right) - \sin\left(\dfrac{\pi}{3} - \theta\right) = \sin \theta$

10. $\dfrac{\cos(x + y) + \cos(x - y)}{\sin(x - y) + \sin(x + y)} = \cot x$

5.5 Double-Angle Identities

Double-Angle Identities ▪ An Application ▪ Product-to-Sum and Sum-to-Product Identities

Double-Angle Identities

When $A = B$ in the identities for the sum of two angles, these identities are called the **double-angle identities.** For example, to derive an expression for $\cos 2A$, we let $B = A$ in the identity $\cos(A + B) = \cos A \cos B - \sin A \sin B$.

$$\cos 2A = \cos(A + A)$$
$$= \cos A \cos A - \sin A \sin A \quad \text{Cosine sum identity (Section 5.3)}$$
$$\mathbf{\cos 2A = \cos^2 A - \sin^2 A}$$

Two other useful forms of this identity can be obtained by substituting either $\cos^2 A = 1 - \sin^2 A$ or $\sin^2 A = 1 - \cos^2 A$. Replace $\cos^2 A$ with the expression $1 - \sin^2 A$ to get

$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= (1 - \sin^2 A) - \sin^2 A \quad \text{Fundamental identity (Section 5.1)}$$
$$\mathbf{\cos 2A = 1 - 2\sin^2 A,}$$

or replace $\sin^2 A$ with $1 - \cos^2 A$ to get

$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= \cos^2 A - (1 - \cos^2 A) \quad \text{Fundamental identity}$$
$$= \cos^2 A - 1 + \cos^2 A$$
$$\mathbf{\cos 2A = 2\cos^2 A - 1.}$$

We find $\sin 2A$ with the identity $\sin(A + B) = \sin A \cos B + \cos A \sin B$, letting $B = A$.

$$\sin 2A = \sin(A + A)$$
$$= \sin A \cos A + \cos A \sin A \quad \text{Sine sum identity (Section 5.4)}$$
$$\mathbf{\sin 2A = 2\sin A \cos A}$$

Using the identity for $\tan(A + B)$, we find $\tan 2A$.

$$\tan 2A = \tan(A + A)$$
$$= \frac{\tan A + \tan A}{1 - \tan A \tan A} \quad \text{Tangent sum identity (Section 5.4)}$$
$$\mathbf{\tan 2A = \frac{2\tan A}{1 - \tan^2 A}}$$

LOOKING AHEAD TO CALCULUS

The identities

$$\cos 2A = 1 - 2\sin^2 A$$

and $\cos 2A = 2\cos^2 A - 1$

can be rewritten as

$$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$$

and $\cos^2 A = \frac{1}{2}(1 + \cos 2A)$.

These identities are used to integrate the functions $f(A) = \sin^2 A$ and $g(A) = \cos^2 A$.

DOUBLE-ANGLE IDENTITIES

$$\cos 2A = \cos^2 A - \sin^2 A \qquad \cos 2A = 1 - 2\sin^2 A$$
$$\cos 2A = 2\cos^2 A - 1 \qquad \sin 2A = 2\sin A \cos A$$
$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

▶ EXAMPLE 1 **FINDING FUNCTION VALUES OF 2θ GIVEN INFORMATION ABOUT θ**

Given $\cos\theta = \frac{3}{5}$ and $\sin\theta < 0$, find $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$.

Solution To find $\sin 2\theta$, we must first find the value of $\sin\theta$.

$$\sin^2\theta + \left(\frac{3}{5}\right)^2 = 1 \qquad \sin^2\theta + \cos^2\theta = 1;\ \cos\theta = \tfrac{3}{5}$$

$$\sin^2\theta = \frac{16}{25} \qquad \text{Simplify.}$$

Pay attention to signs here.

$$\sin\theta = -\frac{4}{5} \qquad \text{Take square roots (Appendix A); choose the negative square root since } \sin\theta < 0.$$

Using the double-angle identity for sine,

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(-\frac{4}{5}\right)\left(\frac{3}{5}\right) = -\frac{24}{25}. \qquad \sin\theta = -\tfrac{4}{5};\ \cos\theta = \tfrac{3}{5}$$

Now we find $\cos 2\theta$, using the first of the double-angle identities for cosine.

Any of the three forms may be used.

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \frac{9}{25} - \frac{16}{25} = -\frac{7}{25}$$

The value of $\tan 2\theta$ can be found in either of two ways. We can use the double-angle identity and the fact that $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{4}{5}}{\frac{3}{5}} = -\frac{4}{5} \div \frac{3}{5} = -\frac{4}{5}\cdot\frac{5}{3} = -\frac{4}{3}$.

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\left(-\frac{4}{3}\right)}{1-\left(-\frac{4}{3}\right)^2} = \frac{-\frac{8}{3}}{-\frac{7}{9}} = \frac{24}{7}$$

Alternatively, we can find $\tan 2\theta$ by finding the quotient of $\sin 2\theta$ and $\cos 2\theta$.

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{-\frac{24}{25}}{-\frac{7}{25}} = \frac{24}{7}$$

NOW TRY EXERCISE 15. ◀

▶ EXAMPLE 2 **FINDING FUNCTION VALUES OF θ GIVEN INFORMATION ABOUT 2θ**

Find the values of the six trigonometric functions of θ if $\cos 2\theta = \frac{4}{5}$ and $90° < \theta < 180°$.

Solution We must obtain a trigonometric function value of θ alone.

$$\cos 2\theta = 1 - 2\sin^2\theta \qquad \text{Double-angle identity}$$

$$\frac{4}{5} = 1 - 2\sin^2\theta \qquad \cos 2\theta = \tfrac{4}{5}$$

$$-\frac{1}{5} = -2\sin^2\theta \qquad \text{Subtract 1.}$$

$$\frac{1}{10} = \sin^2 \theta$$ Multiply by $-\frac{1}{2}$.

$$\sin \theta = \sqrt{\frac{1}{10}}$$ Take square roots; choose the positive square root since θ terminates in quadrant II.

$$\sin \theta = \frac{\sqrt{1}}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} = \frac{\sqrt{10}}{10}$$ Quotient rule; rationalize the denominator.

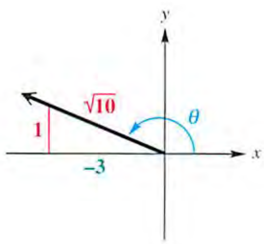


Figure 7

Now find values of $\cos \theta$ and $\tan \theta$ by sketching and labeling a right triangle in quadrant II. Since $\sin \theta = \frac{1}{\sqrt{10}}$, the triangle in Figure 7 is labeled accordingly. The Pythagorean theorem is used to find the remaining leg. Now,

$$\cos \theta = \frac{-3}{\sqrt{10}} = -\frac{3\sqrt{10}}{10}, \quad \text{and} \quad \tan \theta = \frac{1}{-3} = -\frac{1}{3}.$$ (Section 1.3)

Find the other three functions using reciprocals.

$$\csc \theta = \frac{1}{\sin \theta} = \sqrt{10}, \quad \sec \theta = \frac{1}{\cos \theta} = -\frac{\sqrt{10}}{3}, \quad \cot \theta = \frac{1}{\tan \theta} = -3$$

NOW TRY EXERCISE 9. ◀

▶ EXAMPLE 3 VERIFYING A DOUBLE-ANGLE IDENTITY

Verify that the following equation is an identity.

$$\cot x \sin 2x = 1 + \cos 2x$$

Solution We start by working on the left side, using the hint from **Section 5.2** about writing all functions in terms of sine and cosine.

$$\cot x \sin 2x = \frac{\cos x}{\sin x} \cdot \sin 2x$$ Quotient identity

$$= \frac{\cos x}{\sin x}(2 \sin x \cos x)$$ Double-angle identity

$$= 2 \cos^2 x$$

$$= 1 + \cos 2x$$ $\cos 2x = 2\cos^2 x - 1$, so $2\cos^2 x = 1 + \cos 2x$

Be able to recognize alternative forms of identities.

NOW TRY EXERCISE 37. ◀

▶ EXAMPLE 4 SIMPLIFYING EXPRESSIONS USING DOUBLE-ANGLE IDENTITIES

Simplify each expression.

(a) $\cos^2 7x - \sin^2 7x$ **(b)** $\sin 15° \cos 15°$

Solution

(a) This expression suggests one of the double-angle identities for cosine: $\cos 2A = \cos^2 A - \sin^2 A$. Substituting $7x$ for A gives

$$\cos^2 7x - \sin^2 7x = \cos 2(7x) = \cos 14x.$$

(b) If the expression $\sin 15^\circ \cos 15^\circ$ were $2 \sin 15^\circ \cos 15^\circ$, we could apply the identity for $\sin 2A$ directly since

$$\sin 2A = 2 \sin A \cos A.$$

We can still apply the identity with $A = 15^\circ$ by writing the multiplicative identity element 1 as $\frac{1}{2}(2)$.

$$\sin 15^\circ \cos 15^\circ = \frac{1}{2}(2) \sin 15^\circ \cos 15^\circ \quad \text{Multiply by 1 in the form } \tfrac{1}{2}(2).$$

This is not an obvious way to begin, but it is indeed valid.

$$= \frac{1}{2}(2 \sin 15^\circ \cos 15^\circ) \quad \text{Associative property}$$

$$= \frac{1}{2} \sin(2 \cdot 15^\circ) \quad 2 \sin A \cos A = \sin 2A, \text{ with } A = 15^\circ$$

$$= \frac{1}{2} \sin 30^\circ \quad \text{Multiply.}$$

$$= \frac{1}{2} \cdot \frac{1}{2} \quad \sin 30^\circ = \tfrac{1}{2} \text{ (Section 2.1)}$$

$$= \frac{1}{4} \quad \text{Multiply.}$$

NOW TRY EXERCISES 17 AND 19.

Identities involving larger multiples of the variable can be derived by repeated use of the double-angle identities and other identities.

EXAMPLE 5 DERIVING A MULTIPLE-ANGLE IDENTITY

Write $\sin 3x$ in terms of $\sin x$.

Solution

$$\sin 3x = \sin(2x + x)$$

Use the simple fact that 3 = 2 + 1 here.

$$= \sin 2x \cos x + \cos 2x \sin x \quad \text{Sine sum identity (Section 5.4)}$$

$$= (2 \sin x \cos x)\cos x + (\cos^2 x - \sin^2 x)\sin x \quad \text{Double-angle identities}$$

$$= 2 \sin x \cos^2 x + \cos^2 x \sin x - \sin^3 x \quad \text{Multiply.}$$

$$= 2 \sin x(1 - \sin^2 x) + (1 - \sin^2 x)\sin x - \sin^3 x \quad \cos^2 x = 1 - \sin^2 x$$

$$= 2 \sin x - 2 \sin^3 x + \sin x - \sin^3 x - \sin^3 x \quad \text{Distributive property}$$

$$= 3 \sin x - 4 \sin^3 x \quad \text{Combine terms.}$$

NOW TRY EXERCISE 29.

An Application

▶ EXAMPLE 6 DETERMINING WATTAGE CONSUMPTION

If a toaster is plugged into a common household outlet, the wattage consumed is not constant. Instead, it varies at a high frequency according to the model

$$W = \frac{V^2}{R},$$

where V is the voltage and R is a constant that measures the resistance of the toaster in ohms. (*Source:* Bell, D., *Fundamentals of Electric Circuits*, Fourth Edition, Prentice-Hall, 1998.) Graph the wattage W consumed by a typical toaster with $R = 15$ and $V = 163 \sin 120\pi t$ in the window $[0, .05]$ by $[-500, 2000]$. How many oscillations are there?

Solution Substituting the given values into the wattage equation gives

$$W = \frac{V^2}{R} = \frac{(163 \sin 120\pi t)^2}{15}.$$

To determine the range of W, we note that $\sin 120\pi t$ has maximum value 1, so the expression for W has maximum value $\frac{163^2}{15} \approx 1771$. The minimum value is 0. The graph in Figure 8 shows that there are six oscillations.

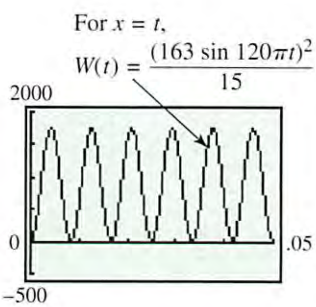


Figure 8

NOW TRY EXERCISE 67. ◀

Product-to-Sum and Sum-to-Product Identities

The identities for $\cos(A + B)$ and $\cos(A - B)$ can be added to derive an identity useful in calculus.

$$\begin{aligned} \cos(A + B) &= \cos A \cos B - \sin A \sin B \\ \cos(A - B) &= \cos A \cos B + \sin A \sin B \\ \hline \cos(A + B) + \cos(A - B) &= 2 \cos A \cos B \end{aligned}$$ Add.

or

$$\mathbf{\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]}$$

Similarly, subtracting $\cos(A + B)$ from $\cos(A - B)$ gives

$$\mathbf{\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)].}$$

Using the identities for $\sin(A + B)$ and $\sin(A - B)$ in the same way, we get two more identities. Those and the previous ones are now summarized.

▼ LOOKING AHEAD TO CALCULUS

The product-to-sum identities are used in calculus to find **integrals** of functions that are products of trigonometric functions. One classic calculus text includes the following example:

Evaluate $\int \cos 5x \cos 3x\, dx$.

The first solution line reads:
"We may write

$\cos 5x \cos 3x = \frac{1}{2}[\cos 8x + \cos 2x]$."

PRODUCT-TO-SUM IDENTITIES

$$\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$$

$$\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$$

$$\sin A \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$$

$$\cos A \sin B = \frac{1}{2}[\sin(A + B) - \sin(A - B)]$$

▶ EXAMPLE 7 USING A PRODUCT-TO-SUM IDENTITY

Write 4 cos 75° sin 25° as the sum or difference of two functions.

Solution Use the identity for cos A sin B, with $A = 75°$ and $B = 25°$.

$$4\cos 75^\circ \sin 25^\circ = 4\left[\frac{1}{2}\left(\sin(75^\circ + 25^\circ) - \sin(75^\circ - 25^\circ)\right)\right]$$
$$= 2\sin 100^\circ - 2\sin 50^\circ$$

NOW TRY EXERCISE 55. ◀

Another group of identities allows us to write sums as products.

SUM-TO-PRODUCT IDENTITIES

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

▶ EXAMPLE 8 USING A SUM-TO-PRODUCT IDENTITY

Write $\sin 2\theta - \sin 4\theta$ as a product of two functions.

Solution Use the identity for $\sin A - \sin B$, with $2\theta = A$ and $4\theta = B$.

$$\sin 2\theta - \sin 4\theta = 2\cos\left(\frac{2\theta + 4\theta}{2}\right)\sin\left(\frac{2\theta - 4\theta}{2}\right)$$
$$= 2\cos\frac{6\theta}{2}\sin\left(\frac{-2\theta}{2}\right)$$
$$= 2\cos 3\theta \sin(-\theta)$$
$$= -2\cos 3\theta \sin\theta \qquad \sin(-\theta) = -\sin\theta \text{ (Section 5.1)}$$

NOW TRY EXERCISE 61. ◀

5.5 Exercises

Concept Check *Match each expression in Column I with its value in Column II.*

I	II
1. $2\cos^2 15° - 1$	**A.** $\frac{1}{2}$
2. $\frac{2\tan 15°}{1-\tan^2 15°}$	**B.** $\frac{\sqrt{2}}{2}$
3. $2\sin 22.5° \cos 22.5°$	**C.** $\frac{\sqrt{3}}{2}$
4. $\cos^2 \frac{\pi}{6} - \sin^2 \frac{\pi}{6}$	**D.** $-\sqrt{3}$
5. $2\sin \frac{\pi}{3} \cos \frac{\pi}{3}$	**E.** $\frac{\sqrt{3}}{3}$
6. $\frac{2\tan \frac{\pi}{3}}{1-\tan^2 \frac{\pi}{3}}$	

Use identities to find values of the sine and cosine functions for each angle measure. See Examples 1 and 2.

7. θ, given $\cos 2\theta = \frac{3}{5}$ and θ terminates in quadrant I

8. θ, given $\cos 2\theta = \frac{3}{4}$ and θ terminates in quadrant III

9. θ, given $\cos 2\theta = -\frac{5}{12}$ and $\frac{\pi}{2} < \theta < \pi$

10. x, given $\cos 2x = \frac{2}{3}$ and $\frac{\pi}{2} < x < \pi$

11. 2θ, given $\sin \theta = \frac{2}{5}$ and $\cos \theta < 0$

12. 2θ, given $\cos \theta = -\frac{12}{13}$ and $\sin \theta > 0$

13. $2x$, given $\tan x = 2$ and $\cos x > 0$

14. $2x$, given $\tan x = \frac{5}{3}$ and $\sin x < 0$

15. 2θ, given $\sin \theta = -\frac{\sqrt{5}}{7}$ and $\cos \theta > 0$

16. 2θ, given $\cos \theta = \frac{\sqrt{3}}{5}$ and $\sin \theta > 0$

Use an identity to write each expression as a single trigonometric function value or as a single number. See Example 4.

17. $\cos^2 15° - \sin^2 15°$

18. $\frac{2\tan 15°}{1-\tan^2 15°}$

19. $1 - 2\sin^2 15°$

20. $1 - 2\sin^2 22\frac{1}{2}°$

21. $2\cos^2 67\frac{1}{2}° - 1$

22. $\cos^2 \frac{\pi}{8} - \frac{1}{2}$

23. $\dfrac{\tan 51°}{1 - \tan^2 51°}$ 24. $\dfrac{\tan 34°}{2(1 - \tan^2 34°)}$ 25. $\dfrac{1}{4} - \dfrac{1}{2}\sin^2 47.1°$

26. $\dfrac{1}{8}\sin 29.5° \cos 29.5°$ 27. $\sin^2\dfrac{2\pi}{5} - \cos^2\dfrac{2\pi}{5}$ 28. $\cos^2 2x - \sin^2 2x$

Express each function as a trigonometric function of x. See Example 5.

29. $\sin 4x$ 30. $\cos 3x$ 31. $\tan 3x$ 32. $\cos 4x$

Graph each expression and use the graph to make a conjecture as to what might be an identity. Then verify your conjecture algebraically.

33. $\cos^4 x - \sin^4 x$ 34. $\dfrac{4\tan x\cos^2 x - 2\tan x}{1 - \tan^2 x}$

35. $\dfrac{2\tan x}{2 - \sec^2 x}$ 36. $\dfrac{\cot^2 x - 1}{2\cot x}$

Verify that each equation is an identity. See Example 3.

37. $(\sin x + \cos x)^2 = \sin 2x + 1$ 38. $\sec 2x = \dfrac{\sec^2 x + \sec^4 x}{2 + \sec^2 x - \sec^4 x}$

39. $\tan 8\theta - \tan 8\theta \tan^2 4\theta = 2\tan 4\theta$ 40. $\sin 2x = \dfrac{2\tan x}{1 + \tan^2 x}$

41. $\cos 2\theta = \dfrac{2 - \sec^2\theta}{\sec^2\theta}$ 42. $-\tan 2\theta = \dfrac{2\tan\theta}{\sec^2\theta - 2}$

43. $\sin 4x = 4\sin x\cos x\cos 2x$ 44. $\dfrac{1 + \cos 2x}{\sin 2x} = \cot x$

45. $\dfrac{2\cos 2\theta}{\sin 2\theta} = \cot\theta - \tan\theta$ 46. $\cot 4\theta = \dfrac{1 - \tan^2 2\theta}{2\tan 2\theta}$

47. $\tan x + \cot x = 2\csc 2x$ 48. $\cos 2x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$

49. $1 + \tan x\tan 2x = \sec 2x$ 50. $\dfrac{\cot A - \tan A}{\cot A + \tan A} = \cos 2A$

51. $\sin 2A\cos 2A = \sin 2A - 4\sin^3 A\cos A$

52. $\sin 4x = 4\sin x\cos x - 8\sin^3 x\cos x$

53. $\tan(\theta - 45°) + \tan(\theta + 45°) = 2\tan 2\theta$

54. $\cot\theta\tan(\theta + \pi) - \sin(\pi - \theta)\cos\left(\dfrac{\pi}{2} - \theta\right) = \cos^2\theta$

Write each expression as a sum or difference of trigonometric functions. See Example 7.

55. $2\sin 58°\cos 102°$ 56. $2\cos 85°\sin 140°$ 57. $2\sin\dfrac{\pi}{6}\cos\dfrac{\pi}{3}$

58. $5\cos 3x\cos 2x$ 59. $6\sin 4x\sin 5x$ 60. $8\sin 7x\sin 9x$

Write each expression as a product of trigonometric functions. See Example 8.

61. $\cos 4x - \cos 2x$ 62. $\cos 5x + \cos 8x$ 63. $\sin 25° + \sin(-48°)$

64. $\sin 102° - \sin 95°$ 65. $\cos 4x + \cos 8x$ 66. $\sin 9x - \sin 3x$

(Modeling) *Solve each problem.*

67. *Wattage Consumption* Refer to Example 6. Use an identity to determine values of a, c, and ω so that $W = a\cos(\omega t) + c$. Check your answer by graphing both expressions for W on the same coordinate axes.

68. *Amperage, Wattage, and Voltage* Amperage is a measure of the amount of electricity that is moving through a circuit, whereas voltage is a measure of the force pushing the electricity. The wattage W consumed by an electrical device can be determined by calculating the product of the amperage I and voltage V. (*Source:* Wilcox, G. and C. Hesselberth, *Electricity for Engineering Technology*, Allyn & Bacon, 1970.)

(a) A household circuit has voltage

$$V = 163 \sin 120\pi t$$

when an incandescent light bulb is turned on with amperage

$$I = 1.23 \sin 120\pi t.$$

Graph the wattage $W = VI$ consumed by the light bulb in the window $[0, .05]$ by $[-50, 300]$.

(b) Determine the maximum and minimum wattages used by the light bulb.

(c) Use identities to determine values for a, c, and ω so that $W = a\cos(\omega t) + c$.

(d) Check your answer by graphing both expressions for W on the same coordinate axes.

(e) Use the graph to estimate the average wattage used by the light. For how many watts (to the nearest integer) do you think this incandescent light bulb is rated?

5.6 Half-Angle Identities

Half-Angle Identities ▪ Applying the Half-Angle Identities

Half-Angle Identities

From the alternative forms of the identity for $\cos 2A$, we derive three additional identities for $\sin \frac{A}{2}$, $\cos \frac{A}{2}$, and $\tan \frac{A}{2}$. These are known as **half-angle identities.**

To derive the identity for $\sin \frac{A}{2}$, start with the following double-angle identity for cosine and solve for $\sin x$.

$$\cos 2x = 1 - 2\sin^2 x$$

$$2\sin^2 x = 1 - \cos 2x \quad \text{Add } 2\sin^2 x\text{; subtract } \cos 2x.$$

$$\sin x = \pm\sqrt{\frac{1 - \cos 2x}{2}} \quad \text{Divide by 2; take square roots. (Appendix A)}$$

Remember both the positive and negative square roots.

$$\mathbf{\sin \frac{A}{2} = \pm\sqrt{\frac{1 - \cos A}{2}}} \quad \text{Let } 2x = A\text{, so } x = \tfrac{A}{2}\text{; substitute.}$$

The $\pm$ sign in this identity indicates that the appropriate sign is chosen depending on the quadrant of $\frac{A}{2}$. For example, if $\frac{A}{2}$ is a quadrant III angle, we choose the negative sign since the sine function is negative in quadrant III.

We derive the identity for $\cos \frac{A}{2}$ using the double-angle identity $\cos 2x = 2\cos^2 x - 1$.

$$1 + \cos 2x = 2\cos^2 x \qquad \text{Add 1.}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2} \qquad \text{Rewrite; divide by 2.}$$

$$\cos x = \pm\sqrt{\frac{1 + \cos 2x}{2}} \qquad \text{Take square roots.}$$

$$\mathbf{\cos \frac{A}{2} = \pm\sqrt{\frac{1 + \cos A}{2}}} \qquad \text{Replace } x \text{ with } \tfrac{A}{2}.$$

An identity for $\tan \frac{A}{2}$ comes from the identities for $\sin \frac{A}{2}$ and $\cos \frac{A}{2}$.

$$\mathbf{\tan \frac{A}{2}} = \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} = \frac{\pm\sqrt{\frac{1 - \cos A}{2}}}{\pm\sqrt{\frac{1 + \cos A}{2}}} = \mathbf{\pm\sqrt{\frac{1 - \cos A}{1 + \cos A}}}$$

We derive an alternative identity for $\tan \frac{A}{2}$ using double-angle identities.

$$\tan \frac{A}{2} = \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} = \frac{2 \sin \frac{A}{2} \cos \frac{A}{2}}{2\cos^2 \frac{A}{2}} \qquad \text{Multiply by } 2\cos \tfrac{A}{2} \text{ in numerator and denominator.}$$

$$= \frac{\sin 2\left(\frac{A}{2}\right)}{1 + \cos 2\left(\frac{A}{2}\right)} \qquad \text{Double-angle identities (Section 5.5)}$$

$$\mathbf{\tan \frac{A}{2} = \frac{\sin A}{1 + \cos A}} \qquad \text{Simplify.}$$

From the identity $\tan \frac{A}{2} = \frac{\sin A}{1 + \cos A}$, we can also derive $\mathbf{\tan \frac{A}{2} = \frac{1 - \cos A}{\sin A}}$.

HALF-ANGLE IDENTITIES

$$\cos \frac{A}{2} = \pm\sqrt{\frac{1 + \cos A}{2}} \qquad \sin \frac{A}{2} = \pm\sqrt{\frac{1 - \cos A}{2}}$$

$$\tan \frac{A}{2} = \pm\sqrt{\frac{1 - \cos A}{1 + \cos A}} \qquad \tan \frac{A}{2} = \frac{\sin A}{1 + \cos A} \qquad \tan \frac{A}{2} = \frac{1 - \cos A}{\sin A}$$

▶ **Note** The final two identities for $\tan \frac{A}{2}$ do not require a sign choice. When using the other half-angle identities, select the plus or minus sign according to the quadrant in which $\frac{A}{2}$ terminates. For example, if an angle $A = 324°$, then $\frac{A}{2} = 162°$, which lies in quadrant II. So when $A = 324°$, $\cos \frac{A}{2}$ and $\tan \frac{A}{2}$ are negative, while $\sin \frac{A}{2}$ is positive.

Applying the Half-Angle Identities

▶ EXAMPLE 1 **USING A HALF-ANGLE IDENTITY TO FIND AN EXACT VALUE**

Find the exact value of cos 15° using the half-angle identity for cosine.

Solution

$$\cos 15^\circ = \cos \frac{1}{2}(30^\circ) = \sqrt{\frac{1 + \cos 30^\circ}{2}}$$

Choose the positive square root.

$$= \sqrt{\frac{1 + \frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{\left(1 + \frac{\sqrt{3}}{2}\right) \cdot 2}{2 \cdot 2}} = \frac{\sqrt{2 + \sqrt{3}}}{2}$$

Simplify the radicals.

NOW TRY EXERCISE 11. ◀

▶ EXAMPLE 2 **USING A HALF-ANGLE IDENTITY TO FIND AN EXACT VALUE**

Find the exact value of tan 22.5° using the identity $\tan \frac{A}{2} = \frac{\sin A}{1 + \cos A}$.

Solution Since $22.5^\circ = \frac{1}{2}(45^\circ)$, replace A with 45°.

$$\tan 22.5^\circ = \tan \frac{45^\circ}{2} = \frac{\sin 45^\circ}{1 + \cos 45^\circ} = \frac{\frac{\sqrt{2}}{2}}{1 + \frac{\sqrt{2}}{2}} = \frac{\frac{\sqrt{2}}{2}}{1 + \frac{\sqrt{2}}{2}} \cdot \frac{2}{2}$$

$$= \frac{\sqrt{2}}{2 + \sqrt{2}} = \frac{\sqrt{2}}{2 + \sqrt{2}} \cdot \frac{2 - \sqrt{2}}{2 - \sqrt{2}} = \frac{2\sqrt{2} - 2}{2}$$

Rationalize the denominator.

$$= \frac{2(\sqrt{2} - 1)}{2} = \sqrt{2} - 1$$

Factor out 2.

Factor first; then divide out the common factor.

NOW TRY EXERCISE 13. ◀

▶ EXAMPLE 3 **FINDING FUNCTION VALUES OF $\frac{s}{2}$ GIVEN INFORMATION ABOUT s**

Given $\cos s = \frac{2}{3}$, with $\frac{3\pi}{2} < s < 2\pi$, find $\cos \frac{s}{2}$, $\sin \frac{s}{2}$, and $\tan \frac{s}{2}$.

Solution The angle associated with $\frac{s}{2}$ terminates in quadrant II, since

$$\frac{3\pi}{2} < s < 2\pi$$

and

$$\frac{3\pi}{4} < \frac{s}{2} < \pi.$$

Divide by 2. (Appendix A)

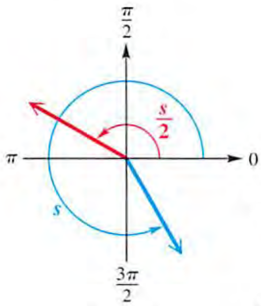


Figure 9

See Figure 9. In quadrant II, the values of $\cos \frac{s}{2}$ and $\tan \frac{s}{2}$ are negative and the value of $\sin \frac{s}{2}$ is positive. Now use the appropriate half-angle identities and simplify the radicals.

$$\sin \frac{s}{2} = \sqrt{\frac{1 - \frac{2}{3}}{2}} = \sqrt{\frac{1}{6}} = \frac{\sqrt{1}}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{\sqrt{6}}{6}$$

$$\cos \frac{s}{2} = -\sqrt{\frac{1 + \frac{2}{3}}{2}} = -\sqrt{\frac{5}{6}} = -\frac{\sqrt{5}}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = -\frac{\sqrt{30}}{6}$$

$$\tan \frac{s}{2} = \frac{\sin \frac{s}{2}}{\cos \frac{s}{2}} = \frac{\frac{\sqrt{6}}{6}}{-\frac{\sqrt{30}}{6}} = \frac{\sqrt{6}}{-\sqrt{30}} = -\frac{\sqrt{6}}{\sqrt{30}} \cdot \frac{\sqrt{30}}{\sqrt{30}} = -\frac{\sqrt{180}}{30} = -\frac{6\sqrt{5}}{6 \cdot 5} = -\frac{\sqrt{5}}{5}$$

Notice that it is not necessary to use a half-angle identity for $\tan \frac{s}{2}$ once we find $\sin \frac{s}{2}$ and $\cos \frac{s}{2}$. However, using this identity would provide an excellent check.

NOW TRY EXERCISE 19. ◀

▶ EXAMPLE 4 SIMPLIFYING EXPRESSIONS USING THE HALF-ANGLE IDENTITIES

Simplify each expression.

(a) $\pm\sqrt{\frac{1 + \cos 12x}{2}}$ **(b)** $\frac{1 - \cos 5\alpha}{\sin 5\alpha}$

Solution

(a) This matches part of the identity for $\cos \frac{A}{2}$. Replace A with $12x$ to get

$$\cos \frac{A}{2} = \pm\sqrt{\frac{1 + \cos A}{2}} = \pm\sqrt{\frac{1 + \cos 12x}{2}} = \cos \frac{12x}{2} = \cos 6x.$$

(b) Use the third identity for $\tan \frac{A}{2}$ given earlier with $A = 5\alpha$ to get

$$\frac{1 - \cos 5\alpha}{\sin 5\alpha} = \tan \frac{5\alpha}{2}.$$

NOW TRY EXERCISES 37 AND 39. ◀

▶ EXAMPLE 5 VERIFYING AN IDENTITY

Verify that the following equation is an identity.

$$\left(\sin \frac{x}{2} + \cos \frac{x}{2}\right)^2 = 1 + \sin x$$

Solution We work on the more complicated left side.

Remember the term $2ab$ when squaring a binomial.

$$\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2$$

$$=\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}+\cos^2\frac{x}{2} \qquad (a+b)^2=a^2+2ab+b^2$$

$$=1+2\sin\frac{x}{2}\cos\frac{x}{2} \qquad \sin^2\tfrac{x}{2}+\cos^2\tfrac{x}{2}=1$$

$$=1+\sin 2\left(\frac{x}{2}\right) \qquad 2\sin\tfrac{x}{2}\cos\tfrac{x}{2}=\sin 2\left(\tfrac{x}{2}\right)$$

$$=1+\sin x \qquad \text{Multiply.}$$

NOW TRY EXERCISE 51.

5.6 Exercises

Concept Check *Determine whether the positive or negative square root should be selected.*

1. $\sin 195° = \pm\sqrt{\dfrac{1-\cos 390°}{2}}$

2. $\cos 58° = \pm\sqrt{\dfrac{1+\cos 116°}{2}}$

3. $\tan 225° = \pm\sqrt{\dfrac{1-\cos 450°}{1+\cos 450°}}$

4. $\sin(-10°) = \pm\sqrt{\dfrac{1-\cos(-20°)}{2}}$

Match each expression in Column I with its value in Column II. See Examples 1 and 2.

I		II	
5. $\sin 15°$	**6.** $\tan 15°$	**A.** $2-\sqrt{3}$	**B.** $\dfrac{\sqrt{2-\sqrt{2}}}{2}$
7. $\cos\dfrac{\pi}{8}$	**8.** $\tan\left(-\dfrac{\pi}{8}\right)$	**C.** $\dfrac{\sqrt{2-\sqrt{3}}}{2}$	**D.** $\dfrac{\sqrt{2+\sqrt{2}}}{2}$
9. $\tan 67.5°$	**10.** $\cos 67.5°$	**E.** $1-\sqrt{2}$	**F.** $1+\sqrt{2}$

Use a half-angle identity to find each exact value. See Examples 1 and 2.

11. $\sin 67.5°$ **12.** $\sin 195°$ **13.** $\cos 195°$

14. $\tan 195°$ **15.** $\cos 165°$ **16.** $\sin 165°$

17. Explain how you could use an identity of this section to find the exact value of $\sin 7.5°$. (*Hint:* $7.5 = \frac{1}{2}\left(\frac{1}{2}\right)(30)$.)

18. The identity $\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$ can be used to find $\tan 22.5° = \sqrt{3-2\sqrt{2}}$, and the identity $\tan\frac{A}{2} = \frac{\sin A}{1+\cos A}$ can be used to find $\tan 22.5° = \sqrt{2}-1$. Show that these answers are the same, without using a calculator. (*Hint:* If $a>0$ and $b>0$ and $a^2=b^2$, then $a=b$.)

Find each of the following. See Example 3.

19. $\cos\dfrac{x}{2}$, given $\cos x = \dfrac{1}{4}$, with $0 < x < \dfrac{\pi}{2}$

20. $\sin \frac{x}{2}$, given $\cos x = -\frac{5}{8}$, with $\frac{\pi}{2} < x < \pi$

21. $\tan \frac{\theta}{2}$, given $\sin \theta = \frac{3}{5}$, with $90° < \theta < 180°$

22. $\cos \frac{\theta}{2}$, given $\sin \theta = -\frac{4}{5}$, with $180° < \theta < 270°$

23. $\sin \frac{x}{2}$, given $\tan x = 2$, with $0 < x < \frac{\pi}{2}$

24. $\cos \frac{x}{2}$, given $\cot x = -3$, with $\frac{\pi}{2} < x < \pi$

25. $\tan \frac{\theta}{2}$, given $\tan \theta = \frac{\sqrt{7}}{3}$, with $180° < \theta < 270°$

26. $\cot \frac{\theta}{2}$, given $\tan \theta = -\frac{\sqrt{5}}{2}$, with $90° < \theta < 180°$

27. $\sin \theta$, given $\cos 2\theta = \frac{3}{5}$ and θ terminates in quadrant I

28. $\cos \theta$, given $\cos 2\theta = \frac{1}{2}$ and θ terminates in quadrant II

29. $\cos x$, given $\cos 2x = -\frac{5}{12}$, with $\frac{\pi}{2} < x < \pi$

30. $\sin x$, given $\cos 2x = \frac{2}{3}$, with $\pi < x < \frac{3\pi}{2}$

31. ***Concept Check*** If $\cos x \approx .9682$ and $\sin x = .25$, then $\tan \frac{x}{2} \approx$ _______.

32. ***Concept Check*** If $\cos x = -.75$ and $\sin x \approx .6614$, then $\tan \frac{x}{2} \approx$ _______.

Use an identity to write each expression as a single trigonometric function. See Example 4.

33. $\sqrt{\frac{1 - \cos 40°}{2}}$

34. $\sqrt{\frac{1 + \cos 76°}{2}}$

35. $\sqrt{\frac{1 - \cos 147°}{1 + \cos 147°}}$

36. $\sqrt{\frac{1 + \cos 165°}{1 - \cos 165°}}$

37. $\frac{1 - \cos 59.74°}{\sin 59.74°}$

38. $\frac{\sin 158.2°}{1 + \cos 158.2°}$

39. $\pm\sqrt{\frac{1 + \cos 18x}{2}}$

40. $\pm\sqrt{\frac{1 + \cos 20\alpha}{2}}$

41. $\pm\sqrt{\frac{1 - \cos 8\theta}{1 + \cos 8\theta}}$

42. $\pm\sqrt{\frac{1 - \cos 5A}{1 + \cos 5A}}$

43. $\pm\sqrt{\frac{1 + \cos \frac{x}{4}}{2}}$

44. $\pm\sqrt{\frac{1 - \cos \frac{3\theta}{5}}{2}}$

Graph each expression and use the graph to make a conjecture as to what might be an identity. Then verify your conjecture algebraically.

45. $\frac{\sin x}{1 + \cos x}$

46. $\frac{1 - \cos x}{\sin x}$

47. $\frac{\tan \frac{x}{2} + \cot \frac{x}{2}}{\cot \frac{x}{2} - \tan \frac{x}{2}}$

48. $1 - 8 \sin^2 \frac{x}{2} \cos^2 \frac{x}{2}$

Verify that each equation is an identity. See Example 5.

49. $\sec^2 \frac{x}{2} = \frac{2}{1 + \cos x}$

50. $\cot^2 \frac{x}{2} = \frac{(1 + \cos x)^2}{\sin^2 x}$

51. $\sin^2 \frac{x}{2} = \frac{\tan x - \sin x}{2 \tan x}$

52. $\frac{\sin 2x}{2 \sin x} = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}$

53. $\frac{2}{1 + \cos x} - \tan^2 \frac{x}{2} = 1$

54. $\tan \frac{\theta}{2} = \csc \theta - \cot \theta$

55. $1 - \tan^2 \frac{\theta}{2} = \frac{2 \cos \theta}{1 + \cos \theta}$

56. $\cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$

57. Use the identity $\tan \frac{A}{2} = \frac{\sin A}{1 + \cos A}$ to derive the equivalent identity $\tan \frac{A}{2} = \frac{1 - \cos A}{\sin A}$ by multiplying both the numerator and denominator by $1 - \cos A$.

(Modeling) Mach Number *An airplane flying faster than sound sends out sound waves that form a cone, as shown in the figure. The cone intersects the ground to form a* ***hyperbola.*** *As this hyperbola passes over a particular point on the ground, a sonic boom is heard at that point. If θ is the angle at the vertex of the cone, then*

$$\sin \frac{\theta}{2} = \frac{1}{m},$$

where m is the Mach number for the speed of the plane. (We assume $m > 1$.) The Mach number is the ratio of the speed of the plane and the speed of sound. Thus, a speed of Mach 1.4 means that the plane is flying at 1.4 times the speed of sound. In Exercises 58–61, one of the values θ or m is given. Find the other value.

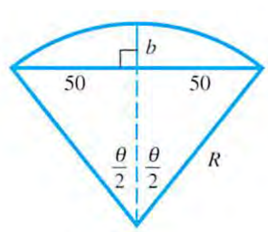
58. $m = \frac{3}{2}$

59. $m = \frac{5}{4}$

60. $\theta = 30°$

61. $\theta = 60°$

62. ***(Modeling) Railroad Curves*** In the United States, circular railroad curves are designated by the **degree of curvature,** the central angle subtended by a chord of 100 ft. See the figure. (*Source:* Hay, W. W., *Railroad Engineering,* John Wiley and Sons, 1982.)



(a) Use the figure to write an expression for $\cos \frac{\theta}{2}$.

(b) Use the result of part (a) and the third half-angle identity for tangent to write an expression for $\tan \frac{\theta}{4}$.

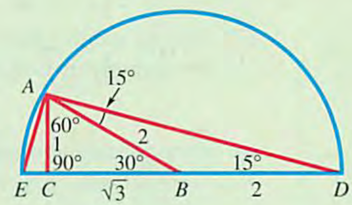
(c) If $b = 12$, what is the measure of angle θ to the nearest degree?

RELATING CONCEPTS

For individual or collaborative investigation
(Exercises 63–70)

These exercises use results from plane geometry, instead of the half-angle formulas, to obtain exact values of the trigonometric functions of 15°. ***Work Exercises 63–70 in order.***

Start with a right triangle ACB having a 60° *angle at A and a* 30° *angle at B. Let the hypotenuse of this triangle have length* 2. *Extend side BC and draw a semicircle with diameter along BC extended, center at B, and radius AB. Draw segment AE. (See the figure.) Since any angle inscribed in a semicircle is a right angle, triangle EAD is a right triangle.*



63. Why does $AB = BD$? Conclude that triangle ABD is isosceles.

64. Why does angle ABD have measure 150°?

65. Why do angles DAB and ADB both have measures of 15°?

66. What is the length DC?

67. Use the Pythagorean theorem to show that the length AD is $\sqrt{6} + \sqrt{2}$. (*Note:* $(\sqrt{6} + \sqrt{2})^2 = 8 + 4\sqrt{3}$.)

68. Use angle ADB of triangle EAD to find cos 15°.

69. Show that AE has length $\sqrt{6} - \sqrt{2}$ and find sin 15°.

70. Use triangle ACD to find tan 15°.

Summary Exercises on Verifying Trigonometric Identities

These summary exercises provide practice with the various types of trigonometric identities presented in this chapter. Verify that each equation is an identity.

1. $\tan\theta + \cot\theta = \sec\theta\csc\theta$

2. $\csc\theta\cos^2\theta + \sin\theta = \csc\theta$

3. $\tan\dfrac{x}{2} = \csc x - \cot x$

4. $\sec(\pi - x) = -\sec x$

5. $\dfrac{\sin t}{1 + \cos t} = \dfrac{1 - \cos t}{\sin t}$

6. $\dfrac{1 - \sin t}{\cos t} = \dfrac{1}{\sec t + \tan t}$

7. $\sin 2\theta = \dfrac{2\tan\theta}{1 + \tan^2\theta}$

8. $\dfrac{2}{1 + \cos x} - \tan^2\dfrac{x}{2} = 1$

9. $\cot\theta - \tan\theta = \dfrac{2\cos^2\theta - 1}{\sin\theta\cos\theta}$

10. $\dfrac{1}{\sec t - 1} + \dfrac{1}{\sec t + 1} = 2\cot t\csc t$

11. $\dfrac{\sin(x + y)}{\cos(x - y)} = \dfrac{\cot x + \cot y}{1 + \cot x\cot y}$

12. $1 - \tan^2\dfrac{\theta}{2} = \dfrac{2\cos\theta}{1 + \cos\theta}$

13. $\dfrac{\sin\theta + \tan\theta}{1 + \cos\theta} = \tan\theta$

14. $\csc^4 x - \cot^4 x = \dfrac{1 + \cos^2 x}{1 - \cos^2 x}$

15. $\cos x = \dfrac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$

16. $\cos 2x = \dfrac{2 - \sec^2 x}{\sec^2 x}$

17. $\dfrac{\tan^2 t + 1}{\tan t \csc^2 t} = \tan t$

18. $\dfrac{\sin s}{1 + \cos s} + \dfrac{1 + \cos s}{\sin s} = 2 \csc s$

19. $\tan 4\theta = \dfrac{2 \tan 2\theta}{2 - \sec^2 2\theta}$

20. $\tan\left(\dfrac{x}{2} + \dfrac{\pi}{4}\right) = \sec x + \tan x$

21. $\dfrac{\cot s - \tan s}{\cos s + \sin s} = \dfrac{\cos s - \sin s}{\sin s \cos s}$

22. $\dfrac{\tan\theta - \cot\theta}{\tan\theta + \cot\theta} = 1 - 2\cos^2\theta$

23. $\dfrac{\tan(x + y) - \tan y}{1 + \tan(x + y)\tan y} = \tan x$

24. $2\cos^2 \dfrac{x}{2} \tan x = \tan x + \sin x$

25. $\dfrac{\cos^4 x - \sin^4 x}{\cos^2 x} = 1 - \tan^2 x$

26. $\dfrac{\csc t + 1}{\csc t - 1} = (\sec t + \tan t)^2$

27. $\dfrac{2(\sin x - \sin^3 x)}{\cos x} = \sin 2x$

28. $\dfrac{1}{2}\cot\dfrac{x}{2} - \dfrac{1}{2}\tan\dfrac{x}{2} = \cot x$

29. $\sin(60° + x) + \sin(60° - x) = \sqrt{3}\cos x$

30. $\sin(60° - x) - \sin(60° + x) = -\sin x$

31. $\dfrac{\cos(x + y) + \cos(y - x)}{\sin(x + y) - \sin(y - x)} = \cot x$

32. $\sin x + \sin 3x + \sin 5x + \sin 7x = 4\cos x \cos 2x \sin 4x$

33. $\sin^3\theta + \cos^3\theta + \sin\theta\cos^2\theta + \sin^2\theta\cos\theta = \sin\theta + \cos\theta$

34. $\dfrac{\cos x + \sin x}{\cos x - \sin x} - \dfrac{\cos x - \sin x}{\cos x + \sin x} = 2\tan 2x$

Chapter 5 Summary

QUICK REVIEW

CONCEPTS	EXAMPLES

5.1 Fundamental Identities

Reciprocal Identities

$$\cot\theta = \frac{1}{\tan\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

Negative-Angle Identities

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$
$$\csc(-\theta) = -\csc\theta \qquad \sec(-\theta) = \sec\theta \qquad \cot(-\theta) = -\cot\theta$$

If θ is in quadrant IV and $\sin\theta = -\frac{3}{5}$, find $\csc\theta$, $\cos\theta$, and $\sin(-\theta)$.

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{-\frac{3}{5}} = -\frac{5}{3}$$

$$\sin^2\theta + \cos^2\theta = 1$$
$$\left(-\frac{3}{5}\right)^2 + \cos^2\theta = 1$$
$$\cos^2\theta = 1 - \frac{9}{25} = \frac{16}{25}$$
$$\cos\theta = +\sqrt{\frac{16}{25}} = \frac{4}{5}$$

$\cos\theta$ is positive in quadrant IV.

$$\sin(-\theta) = -\sin\theta = \frac{3}{5}$$

5.2 Verifying Trigonometric Identities

See the box titled Hints for Verifying Identities on page 207.

5.3 Sum and Difference Identities for Cosine

5.4 Sum and Difference Identities for Sine and Tangent

Cofunction Identities

$$\cos(90° - \theta) = \sin\theta \qquad \cot(90° - \theta) = \tan\theta$$
$$\sin(90° - \theta) = \cos\theta \qquad \sec(90° - \theta) = \csc\theta$$
$$\tan(90° - \theta) = \cot\theta \qquad \csc(90° - \theta) = \sec\theta$$

Find a value of θ such that $\tan\theta = \cot 78°$.

$$\tan\theta = \cot 78°$$
$$\cot(90° - \theta) = \cot 78°$$
$$90° - \theta = 78°$$
$$\theta = 12°$$

Sum and Difference Identities

$$\cos(A - B) = \cos A\cos B + \sin A\sin B$$
$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$
$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$
$$\sin(A - B) = \sin A\cos B - \cos A\sin B$$
$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$
$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

Find the exact value of $\cos(-15°)$.

$$\cos(-15°) = \cos(30° - 45°)$$
$$= \cos 30°\cos 45° + \sin 30°\sin 45°$$
$$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + \frac{1}{2}\cdot\frac{\sqrt{2}}{2}$$
$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

Write $\tan\left(\frac{\pi}{4} + \theta\right)$ in terms of $\tan\theta$.

$$\tan\left(\frac{\pi}{4} + \theta\right) = \frac{\tan\frac{\pi}{4} + \tan\theta}{1 - \tan\frac{\pi}{4}\tan\theta} = \frac{1 + \tan\theta}{1 - \tan\theta} \qquad \tan\frac{\pi}{4} = 1$$

CONCEPTS | EXAMPLES

5.5 Double-Angle Identities

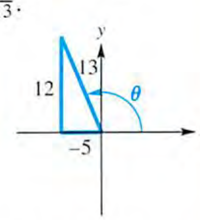
Double-Angle Identities

$$\cos 2A = \cos^2 A - \sin^2 A \qquad \cos 2A = 1 - 2\sin^2 A$$
$$\cos 2A = 2\cos^2 A - 1 \qquad \sin 2A = 2 \sin A \cos A$$
$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Given $\cos \theta = -\frac{5}{13}$ and $\sin \theta > 0$, find $\sin 2\theta$.

Sketch a triangle in quadrant II since $\cos \theta < 0$ and $\sin \theta > 0$. Use it to find that $\sin \theta = \frac{12}{13}$.

$$\sin 2\theta = 2 \sin \theta \cos \theta$$
$$= 2\left(\frac{12}{13}\right)\left(-\frac{5}{13}\right) = -\frac{120}{169}$$



Product-to-Sum Identities

$$\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$$
$$\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$$
$$\sin A \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$$
$$\cos A \sin B = \frac{1}{2}[\sin(A + B) - \sin(A - B)]$$

Write $\sin(-\theta) \sin 2\theta$ as the difference of two functions.

$$\sin(-\theta)\sin 2\theta = \frac{1}{2}[\cos(-\theta - 2\theta) - \cos(-\theta + 2\theta)]$$
$$= \frac{1}{2}[\cos(-3\theta) - \cos \theta]$$
$$= \frac{1}{2}\cos(-3\theta) - \frac{1}{2}\cos\theta$$
$$= \frac{1}{2}\cos 3\theta - \frac{1}{2}\cos\theta$$

Sum-to-Product Identities

$$\sin A + \sin B = 2 \sin\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right)$$
$$\sin A - \sin B = 2 \cos\left(\frac{A + B}{2}\right)\sin\left(\frac{A - B}{2}\right)$$
$$\cos A + \cos B = 2 \cos\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right)$$
$$\cos A - \cos B = -2 \sin\left(\frac{A + B}{2}\right)\sin\left(\frac{A - B}{2}\right)$$

Write $\cos\theta + \cos 3\theta$ as a product of two functions.

$$\cos\theta + \cos 3\theta = 2\cos\left(\frac{\theta + 3\theta}{2}\right)\cos\left(\frac{\theta - 3\theta}{2}\right)$$
$$= 2\cos\left(\frac{4\theta}{2}\right)\cos\left(\frac{-2\theta}{2}\right)$$
$$= 2\cos 2\theta \cos(-\theta)$$
$$= 2\cos 2\theta\cos\theta$$

5.6 Half-Angle Identities

Half-Angle Identities

$$\cos\frac{A}{2} = \pm\sqrt{\frac{1 + \cos A}{2}} \qquad \sin\frac{A}{2} = \pm\sqrt{\frac{1 - \cos A}{2}}$$
$$\tan\frac{A}{2} = \pm\sqrt{\frac{1 - \cos A}{1 + \cos A}} \qquad \tan\frac{A}{2} = \frac{\sin A}{1 + \cos A}$$
$$\tan\frac{A}{2} = \frac{1 - \cos A}{\sin A}$$

(In the first three identities, the sign is chosen based on the quadrant of $\frac{A}{2}$.)

Find the exact value of $\tan 67.5°$.

We choose the last form with $A = 135°$.

$$\tan 67.5° = \tan\frac{135°}{2} = \frac{1 - \cos 135°}{\sin 135°} = \frac{1 - \left(-\frac{\sqrt{2}}{2}\right)}{\frac{\sqrt{2}}{2}}$$
$$= \frac{1 + \frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} \cdot \frac{2}{2} = \frac{2 + \sqrt{2}}{\sqrt{2}} \quad \text{or} \quad \sqrt{2} + 1$$

Rationalize the denominator; simplify.

CHAPTER 5 ▶ Review Exercises

Concept Check *For each expression in Column I, choose the expression from Column II that completes an identity.*

I		II	
1. $\sec x =$ ______	2. $\csc x =$ ______	A. $\dfrac{1}{\sin x}$	B. $\dfrac{1}{\cos x}$
3. $\tan x =$ ______	4. $\cot x =$ ______	C. $\dfrac{\sin x}{\cos x}$	D. $\dfrac{1}{\cot^2 x}$
5. $\tan^2 x =$ ______	6. $\sec^2 x =$ ______	E. $\dfrac{1}{\cos^2 x}$	F. $\dfrac{\cos x}{\sin x}$

Use identities to write each expression in terms of $\sin\theta$ *and* $\cos\theta$, *and simplify.*

7. $\sec^2\theta - \tan^2\theta$ **8.** $\dfrac{\cot\theta}{\sec\theta}$ **9.** $\tan^2\theta(1+\cot^2\theta)$

10. $\csc\theta + \cot\theta$ **11.** $\tan\theta - \sec\theta\csc\theta$ **12.** $\csc^2\theta + \sec^2\theta$

13. Use the trigonometric identities to find $\sin x$, $\tan x$, and $\cot(-x)$, given $\cos x = \frac{3}{5}$ and x in quadrant IV.

14. Given $\tan x = -\frac{5}{4}$, where $\frac{\pi}{2} < x < \pi$, use the trigonometric identities to find $\cot x$, $\csc x$, and $\sec x$.

15. Find the exact values of the six trigonometric functions of 165°.

16. Find the exact values of $\sin x$, $\cos x$, and $\tan x$, for $x = \frac{\pi}{12}$, using

(a) difference identities **(b)** half-angle identities.

Concept Check *For each expression in Column I, use an identity to choose an expression from Column II with the same value.*

I		II	
17. $\cos 210°$	18. $\sin 35°$	A. $\sin(-35°)$	B. $\cos 55°$
19. $\tan(-35°)$	20. $-\sin 35°$	C. $\sqrt{\dfrac{1+\cos 150°}{2}}$	D. $2\sin 150°\cos 150°$
21. $\cos 35°$	22. $\cos 75°$	E. $\cos 150°\cos 60° - \sin 150°\sin 60°$	
23. $\sin 75°$	24. $\sin 300°$	F. $\cot(-35°)$	
25. $\cos 300°$	26. $\cos(-55°)$	G. $\cos^2 150° - \sin^2 150°$	
		H. $\sin 15°\cos 60° + \cos 15°\sin 60°$	
		I. $\cos(-35°)$	J. $\cot 125°$

For each of the following, find $\sin(x+y)$, $\cos(x-y)$, $\tan(x+y)$, *and the quadrant of* $x+y$.

27. $\sin x = -\dfrac{3}{5}$, $\cos y = -\dfrac{7}{25}$, x and y in quadrant III

28. $\sin x = \dfrac{3}{5}$, $\cos y = \dfrac{24}{25}$, x in quadrant I, y in quadrant IV

29. $\sin x = -\dfrac{1}{2}$, $\cos y = -\dfrac{2}{5}$, x and y in quadrant III

30. $\sin y = -\dfrac{2}{3}$, $\cos x = -\dfrac{1}{5}$, x in quadrant II, y in quadrant III

31. $\sin x = \frac{1}{10}$, $\cos y = \frac{4}{5}$, x in quadrant I, y in quadrant IV

32. $\cos x = \frac{2}{9}$, $\sin y = -\frac{1}{2}$, x in quadrant IV, y in quadrant III

Find sine and cosine of each of the following.

33. θ, given $\cos 2\theta = -\frac{3}{4}$, $90° < 2\theta < 180°$

34. B, given $\cos 2B = \frac{1}{8}$, B in quadrant IV

35. $2x$, given $\tan x = 3$, $\sin x < 0$

36. $2y$, given $\sec y = -\frac{5}{3}$, $\sin y > 0$

Find each of the following.

37. $\cos \frac{\theta}{2}$, given $\cos \theta = -\frac{1}{2}$, with $90° < \theta < 180°$

38. $\sin \frac{A}{2}$, given $\cos A = -\frac{3}{4}$, with $90° < A < 180°$

39. $\tan x$, given $\tan 2x = 2$, with $\pi < x < \frac{3\pi}{2}$

40. $\sin y$, given $\cos 2y = -\frac{1}{3}$, with $\frac{\pi}{2} < y < \pi$

41. $\tan \frac{x}{2}$, given $\sin x = .8$, with $0 < x < \frac{\pi}{2}$

42. $\sin 2x$, given $\sin x = .6$, with $\frac{\pi}{2} < x < \pi$

Graph each expression and use the graph to make a conjecture as to what might be an identity. Then verify your conjecture algebraically.

43. $-\frac{\sin 2x + \sin x}{\cos 2x - \cos x}$

44. $\frac{1 - \cos 2x}{\sin 2x}$

45. $\frac{\sin x}{1 - \cos x}$

46. $\frac{\cos x \sin 2x}{1 + \cos 2x}$

47. $\frac{2(\sin x - \sin^3 x)}{\cos x}$

48. $\csc x - \cot x$

Verify that each equation is an identity.

49. $\sin^2 x - \sin^2 y = \cos^2 y - \cos^2 x$

50. $2\cos^3 x - \cos x = \frac{\cos^2 x - \sin^2 x}{\sec x}$

51. $\frac{\sin^2 x}{2 - 2\cos x} = \cos^2 \frac{x}{2}$

52. $\frac{\sin 2x}{\sin x} = \frac{2}{\sec x}$

53. $2\cos A - \sec A = \cos A - \frac{\tan A}{\csc A}$

54. $\frac{2\tan B}{\sin 2B} = \sec^2 B$

55. $1 + \tan^2 \alpha = 2\tan \alpha \csc 2\alpha$

56. $\frac{2\cot x}{\tan 2x} = \csc^2 x - 2$

57. $\tan \theta \sin 2\theta = 2 - 2\cos^2 \theta$

58. $\csc A \sin 2A - \sec A = \cos 2A \sec A$

59. $2\tan x \csc 2x - \tan^2 x = 1$

60. $2\cos^2 \theta - 1 = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}$

61. $\tan \theta \cos^2 \theta = \frac{2\tan \theta \cos^2 \theta - \tan \theta}{1 - \tan^2 \theta}$

62. $\sec^2 \alpha - 1 = \frac{\sec 2\alpha - 1}{\sec 2\alpha + 1}$

63. $\dfrac{\sin^2 x - \cos^2 x}{\csc x} = 2\sin^3 x - \sin x$

64. $\sin^3 \theta = \sin \theta - \cos^2 \theta \sin \theta$

65. $\tan 4\theta = \dfrac{2 \tan 2\theta}{2 - \sec^2 2\theta}$

66. $2\cos^2 \dfrac{x}{2} \tan x = \tan x + \sin x$

67. $\tan\left(\dfrac{x}{2} + \dfrac{\pi}{4}\right) = \sec x + \tan x$

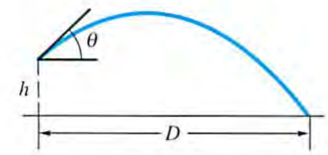
68. $\dfrac{1}{2} \cot \dfrac{x}{2} - \dfrac{1}{2} \tan \dfrac{x}{2} = \cot x$

69. $-\cot \dfrac{x}{2} = \dfrac{\sin 2x + \sin x}{\cos 2x - \cos x}$

70. $\dfrac{\sin 3t + \sin 2t}{\sin 3t - \sin 2t} = \dfrac{\tan \frac{5t}{2}}{\tan \frac{t}{2}}$

(Modeling) *Solve each problem.*

71. ***Distance Traveled by a Stone*** The distance D of an object thrown (or projected) from height h (feet) at angle θ with initial velocity v is modeled by the formula



$$D = \frac{v^2 \sin \theta \cos \theta + v \cos \theta \sqrt{(v \sin \theta)^2 + 64h}}{32}.$$

See the figure. (*Source:* Kreighbaum, E. and K. Barthels, *Biomechanics,* Allyn & Bacon, 1996.) Also see the **Chapter 2** Quantitative Reasoning.

(a) Find D when $h = 0$; that is, when the object is projected from the ground.

(b) Suppose a car driving over loose gravel kicks up a small stone at a velocity of 36 ft per sec (about 25 mph) and an angle $\theta = 30°$. How far will the stone travel?

72. ***Amperage, Wattage, and Voltage*** Suppose that for an electric heater, voltage is given by $V = a \sin 2\pi\omega t$ and amperage by $I = b \sin 2\pi\omega t$, where t is time in seconds.

(a) Find the period of the graph for the voltage.

(b) Show that the graph of the wattage $W = VI$ will have half the period of the voltage. Interpret this result.

CHAPTER 5 ▶ Test

1. If $\cos \theta = \frac{24}{25}$ and θ is in quadrant IV, find the five remaining trigonometric function values of θ.

2. Express $\sec \theta - \sin \theta \tan \theta$ as a single function of θ.

3. Express $\tan^2 x - \sec^2 x$ in terms of $\sin x$ and $\cos x$, and simplify.

4. Find the exact value of $\cos \frac{5\pi}{12}$.

5. Express as a function of x alone.

(a) $\cos(270° - x)$ (b) $\tan(\pi + x)$

6. Use a half-angle identity to find the exact value of $\sin(-22.5°)$.

7. Graph $y = \cot \frac{1}{2}x - \cot x$, and make a conjecture as to what might be an identity. Then verify your conjecture algebraically.

8. Given that $\sin A = \frac{5}{13}$, $\cos B = -\frac{3}{5}$, A is a quadrant I angle, and B is a quadrant II angle, find each of the following.

(a) $\sin(A + B)$ (b) $\cos(A + B)$ (c) $\tan(A - B)$

(d) the quadrant of $A + B$

9. Given that $\cos \theta = -\frac{3}{5}$ and $\frac{\pi}{2} < \theta < \pi$, find each of the following.

(a) $\cos 2\theta$ (b) $\sin 2\theta$ (c) $\tan 2\theta$ (d) $\cos \dfrac{1}{2}\theta$ (e) $\tan \dfrac{1}{2}\theta$

Verify each identity.

10. $\sec^2 B = \dfrac{1}{1 - \sin^2 B}$

11. $\tan^2 x - \sin^2 x = (\tan x \sin x)^2$

12. $\dfrac{\tan x - \cot x}{\tan x + \cot x} = 2\sin^2 x - 1$

13. $\cos 2A = \dfrac{\cot A - \tan A}{\csc A \sec A}$

14. $\dfrac{\sin 2x}{\cos 2x + 1} = \tan x$

15. ***(Modeling) Voltage*** The voltage in common household current is expressed as $V = 163 \sin \omega t$, where ω is the angular speed (in radians per second) of the generator at the electrical plant and t is time (in seconds).

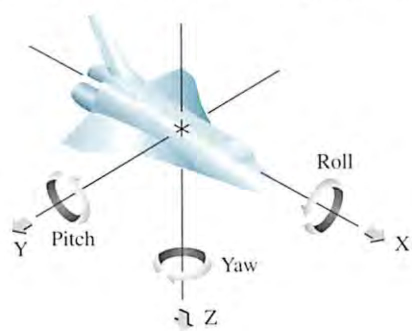
(a) Use an identity to express V in terms of cosine.

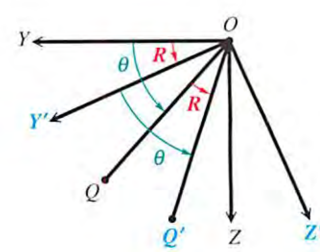
(b) If $\omega = 120\pi$, what is the maximum voltage? Give the least positive value of t when the maximum voltage occurs.

CHAPTER 5 ▶ Quantitative Reasoning

How does NASA calculate rotation and roll of a spacecraft?

With the spectacular success of the Mars explorers *Spirit* and *Opportunity,* interest in space travel has been revived. The drawing on the left below shows the three quantities that determine the motion of a spacecraft. A conventional three-dimensional spacecraft coordinate system is shown on the right below.





Angle $YOQ = \theta$ and $OQ = r$. The coordinates of Q are (x, y, z), where

$$y = r\cos\theta \qquad \text{and} \qquad z = r\sin\theta.$$

When the spacecraft performs a rotation, it is necessary to find the coordinates in the spacecraft system after the rotation takes place. For example, suppose the spacecraft undergoes roll through angle R. The coordinates (x, y, z) of point Q become (x', y', z'), the coordinates of the corresponding point Q'. In the new reference system, $OQ' = r$ and, since the roll is around the x-axis and angle $Y'OQ' = YOQ = \theta$,

$$x' = x, \qquad y' = r\cos(\theta + R), \qquad \text{and} \qquad z' = r\sin(\theta + R).$$

Use the cosine and sine sum identities to write expressions for y' and z' in terms of y, z, and R. (*Source:* Kastner, B., *Space Mathematics,* NASA.)