6 Inverse Circular Functions and Trigonometric Equations

On the night of October 20, 1955, a new sound exploded from a stage at Brooklyn High School in Cleveland, Ohio. The concert showcased rising stars in the music industry, including Elvis Presley and Bill Haley, and ushered in a new era of popular music—*rock and roll*. The distinctive melodic phrases, or *riffs*, generated by the musicians' electric guitars defined the new genre. (*Source:* Rock and Roll Hall of Fame.)

Sound waves, such as those initiated by musical instruments, travel in sinusoidal patterns that can be graphed as sine or cosine functions and described by trigonometric equations. When sound waves are combined and organized to have rhythm, melody, harmony, and dynamics, the brain interprets them as music.

In Section 6.2, Example 6, and Section 6.3, Example 5, various aspects of music are analyzed using trigonometric equations and graphs.

6.1 Inverse Circular Functions

Inverse Functions ▪ Inverse Sine Function ▪ Inverse Cosine Function ▪ Inverse Tangent Function ▪ Remaining Inverse Circular Functions ▪ Inverse Function Values

Inverse Functions ***Recall that for a function f, every element x in the domain corresponds to one and only one element y, or $f(x)$, in the range.*** (Appendix C) This means the following:

1. If point (a, b) lies on the graph of f, then there is no other point on the graph that has a as first coordinate.
2. Other points may have b as second coordinate, however, since the definition of function allows range elements to be used more than once.

If a function is defined so that ***each range element is used only once,*** then it is called a **one-to-one function.** For example, the function

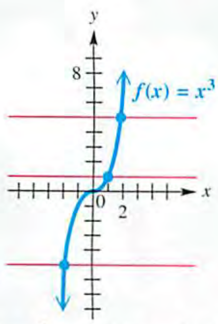
$$f(x) = x^3 \text{ is a one-to-one function}$$

because every real number has exactly one real cube root. On the other hand,

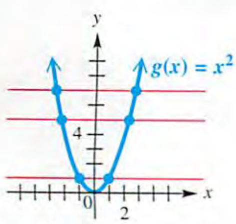
$$g(x) = x^2 \text{ is not a one-to-one function}$$

because, for example, $g(2) = 4$ and $g(-2) = 4$. There are two domain elements, 2 and -2, that correspond to the range element 4.

The **horizontal line test** helps determine graphically whether a function is one-to-one.



$f(x) = x^3$ is a one-to-one function. It satisfies the conditions of the horizontal line test.



$g(x) = x^2$ is not one-to-one. It does not satisfy the conditions of the horizontal line test.

Figure 1

HORIZONTAL LINE TEST

Any horizontal line will intersect the graph of a one-to-one function in at most one point.

This test is applied to the graphs of $f(x) = x^3$ and $g(x) = x^2$ in Figure 1.

By interchanging the components of the ordered pairs of a one-to-one function f, we obtain a new set of ordered pairs that satisfies the definition of function. This new function, called the *inverse function,* or *inverse,* of f, is symbolized f^{-1}.

INVERSE FUNCTION

The **inverse function** of the one-to-one function f is defined as

$$f^{-1} = \{(y, x) \mid (x, y) \text{ belongs to } f\}.$$

▶ **Caution** ***The -1 in $f^{-1}(x)$ is not an exponent.*** That is,

$$f^{-1}(x) \neq \frac{1}{f(x)}.$$

The following statements summarize our discussion of inverse functions.

SUMMARY OF INVERSE FUNCTIONS

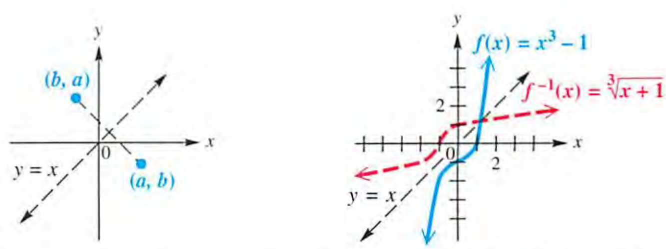
1. In a one-to-one function, each x-value corresponds to only one y-value and each y-value corresponds to only one x-value.
2. If a function f is one-to-one, then f has an inverse function f^{-1}.
3. The domain of f is the range of f^{-1}, and the range of f is the domain of f^{-1}.
4. The graphs of f and f^{-1} are reflections of each other across the line $y = x$.
5. To find $f^{-1}(x)$ from $f(x)$, follow these steps.

   Step 1 Replace $f(x)$ with y and interchange x and y.

   Step 2 Solve for y.

   Step 3 Replace y with $f^{-1}(x)$.

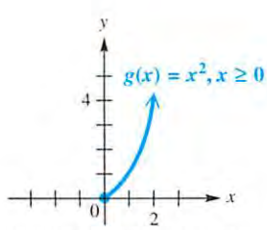
Figure 2 illustrates some of these concepts.



(b, a) is the reflection of (a, b) across the line $y = x$.

The graph of f^{-1} is the reflection of the graph of f across the line $y = x$.

Figure 2



This is a one-to-one function.

Figure 3

We often restrict the domain of a function that is not one-to-one to make it one-to-one, without changing the range. For example, we saw in Figure 1 that $g(x) = x^2$, with its natural domain $(-\infty, \infty)$, is not one-to-one. However, if we restrict its domain to the set of nonnegative numbers $[0, \infty)$, we obtain a new function f that is one-to-one and has the same range as g, $[0, \infty)$. See Figure 3.

▶ **Note** We could have chosen to restrict the domain of $g(x) = x^2$ to $(-\infty, 0]$ to obtain a different one-to-one function. For the trigonometric functions, such choices are made based on general agreement by mathematicians.

▼ **LOOKING AHEAD TO CALCULUS**

The **inverse circular functions** are used in calculus to solve certain types of related-rates problems and to integrate certain rational functions.

Inverse Sine Function

Refer to the graph of the sine function in Figure 4 on the next page. Applying the horizontal line test, we see that $y = \sin x$ does not define a one-to-one function. If we restrict the domain to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, which is the part of the graph in Figure 4 shown in color, this restricted function is one-to-one and has an inverse function. The range of $y = \sin x$ is $[-1, 1]$, so the domain of the inverse function will be $[-1, 1]$, and its range will be $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

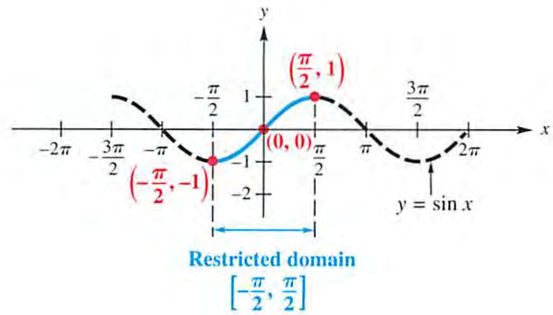


Figure 4

Reflecting the graph of $y = \sin x$ on the restricted domain, shown in Figure 5(a), across the line $y = x$ gives the graph of the inverse function, shown in Figure 5(b). Some key points are labeled on the graph. The equation of the inverse of $y = \sin x$ is found by interchanging x and y to get

$$x = \sin y.$$

This equation is solved for y by writing

$$\boldsymbol{y = \sin^{-1} x} \quad \text{(read \textbf{“inverse sine of } \boldsymbol{x}\textbf{”}).}$$

As Figure 5(b) shows, the domain of $y = \sin^{-1} x$ is $[-1, 1]$, while the restricted domain of $y = \sin x$, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, is the range of $y = \sin^{-1} x$. An alternative notation for $\sin^{-1} x$ is $\arcsin x$.

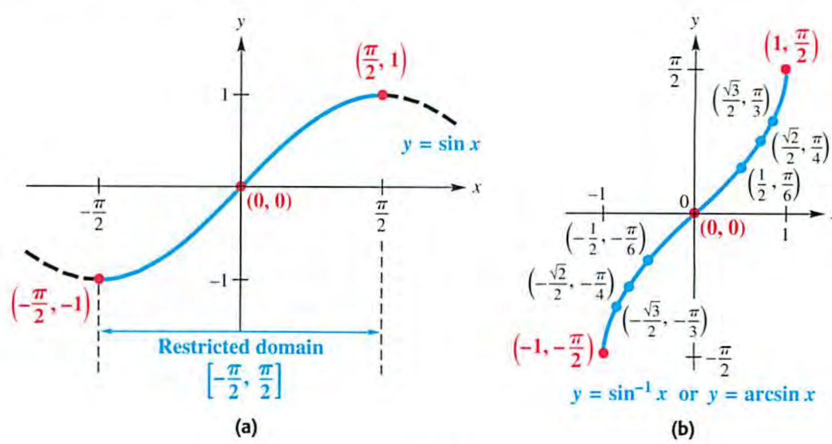


Figure 5

INVERSE SINE FUNCTION

$\boldsymbol{y = \sin^{-1} x}$ or $\boldsymbol{y = \arcsin x}$ means that $x = \sin y$, for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

We can think of $\boldsymbol{y = \sin^{-1} x}$ *or* $\boldsymbol{y = \arcsin x}$ *as*

“y is the number in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ *whose sine is x.”*

Thus, we can write $y = \sin^{-1} x$ as $\sin y = x$ to evaluate it. We must pay close attention to the domain and range intervals.

▶ EXAMPLE 1 FINDING INVERSE SINE VALUES

Find y in each equation.

(a) $y = \arcsin \dfrac{1}{2}$ **(b)** $y = \sin^{-1}(-1)$ **(c)** $y = \sin^{-1}(-2)$

Algebraic Solution

(a) The graph of the function defined by $y = \arcsin x$ (Figure 5(b)) includes the point $\left(\frac{1}{2}, \frac{\pi}{6}\right)$. Therefore, $\arcsin \frac{1}{2} = \frac{\pi}{6}$.

Alternatively, we can think of $y = \arcsin \frac{1}{2}$ as "y is the number in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is $\frac{1}{2}$." Then we can write the given equation as $\sin y = \frac{1}{2}$. Since $\sin \frac{\pi}{6} = \frac{1}{2}$ and $\frac{\pi}{6}$ is in the range of the arcsine function, $y = \frac{\pi}{6}$.

(b) Writing the equation $y = \sin^{-1}(-1)$ in the form $\sin y = -1$ shows that $y = -\frac{\pi}{2}$. Notice that the point $\left(-1, -\frac{\pi}{2}\right)$ is on the graph of $y = \sin^{-1} x$.

(c) Because -2 is not in the domain of the inverse sine function, $\sin^{-1}(-2)$ does not exist.

Graphing Calculator Solution

We graph $Y_1 = \sin^{-1} X$ and find the points with X-values $\frac{1}{2} = .5$ and -1. For these X-values, Figure 6 shows that $Y = \frac{\pi}{6} \approx .52359878$ and $Y = -\frac{\pi}{2} \approx -1.570796$.

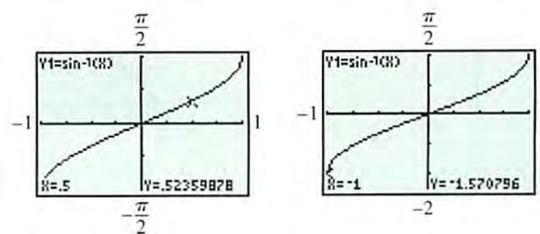


Figure 6

Since $\sin^{-1}(-2)$ does not exist, a calculator will give an error message for this input.

NOW TRY EXERCISES 13 AND 23. ◀

▶ **Caution** In Example 1(b), it is tempting to give the value of $\sin^{-1}(-1)$ as $\frac{3\pi}{2}$, since $\sin \frac{3\pi}{2} = -1$. Notice, however, that $\frac{3\pi}{2}$ is not in the range of the inverse sine function. ***Be certain that the number given for an inverse function value is in the range of the particular inverse function being considered.***

INVERSE SINE FUNCTION $y = \sin^{-1} x$ OR $y = \arcsin x$

Domain: $[-1, 1]$ Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

x	y
-1	$-\frac{\pi}{2}$
$-\frac{\sqrt{2}}{2}$	$-\frac{\pi}{4}$
0	0
$\frac{\sqrt{2}}{2}$	$\frac{\pi}{4}$
1	$\frac{\pi}{2}$

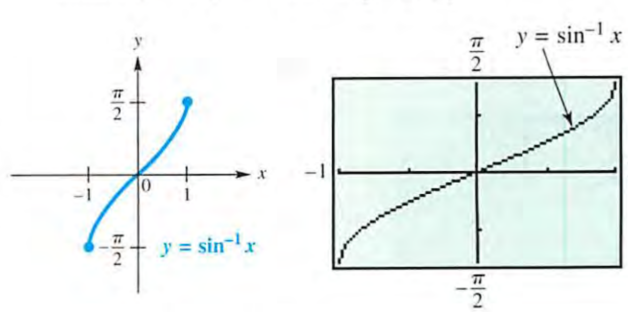


Figure 7

- The inverse sine function is increasing and continuous on its domain $[-1, 1]$.
- Its x-intercept is 0, and its y-intercept is 0.
- Its graph is symmetric with respect to the origin; it is an odd function.

Inverse Cosine Function The function $y = \cos^{-1} x$ (or $y = \arccos x$) is defined by restricting the domain of the function $y = \cos x$ to the interval $[0, \pi]$ as in Figure 8. This restricted function, which is the part of the graph in Figure 8 shown in color, is one-to-one and has an inverse function. The inverse function, $y = \cos^{-1} x$, is found by interchanging the roles of x and y. Reflecting the graph of $y = \cos x$ across the line $y = x$ gives the graph of the inverse function shown in Figure 9. Again, some key points are shown on the graph.

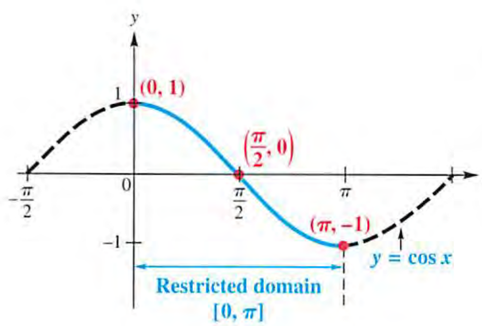


Figure 8

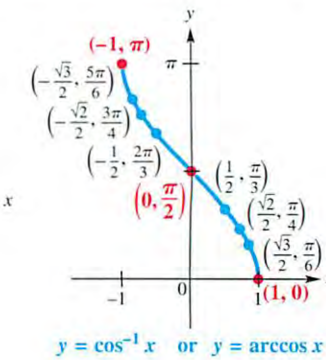


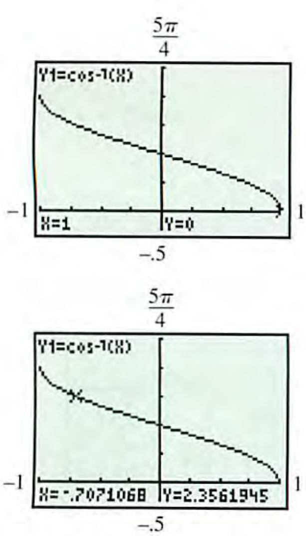
Figure 9

INVERSE COSINE FUNCTION

$y = \cos^{-1} x$ or $y = \arccos x$ means that $x = \cos y$, for $0 \le y \le \pi$.

We can think of $y = \cos^{-1} x$ *or* $y = \arccos x$ *as*

"y is the number in the interval $[0, \pi]$ *whose cosine is x."*



These screens support the results of Example 2, since $-\frac{\sqrt{2}}{2} \approx -.7071068$ and $\frac{3\pi}{4} \approx 2.3561945$.

▶ EXAMPLE 2 FINDING INVERSE COSINE VALUES

Find y in each equation.

(a) $y = \arccos 1$ **(b)** $y = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

Solution

(a) Since the point $(1, 0)$ lies on the graph of $y = \arccos x$ in Figure 9, the value of y is 0. Alternatively, we can think of $y = \arccos 1$ as

"y is the number in $[0, \pi]$ whose cosine is 1," or $\cos y = 1$.

Thus, $y = 0$, since $\cos 0 = 1$ and 0 is in the range of the arccosine function.

(b) We must find the value of y that satisfies $\cos y = -\frac{\sqrt{2}}{2}$, where y is in the interval $[0, \pi]$, the range of the function $y = \cos^{-1} x$. The only value for y that satisfies these conditions is $\frac{3\pi}{4}$. Again, this can be verified from the graph in Figure 9.

NOW TRY EXERCISES 15 AND 25. ◀

Our observations about the inverse cosine function lead to the following generalizations.

INVERSE COSINE FUNCTION $y = \cos^{-1} x$ OR $y = \arccos x$

Domain: $[-1, 1]$ Range: $[0, \pi]$

x	y
-1	π
$-\frac{\sqrt{2}}{2}$	$\frac{3\pi}{4}$
0	$\frac{\pi}{2}$
$\frac{\sqrt{2}}{2}$	$\frac{\pi}{4}$
1	0

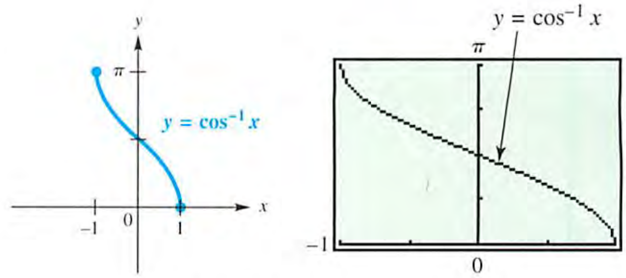


Figure 10

- The inverse cosine function is decreasing and continuous on its domain $[-1, 1]$.
- Its x-intercept is 1, and its y-intercept is $\frac{\pi}{2}$.
- Its graph is neither symmetric with respect to the y-axis nor the origin.

Inverse Tangent Function Restricting the domain of the function $y = \tan x$ to the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ yields a one-to-one function. By interchanging the roles of x and y, we obtain the inverse tangent function given by $y = \tan^{-1} x$ or $y = \arctan x$. Figure 11 shows the graph of the restricted tangent function. Figure 12 gives the graph of $y = \tan^{-1} x$.

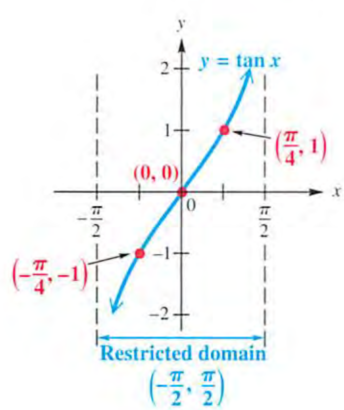


Figure 11

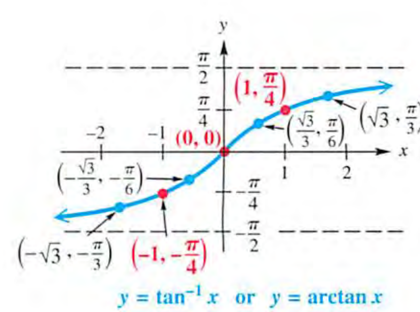


Figure 12

INVERSE TANGENT FUNCTION

$y = \tan^{-1} x$ or $y = \arctan x$ means that $x = \tan y$, for $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

INVERSE TANGENT FUNCTION
$y = \tan^{-1} x$ OR $y = \arctan x$

Domain: $(-\infty, \infty)$ Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

x	y
-1	$-\frac{\pi}{4}$
$-\frac{\sqrt{3}}{3}$	$-\frac{\pi}{6}$
0	0
$\frac{\sqrt{3}}{3}$	$\frac{\pi}{6}$
1	$\frac{\pi}{4}$

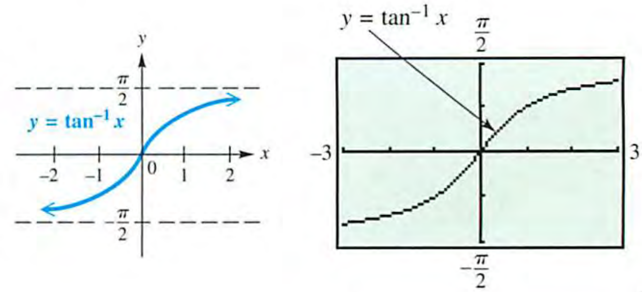
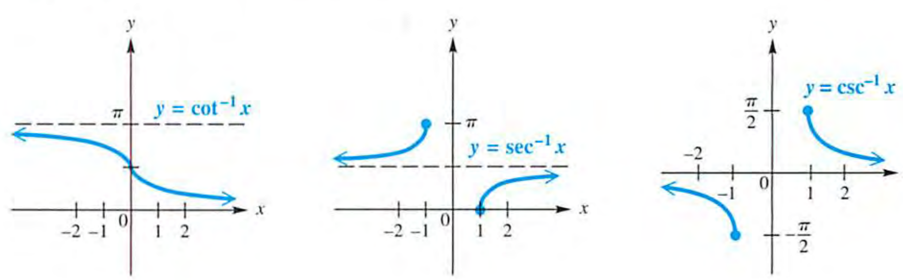


Figure 13

- The inverse tangent function is increasing and continuous on its domain $(-\infty, \infty)$.
- Its x-intercept is 0, and its y-intercept is 0.
- Its graph is symmetric with respect to the origin; it is an odd function.
- The lines $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$ are horizontal asymptotes.

Remaining Inverse Circular Functions The remaining three inverse trigonometric functions are defined similarly; their graphs are shown in Figure 14.



(a) **(b)** **(c)**

Figure 14

INVERSE COTANGENT, SECANT, AND COSECANT FUNCTIONS*

$\mathbf{y = \cot^{-1} x}$ or $\mathbf{y = \text{arccot}\, x}$ means that $x = \cot y$, for $0 < y < \pi$.

$\mathbf{y = \sec^{-1} x}$ or $\mathbf{y = \text{arcsec}\, x}$ means that $x = \sec y$, for $0 \le y \le \pi$, $y \ne \frac{\pi}{2}$.

$\mathbf{y = \csc^{-1} x}$ or $\mathbf{y = \text{arccsc}\, x}$ means that $x = \csc y$, for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $y \ne 0$.

*The inverse secant and inverse cosecant functions are sometimes defined with different ranges. We use intervals that match their reciprocal functions (except for one missing point).

The table gives all six inverse trigonometric functions with their domains and ranges.

Inverse Function	Domain	Range	
		Interval	Quadrants of the Unit Circle
$y = \sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	I and IV
$y = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$	I and II
$y = \tan^{-1} x$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	I and IV
$y = \cot^{-1} x$	$(-\infty, \infty)$	$(0, \pi)$	I and II
$y = \sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi], y \neq \frac{\pi}{2}$	I and II
$y = \csc^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right], y \neq 0$	I and IV

Inverse Function Values The inverse circular functions are formally defined with real number ranges. However, there are times when it may be convenient to find degree-measured angles equivalent to these real number values. It is also often convenient to think in terms of the unit circle and choose the inverse function values based on the quadrants given in the preceding table.

▶ EXAMPLE 3 FINDING INVERSE FUNCTION VALUES (DEGREE-MEASURED ANGLES)

Find the *degree measure* of θ in the following.

(a) $\theta = \arctan 1$ **(b)** $\theta = \sec^{-1} 2$

Solution

(a) Here θ must be in $(-90°, 90°)$, but since 1 is positive, θ must be in quadrant I. The alternative statement, $\tan \theta = 1$, leads to $\theta = 45°$.

(b) Write the equation as $\sec \theta = 2$. For $\sec^{-1} x$, θ is in quadrant I or II. Because 2 is positive, θ is in quadrant I and $\theta = 60°$, since $\sec 60° = 2$. Note that $60°$ $\left(\text{the degree equivalent of } \frac{\pi}{3}\right)$ is in the range of the inverse secant function.

NOW TRY EXERCISES 35 AND 41. ◀

The inverse trigonometric function keys on a calculator give results in the proper quadrant for the inverse sine, inverse cosine, and inverse tangent functions, according to the definitions of these functions. For example, on a calculator, in degrees, $\sin^{-1} .5 = 30°$, $\sin^{-1}(-.5) = -30°$, $\tan^{-1}(-1) = -45°$, and $\cos^{-1}(-.5) = 120°$.

Finding $\cot^{-1} x$, $\sec^{-1} x$, and $\csc^{-1} x$ with a calculator is not as straightforward, because these functions must be expressed in terms of $\tan^{-1} x$, $\cos^{-1} x$, and $\sin^{-1} x$, respectively. If $y = \sec^{-1} x$, for example, then $\sec y = x$, which must be written as a cosine function as follows:

$$\textbf{If } \sec y = x, \text{ then } \frac{1}{\cos y} = x \quad \text{or} \quad \cos y = \frac{1}{x}, \quad \text{and} \quad y = \cos^{-1} \frac{1}{x}.$$

In summary, to find $\sec^{-1} x$, we find $\cos^{-1} \frac{1}{x}$. Similar statements apply to $\csc^{-1} x$ and $\cot^{-1} x$. There is one additional consideration with $\cot^{-1} x$. Since we take the inverse tangent of the reciprocal to find inverse cotangent, the calculator gives values of inverse cotangent with the same range as inverse tangent, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, which is not the correct range for inverse cotangent. For inverse cotangent, the proper range must be considered and the results adjusted accordingly.

▶ EXAMPLE 4 FINDING INVERSE FUNCTION VALUES WITH A CALCULATOR

(a) Find y in radians if $y = \csc^{-1}(-3)$.

(b) Find θ in degrees if $\theta = \text{arccot}(-.3541)$.

Solution

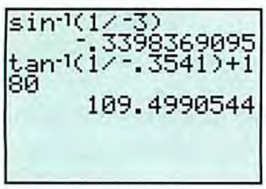


Figure 15

(a) With the calculator in radian mode, enter $\csc^{-1}(-3)$ as $\sin^{-1}\left(-\frac{1}{3}\right)$ to get $y \approx -.3398369095$. See Figure 15.

(b) Now set the calculator to degree mode. A calculator gives the inverse tangent value of a negative number as a quadrant IV angle. The restriction on the range of arccotangent implies that θ must be in quadrant II, so enter

$$\text{arccot}(-.3541) \quad \text{as} \quad \tan^{-1}\left(\frac{1}{-.3541}\right) + 180°.$$

As shown in Figure 15, $\theta \approx 109.4990544°$.

NOW TRY EXERCISES 51 AND 59. ◀

> ▶ **Caution** ***Be careful when using your calculator to evaluate the inverse cotangent of a negative quantity.*** To do this, we must enter the inverse tangent of the *reciprocal* of the negative quantity, which returns an angle in quadrant IV. Since inverse cotangent is negative in quadrant II, adjust your calculator result by adding 180° or π accordingly.

▶ EXAMPLE 5 FINDING FUNCTION VALUES USING DEFINITIONS OF THE TRIGONOMETRIC FUNCTIONS

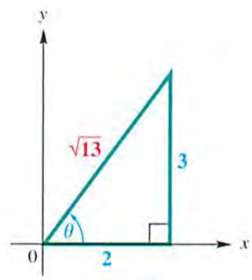
Evaluate each expression without using a calculator.

(a) $\sin\left(\tan^{-1}\frac{3}{2}\right)$ **(b)** $\tan\left(\cos^{-1}\left(-\frac{5}{13}\right)\right)$

Solution

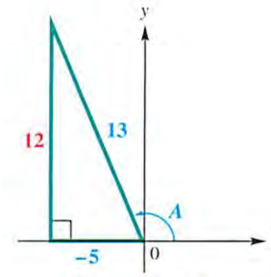
(a) Let $\theta = \tan^{-1}\frac{3}{2}$, so $\tan\theta = \frac{3}{2}$. The inverse tangent function yields values only in quadrants I and IV, and since $\frac{3}{2}$ is positive, θ is in quadrant I. Sketch θ in quadrant I, and label a triangle, as shown in Figure 16 on the next page. By the Pythagorean theorem, the hypotenuse is $\sqrt{13}$. The value of sine is the quotient of the side opposite and the hypotenuse, so

$$\sin\left(\tan^{-1}\frac{3}{2}\right) = \sin\theta = \frac{3}{\sqrt{13}} = \frac{3}{\sqrt{13}} \cdot \frac{\sqrt{13}}{\sqrt{13}} = \frac{3\sqrt{13}}{13}. \quad \text{(Section 2.1)}$$



$\theta = \tan^{-1} \frac{3}{2}$

Figure 16



$A = \cos^{-1}\left(-\frac{5}{13}\right)$

Figure 17

(b) Let $A = \cos^{-1}\left(-\frac{5}{13}\right)$. Then, $\cos A = -\frac{5}{13}$. Since $\cos^{-1} x$ for a negative value of x is in quadrant II, sketch A in quadrant II, as shown in Figure 17.

$$\tan\left(\cos^{-1}\left(-\frac{5}{13}\right)\right) = \tan A = -\frac{12}{5}$$

NOW TRY EXERCISES 77 AND 79.

▶ EXAMPLE 6 FINDING FUNCTION VALUES USING IDENTITIES

Evaluate each expression without using a calculator.

(a) $\cos\left(\arctan \sqrt{3} + \arcsin \frac{1}{3}\right)$ **(b)** $\tan\left(2 \arcsin \frac{2}{5}\right)$

Solution

(a) Let $A = \arctan \sqrt{3}$ and $B = \arcsin \frac{1}{3}$, so $\tan A = \sqrt{3}$ and $\sin B = \frac{1}{3}$. Sketch both A and B in quadrant I, as shown in Figure 18. Now, use the cosine sum identity.

$$\cos(A + B) = \cos A \cos B - \sin A \sin B \quad \text{(Section 5.3)}$$

$$\cos\left(\arctan \sqrt{3} + \arcsin \frac{1}{3}\right) = \cos\left(\arctan \sqrt{3}\right) \cos\left(\arcsin \frac{1}{3}\right) - \sin\left(\arctan \sqrt{3}\right) \sin\left(\arcsin \frac{1}{3}\right) \quad (1)$$

From Figure 18,

$$\cos\left(\arctan \sqrt{3}\right) = \cos A = \frac{1}{2}, \quad \cos\left(\arcsin \frac{1}{3}\right) = \cos B = \frac{2\sqrt{2}}{3},$$

$$\sin\left(\arctan \sqrt{3}\right) = \sin A = \frac{\sqrt{3}}{2}, \quad \sin\left(\arcsin \frac{1}{3}\right) = \sin B = \frac{1}{3}.$$

Substitute these values into equation (1) to get

$$\cos\left(\arctan \sqrt{3} + \arcsin \frac{1}{3}\right) = \frac{1}{2} \cdot \frac{2\sqrt{2}}{3} - \frac{\sqrt{3}}{2} \cdot \frac{1}{3} = \frac{2\sqrt{2} - \sqrt{3}}{6}.$$

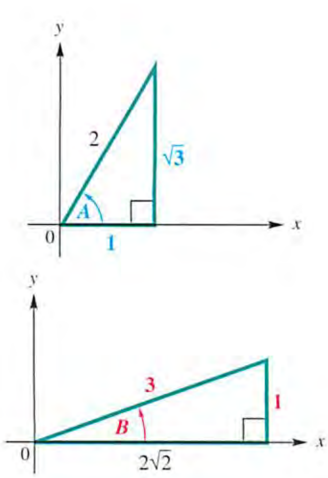


Figure 18

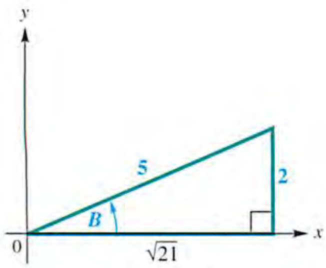


Figure 19

(b) Let $\arcsin \frac{2}{5} = B$. Then, from the double-angle tangent identity,

$$\tan\left(2 \arcsin \frac{2}{5}\right) = \tan 2B = \frac{2 \tan B}{1 - \tan^2 B}. \quad \text{(Section 5.5)}$$

Since $\arcsin \frac{2}{5} = B$, $\sin B = \frac{2}{5}$. Sketch a triangle in quadrant I, find the length of the third side, and then find $\tan B$. From the triangle in Figure 19, $\tan B = \frac{2}{\sqrt{21}}$, and

$$\tan\left(2 \arcsin \frac{2}{5}\right) = \frac{2\left(\frac{2}{\sqrt{21}}\right)}{1 - \left(\frac{2}{\sqrt{21}}\right)^2} = \frac{\frac{4}{\sqrt{21}}}{1 - \frac{4}{21}} = \frac{\frac{4}{\sqrt{21}} \cdot \frac{\sqrt{21}}{\sqrt{21}}}{\frac{17}{21}} = \frac{\frac{4\sqrt{21}}{21}}{\frac{17}{21}} = \frac{4\sqrt{21}}{17}.$$

Be careful simplifying the complex fraction.

NOW TRY EXERCISES 81 AND 89.

While the work shown in Examples 5 and 6 does not rely on a calculator, we can support our algebraic work with one. By entering $\cos\left(\arctan \sqrt{3} + \arcsin \frac{1}{3}\right)$ from Example 6(a) into a calculator, we get the approximation .1827293862, the same approximation as when we enter $\frac{2\sqrt{2} - \sqrt{3}}{6}$ (the exact value we obtained algebraically). Similarly, we obtain the same approximation when we evaluate $\tan\left(2 \arcsin \frac{2}{5}\right)$ and $\frac{4\sqrt{21}}{17}$, supporting our answer in Example 6(b).

EXAMPLE 7 WRITING FUNCTION VALUES IN TERMS OF *u*

Write each trigonometric expression as an algebraic expression in u.

(a) $\sin(\tan^{-1} u)$ **(b)** $\cos(2 \sin^{-1} u)$

Solution

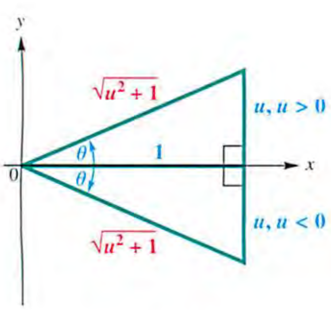


Figure 20

(a) Let $\theta = \tan^{-1} u$, so $\tan \theta = u$. Here, u may be positive or negative. Since $-\frac{\pi}{2} < \tan^{-1} u < \frac{\pi}{2}$, sketch θ in quadrants I and IV and label two triangles, as shown in Figure 20. Since sine is given by the quotient of the side opposite and the hypotenuse,

$$\sin(\tan^{-1} u) = \sin \theta = \frac{u}{\sqrt{u^2 + 1}} = \frac{u}{\sqrt{u^2 + 1}} \cdot \frac{\sqrt{u^2 + 1}}{\sqrt{u^2 + 1}} = \frac{u\sqrt{u^2 + 1}}{u^2 + 1}.$$

Rationalize the denominator.

The result is positive when u is positive and negative when u is negative.

(b) Let $\theta = \sin^{-1} u$, so $\sin \theta = u$. To find $\cos 2\theta$, use the double-angle identity $\cos 2\theta = 1 - 2 \sin^2 \theta$.

$$\cos(2 \sin^{-1} u) = \cos 2\theta = 1 - 2 \sin^2 \theta = 1 - 2u^2 \quad \text{(Section 5.5)}$$

NOW TRY EXERCISES 97 AND 101.

▶ EXAMPLE 8 **FINDING THE OPTIMAL ANGLE OF ELEVATION OF A SHOT PUT**

The optimal angle of elevation θ a shot-putter should aim for to throw the greatest distance depends on the velocity v of the throw and the initial height h of the shot. See Figure 21. One model for θ that achieves this greatest distance is

$$\theta = \arcsin\left(\sqrt{\frac{v^2}{2v^2 + 64h}}\right).$$

(*Source:* Townend, M. S., *Mathematics in Sport,* Chichester, Ellis Horwood Limited, 1984.)

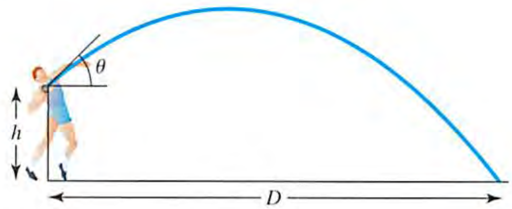


Figure 21

Suppose a shot-putter can consistently throw the steel ball with $h = 6.6$ ft and $v = 42$ ft per sec. At what angle should he throw the ball to maximize distance?

Solution To find this angle, substitute and use a calculator in degree mode.

$$\theta = \arcsin\left(\sqrt{\frac{42^2}{2(42^2) + 64(6.6)}}\right) \approx 42^\circ \quad h = 6.6,\ v = 42$$

NOW TRY EXERCISE 107. ◀

6.1 Exercises

Concept Check *Complete each statement.*

1. For a function to have an inverse, it must be ________.
2. The domain of $y = \arcsin x$ equals the ________ of $y = \sin x$.
3. The range of $y = \cos^{-1} x$ equals the ________ of $y = \cos x$.
4. The point $\left(\frac{\pi}{4}, 1\right)$ lies on the graph of $y = \tan x$. Therefore, the point ________ lies on the graph of ________.
5. If a function f has an inverse and $f(\pi) = -1$, then $f^{-1}(-1) =$ ________.
6. How can the graph of f^{-1} be sketched if the graph of f is known?

Concept Check *In Exercises 7–10, write short answers.*

7. Consider the inverse sine function, defined by $y = \sin^{-1} x$ or $y = \arcsin x$.
   (a) What is its domain? (b) What is its range?
   (c) Is this function increasing or decreasing?
   (d) Why is $\arcsin(-2)$ not defined?

8. Consider the inverse cosine function, defined by $y = \cos^{-1} x$ or $y = \arccos x$.

(a) What is its domain? **(b)** What is its range?

(c) Is this function increasing or decreasing?

(d) $\text{Arccos}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$. Why is $\arccos\left(-\frac{1}{2}\right)$ not equal to $-\frac{4\pi}{3}$?

9. Consider the inverse tangent function, defined by $y = \tan^{-1} x$ or $y = \arctan x$.

(a) What is its domain? **(b)** What is its range?

(c) Is this function increasing or decreasing?

(d) Is there any real number x for which $\arctan x$ is not defined? If so, what is it (or what are they)?

10. Give the domain and range of the three other inverse trigonometric functions, as defined in this section.

(a) inverse cosecant function **(b)** inverse secant function

(c) inverse cotangent function

11. ***Concept Check*** Is $\sec^{-1} a$ calculated as $\cos^{-1} \frac{1}{a}$ or as $\frac{1}{\cos^{-1} a}$?

12. ***Concept Check*** For positive values of a, $\cot^{-1} a$ is calculated as $\tan^{-1} \frac{1}{a}$. How is $\cot^{-1} a$ calculated for negative values of a?

Find the exact value of each real number y. Do not use a calculator. See Examples 1 and 2.

13. $y = \sin^{-1} 0$ **14.** $y = \tan^{-1} 1$ **15.** $y = \cos^{-1}(-1)$

16. $y = \arctan(-1)$ **17.** $y = \sin^{-1}(-1)$ **18.** $y = \cos^{-1} \dfrac{1}{2}$

19. $y = \arctan 0$ **20.** $y = \arcsin\left(-\dfrac{\sqrt{3}}{2}\right)$ **21.** $y = \arccos 0$

22. $y = \tan^{-1}(-1)$ **23.** $y = \sin^{-1} \dfrac{\sqrt{2}}{2}$ **24.** $y = \cos^{-1}\left(-\dfrac{1}{2}\right)$

25. $y = \arccos\left(-\dfrac{\sqrt{3}}{2}\right)$ **26.** $y = \arcsin\left(-\dfrac{\sqrt{2}}{2}\right)$ **27.** $y = \cot^{-1}(-1)$

28. $y = \sec^{-1}\left(-\sqrt{2}\right)$ **29.** $y = \csc^{-1}(-2)$ **30.** $y = \text{arccot}\left(-\sqrt{3}\right)$

31. $y = \text{arcsec} \dfrac{2\sqrt{3}}{3}$ **32.** $y = \csc^{-1} \sqrt{2}$ **33.** $y = \sec^{-1} 1$

34. ***Concept Check*** Is there a value for y such that $y = \sec^{-1} 0$?

Give the degree measure of θ. Do not use a calculator. See Example 3.

35. $\theta = \arctan(-1)$ **36.** $\theta = \arccos\left(-\dfrac{1}{2}\right)$ **37.** $\theta = \arcsin\left(-\dfrac{\sqrt{3}}{2}\right)$

38. $\theta = \arcsin\left(-\dfrac{\sqrt{2}}{2}\right)$ **39.** $\theta = \cot^{-1}\left(-\dfrac{\sqrt{3}}{3}\right)$ **40.** $\theta = \csc^{-1}(-2)$

41. $\theta = \sec^{-1}(-2)$ **42.** $\theta = \csc^{-1}(-1)$ **43.** $\theta = \tan^{-1} \sqrt{3}$

44. $\theta = \cot^{-1} \dfrac{\sqrt{3}}{3}$ **45.** $\theta = \sin^{-1} 2$ **46.** $\theta = \cos^{-1}(-2)$

Use a calculator to give each value in decimal degrees. See Example 4.

47. $\theta = \sin^{-1}(-.13349122)$ **48.** $\theta = \cos^{-1}(-.13348816)$

49. $\theta = \arccos(-.39876459)$ **50.** $\theta = \arcsin .77900016$

51. $\theta = \csc^{-1} 1.9422833$

52. $\theta = \cot^{-1} 1.7670492$

53. $\theta = \cot^{-1}(-.60724226)$

54. $\theta = \cot^{-1}(-2.7733744)$

55. $\theta = \tan^{-1}(-7.7828641)$

56. $\theta = \sec^{-1}(-5.1180378)$

Use a calculator to give each real number value. (Be sure the calculator is in radian mode.) See Example 4.

57. $y = \arctan 1.1111111$

58. $y = \arcsin .81926439$

59. $y = \cot^{-1}(-.92170128)$

60. $y = \sec^{-1}(-1.2871684)$

61. $y = \arcsin .92837781$

62. $y = \arccos .44624593$

63. $y = \cos^{-1}(-.32647891)$

64. $y = \sec^{-1} 4.7963825$

65. $y = \cot^{-1}(-36.874610)$

66. $y = \cot^{-1}(1.0036571)$

The screen here shows how to define the inverse secant, cosecant, and cotangent functions in order to graph them using a TI-83/84 Plus graphing calculator.

Use this information to graph each inverse circular function and compare your graphs to those in Figure 14.

67. $y = \sec^{-1} x$

68. $y = \csc^{-1} x$

69. $y = \cot^{-1} x$

Graph each inverse circular function by hand.

70. $y = \text{arccsc } 2x$

71. $y = \text{arcsec } \frac{1}{2}x$

72. $y = 2 \cot^{-1} x$

73. ***Concept Check*** Explain why attempting to find $\sin^{-1} 1.003$ on your calculator will result in an error message.

RELATING CONCEPTS

For individual or collaborative investigation
(Exercises 74–76)*

74. Consider the function defined by $f(x) = 3x - 2$ and its inverse $f^{-1}(x) = \frac{1}{3}x + \frac{2}{3}$. Simplify $f(f^{-1}(x))$ and $f^{-1}(f(x))$. What do you notice in each case? What would the graph look like in each case?

75. Use a graphing calculator to graph $y = \tan(\tan^{-1} x)$ in the standard viewing window, using radian mode. How does this compare to the graph you described in Exercise 74?

76. Use a graphing calculator to graph $y = \tan^{-1}(\tan x)$ in the standard viewing window, using radian and dot modes. Why does this graph not agree with the graph you found in Exercise 75?

*The authors wish to thank Carol Walker of Hinds Community College for making a suggestion on which these exercises are based.

Give the exact value of each expression without using a calculator. See Examples 5 and 6.

77. $\tan\left(\arccos \frac{3}{4}\right)$ **78.** $\sin\left(\arccos \frac{1}{4}\right)$ **79.** $\cos(\tan^{-1}(-2))$

80. $\sec\left(\sin^{-1}\left(-\frac{1}{5}\right)\right)$ **81.** $\sin\left(2 \tan^{-1} \frac{12}{5}\right)$ **82.** $\cos\left(2 \sin^{-1} \frac{1}{4}\right)$

83. $\cos\left(2 \arctan \frac{4}{3}\right)$ **84.** $\tan\left(2 \cos^{-1} \frac{1}{4}\right)$ **85.** $\sin\left(2 \cos^{-1} \frac{1}{5}\right)$

86. $\cos(2 \tan^{-1}(-2))$ **87.** $\sec(\sec^{-1} 2)$ **88.** $\csc(\csc^{-1} \sqrt{2})$

89. $\cos\left(\tan^{-1} \frac{5}{12} - \tan^{-1} \frac{3}{4}\right)$ **90.** $\cos\left(\sin^{-1} \frac{3}{5} + \cos^{-1} \frac{5}{13}\right)$

91. $\sin\left(\sin^{-1} \frac{1}{2} + \tan^{-1}(-3)\right)$ **92.** $\tan\left(\cos^{-1} \frac{\sqrt{3}}{2} - \sin^{-1}\left(-\frac{3}{5}\right)\right)$

Use a calculator to find each value. Give answers as real numbers.

93. $\cos(\tan^{-1} .5)$ **94.** $\sin(\cos^{-1} .25)$

95. $\tan(\arcsin .12251014)$ **96.** $\cot(\arccos .58236841)$

Write each expression as an algebraic (nontrigonometric) expression in u, $u > 0$. See Example 7.

97. $\sin(\arccos u)$ **98.** $\tan(\arccos u)$ **99.** $\cos(\arcsin u)$

100. $\cot(\arcsin u)$ **101.** $\sin\left(2 \sec^{-1} \frac{u}{2}\right)$ **102.** $\cos\left(2 \tan^{-1} \frac{3}{u}\right)$

103. $\tan\left(\sin^{-1} \frac{u}{\sqrt{u^2 + 2}}\right)$ **104.** $\sec\left(\cos^{-1} \frac{u}{\sqrt{u^2 + 5}}\right)$

105. $\sec\left(\operatorname{arccot} \frac{\sqrt{4 - u^2}}{u}\right)$ **106.** $\csc\left(\arctan \frac{\sqrt{9 - u^2}}{u}\right)$

(Modeling) *Solve each problem.*

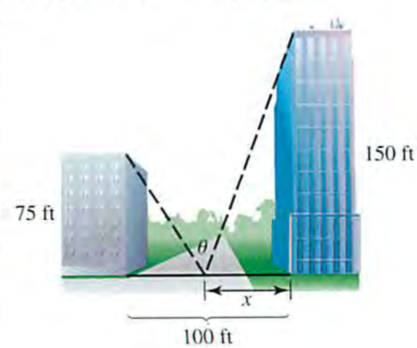
107. ***Angle of Elevation of a Shot Put*** Refer to Example 8.

(a) What is the optimal angle when $h = 0$?

(b) Fix h at 6 ft and regard θ as a function of v. As v gets larger and larger, the graph approaches an asymptote. Find the equation of that asymptote.

108. ***Landscaping Formula*** A shrub is planted in a 100-ft-wide space between buildings measuring 75 ft and 150 ft tall. The location of the shrub determines how much sun it receives each day. Show that if θ is the angle in the figure and x is the distance of the shrub from the taller building, then the value of θ (in radians) is given by

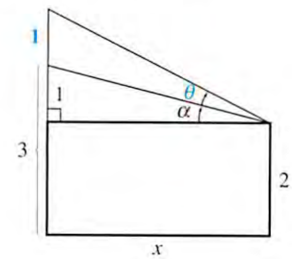
$$\theta = \pi - \arctan\left(\frac{75}{100 - x}\right) - \arctan\left(\frac{150}{x}\right).$$



109. ***Observation of a Painting*** A painting 1 m high and 3 m from the floor will cut off an angle θ to an observer, where

$$\theta = \tan^{-1}\left(\frac{x}{x^2 + 2}\right).$$

Assume that the observer is x meters from the wall where the painting is displayed and that the eyes of the observer are 2 m above the ground. (See the figure.) Find the value of θ for the following values of x. Round to the nearest degree.



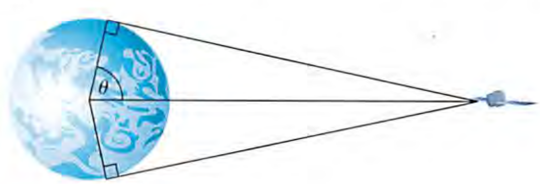
(a) 1 **(b)** 2 **(c)** 3

(d) Derive the formula given above. (*Hint:* Use the identity for $\tan(\theta + \alpha)$. Use right triangles.)

(e) Graph the function for θ with a graphing calculator, and determine the distance that maximizes the angle.

(f) The idea in part (e) was first investigated in 1471 by the astronomer Regiomontanus. (*Source:* Maor, E., *Trigonometric Delights,* Princeton University Press, 1998.) If the bottom of the picture is a meters above eye level and the top of the picture is b meters above eye level, then the optimum value of x is $\sqrt{ab}$ meters. Use this result to find the exact answer to part (e).

110. ***Communications Satellite Coverage*** The figure shows a stationary communications satellite positioned 20,000 mi above the equator. What percent, to the nearest tenth, of the equator can be seen from the satellite? The diameter of Earth is 7927 mi at the equator.



6.2 Trigonometric Equations I

Solving by Linear Methods ▪ Solving by Factoring ▪ Solving by Quadratic Methods ▪ Solving by Using Trigonometric Identities

▼ LOOKING AHEAD TO CALCULUS
There are many instances in calculus where it is necessary to solve trigonometric equations. Examples include solving related-rates problems and optimization problems.

In **Chapter 5,** we studied trigonometric equations that were identities. We now consider trigonometric equations that are *conditional;* that is, equations that are satisfied by some values but not others. (Appendix A)

Solving by Linear Methods

Conditional equations with trigonometric (or circular) functions can usually be solved using algebraic methods and trigonometric identities.

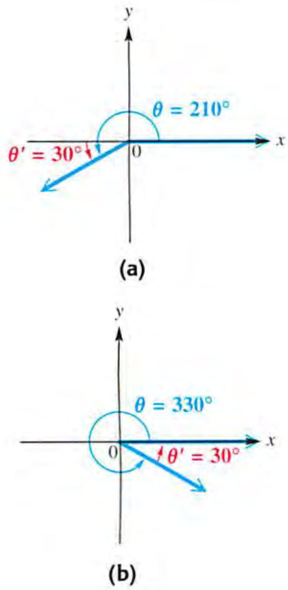


Figure 22

▶ EXAMPLE 1 SOLVING A TRIGONOMETRIC EQUATION BY LINEAR METHODS

Solve $2 \sin \theta + 1 = 0$ over the interval $[0°, 360°)$.

Solution Because $\sin \theta$ is the first power of a trigonometric function, we use the same method as we would to solve the linear equation $2x + 1 = 0$.

$$2 \sin \theta + 1 = 0$$

$$2 \sin \theta = -1 \quad \text{Subtract 1. (Appendix A)}$$

$$\sin \theta = -\frac{1}{2} \quad \text{Divide by 2.}$$

To find values of θ that satisfy $\sin \theta = -\frac{1}{2}$, we observe that θ must be in either quadrant III or IV since the sine function is negative only in these two quadrants. Furthermore, the reference angle must be 30° since $\sin 30° = \frac{1}{2}$. The graphs in Figure 22 show the two possible values of θ, 210° and 330°. The solution set is $\{210°, 330°\}$.

Alternatively, we could determine the solutions by referring to Figure 12 in **Section 3.3** on page 119.

NOW TRY EXERCISE 11. ◀

Solving by Factoring

▶ EXAMPLE 2 SOLVING A TRIGONOMETRIC EQUATION BY FACTORING

Solve $\sin \theta \tan \theta = \sin \theta$ over the interval $[0°, 360°)$.

Solution

$$\sin \theta \tan \theta = \sin \theta$$

$$\sin \theta \tan \theta - \sin \theta = 0 \quad \text{Subtract } \sin \theta.$$

$$\sin \theta(\tan \theta - 1) = 0 \quad \text{Factor out } \sin \theta.$$

$$\sin \theta = 0 \quad \text{or} \quad \tan \theta - 1 = 0 \quad \text{Zero-factor property (Appendix A)}$$

$$\tan \theta = 1$$

$$\theta = 0° \quad \text{or} \quad \theta = 180° \qquad \theta = 45° \quad \text{or} \quad \theta = 225°$$

The solution set is $\{0°, 45°, 180°, 225°\}$.

NOW TRY EXERCISE 31. ◀

▶ **Caution** There are four solutions in Example 2. Trying to solve the equation by dividing each side by $\sin \theta$ would lead to just $\tan \theta = 1$, which would give $\theta = 45°$ or $\theta = 225°$. The other two solutions would not appear. The missing solutions are the ones that make the divisor, $\sin \theta$, equal 0. ***For this reason, we avoid dividing by a variable expression.***

Solving by Quadratic Methods An equation in the form $au^2 + bu + c = 0$, where u is an algebraic expression, is solved by quadratic methods. (Appendix A) The expression u may be a trigonometric function, as in the equation $\tan^2 x + \tan x - 2 = 0$ which we solve in the next example.

▶ EXAMPLE 3 SOLVING A TRIGONOMETRIC EQUATION BY FACTORING

Solve $\tan^2 x + \tan x - 2 = 0$ over the interval $[0, 2\pi)$.

Solution This equation is quadratic in form and can be solved by factoring.

$$\tan^2 x + \tan x - 2 = 0$$

$$(\tan x - 1)(\tan x + 2) = 0 \quad \text{Factor.}$$

$$\tan x - 1 = 0 \quad \text{or} \quad \tan x + 2 = 0 \quad \text{Zero-factor property}$$

$$\tan x = 1 \quad \text{or} \quad \tan x = -2 \quad \text{Solve each equation.}$$

The solutions for $\tan x = 1$ over the interval $[0, 2\pi)$ are $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.

To solve $\tan x = -2$ over that interval, we use a scientific calculator set in *radian* mode. We find that $\tan^{-1}(-2) \approx -1.1071487$. This is a quadrant IV number, based on the range of the inverse tangent function. (Refer to Figure 12 in **Section 3.3** on page 119.) However, since we want solutions over the interval $[0, 2\pi)$, we must first add π to -1.1071487, and then add 2π.

$$x \approx -1.1071487 + \pi \approx 2.0344439$$

$$x \approx -1.1071487 + 2\pi \approx 5.1760366$$

The solutions over the required interval form the solution set

$$\left\{ \underbrace{\frac{\pi}{4}, \frac{5\pi}{4}}_{\text{Exact values}}, \underbrace{2.0344, \quad 5.1760}_{\text{Approximate values to four decimal places}} \right\}.$$

NOW TRY EXERCISE 21. ◀

▶ EXAMPLE 4 SOLVING A TRIGONOMETRIC EQUATION USING THE QUADRATIC FORMULA

Find all solutions of $\cot x(\cot x + 3) = 1$. Write the solution set.

Solution We multiply the factors on the left and subtract 1 to get the equation in standard quadratic form.

$$\cot^2 x + 3 \cot x - 1 = 0 \quad \text{(Appendix A)}$$

Since this equation cannot be solved by factoring, we use the quadratic formula, with $a = 1$, $b = 3$, $c = -1$, and $\cot x$ as the variable.

$$\cot x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{Quadratic formula (Appendix A)}$$

$$= \frac{-3 \pm \sqrt{3^2 - 4(1)(-1)}}{2(1)} \quad a = 1, b = 3, c = -1$$

Be careful with signs.

$$= \frac{-3 \pm \sqrt{9 + 4}}{2} \quad \text{Simplify.}$$

$$= \frac{-3 \pm \sqrt{13}}{2}$$

$$\cot x \approx -3.302775638 \quad \text{or} \quad \cot x \approx .3027756377 \quad \text{Use a calculator.}$$

We cannot find inverse cotangent values directly on a calculator, so we use the fact that $\cot x = \frac{1}{\tan x}$, and take reciprocals to get

$$\tan x \approx \frac{1}{-3.302775638} \quad \text{or} \quad \tan x \approx \frac{1}{.3027756377}$$
$$\tan x \approx -.3027756377 \quad \text{or} \quad \tan x \approx 3.302775638$$
$$x \approx -.2940013018 \quad \text{or} \quad x \approx 1.276795025.$$

To find *all* solutions, we add integer multiples of the period of the tangent function, which is π, to each solution found previously. Thus, the solution set of the equation is written as

$$\{-.2940 + n\pi, 1.2768 + n\pi, \text{ where } n \text{ is any integer}\}.$$

NOW TRY EXERCISE 43.

Solving by Using Trigonometric Identities

Recall that squaring both sides of an equation, such as $\sqrt{x+4} = x + 2$, will yield all solutions but may also give extraneous values. (In this equation, 0 is a solution, while -3 is extraneous. Verify this.)

▶ EXAMPLE 5 SOLVING A TRIGONOMETRIC EQUATION BY SQUARING

Solve $\tan x + \sqrt{3} = \sec x$ over the interval $[0, 2\pi)$.

Solution Since the tangent and secant functions are related by the identity $1 + \tan^2 x = \sec^2 x$, square both sides and express $\sec^2 x$ in terms of $\tan^2 x$.

$$(\tan x + \sqrt{3})^2 = (\sec x)^2$$

Don't forget the middle term.

$$\tan^2 x + 2\sqrt{3}\tan x + 3 = \sec^2 x \qquad (x+y)^2 = x^2 + 2xy + y^2$$

$$\tan^2 x + 2\sqrt{3}\tan x + 3 = 1 + \tan^2 x \qquad \text{Pythagorean identity (Section 5.1)}$$

$$2\sqrt{3}\tan x = -2 \qquad \text{Subtract } 3 + \tan^2 x.$$

$$\tan x = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3} \qquad \text{Divide by } 2\sqrt{3}\text{; rationalize the denominator.}$$

The possible solutions are $\frac{5\pi}{6}$ and $\frac{11\pi}{6}$. Now check them. Try $\frac{5\pi}{6}$ first.

Left side: $\tan x + \sqrt{3} = \tan\frac{5\pi}{6} + \sqrt{3} = -\frac{\sqrt{3}}{3} + \sqrt{3} = \frac{2\sqrt{3}}{3}$

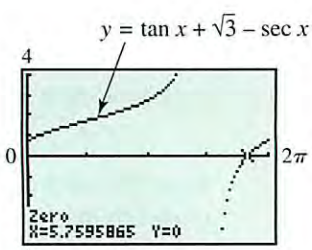
Right side: $\sec x = \sec\frac{5\pi}{6} = -\frac{2\sqrt{3}}{3}$ ← Not equal

The check shows that $\frac{5\pi}{6}$ is not a solution. Now check $\frac{11\pi}{6}$.

Left side: $\tan\frac{11\pi}{6} + \sqrt{3} = -\frac{\sqrt{3}}{3} + \sqrt{3} = \frac{2\sqrt{3}}{3}$

Right side: $\sec\frac{11\pi}{6} = \frac{2\sqrt{3}}{3}$ ← Equal

This solution satisfies the equation, so $\left\{\frac{11\pi}{6}\right\}$ is the solution set.



Dot mode; radian mode

The graph shows that on the interval $[0, 2\pi)$, the only x-intercept of the graph of $y = \tan x + \sqrt{3} - \sec x$ is 5.7595865, which is an approximation for $\frac{11\pi}{6}$, the solution found in Example 5.

NOW TRY EXERCISE 41.

Methods for solving trigonometric equations can be summarized as follows.

SOLVING A TRIGONOMETRIC EQUATION

1. Decide whether the equation is linear or quadratic in form, so you can determine the solution method.
2. If only one trigonometric function is present, solve the equation for that function.
3. If more than one trigonometric function is present, rearrange the equation so that one side equals 0. Then try to factor and set each factor equal to 0 to solve.
4. If the equation is quadratic in form, but not factorable, use the quadratic formula. Check that solutions are in the desired interval.
5. Try using identities to change the form of the equation. It may be helpful to square both sides of the equation first. If this is done, check for extraneous solutions.

▶ EXAMPLE 6 DESCRIBING A MUSICAL TONE FROM A GRAPH

A basic component of music is a pure tone. The graph in Figure 23 models the sinusoidal pressure $y = P$ in pounds per square foot from a pure tone at time $x = t$ in seconds.

(a) The frequency of a pure tone is often measured in hertz. One hertz is equal to one cycle per second and is abbreviated Hz. What is the frequency f in hertz of the pure tone shown in the graph?

(b) The time for the tone to produce one complete cycle is called the **period.** Approximate the period T in seconds of the pure tone.

(c) An equation for the graph is $y = .004 \sin 300\pi x$. Use a calculator to estimate all solutions to the equation that make $y = .004$ over the interval $[0, .02]$.

Solution

(a) From the graph in Figure 23, we see that there are 6 cycles in .04 sec. This is equivalent to $\frac{6}{.04} = 150$ cycles per sec. The pure tone has a frequency of $f = 150$ Hz.

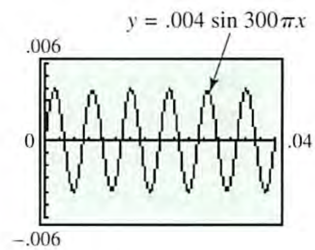


Figure 23

(b) Six periods cover a time of .04 sec. One period would be equal to $T = \frac{.04}{6} = \frac{1}{150}$, or $.00\overline{6}$ sec.

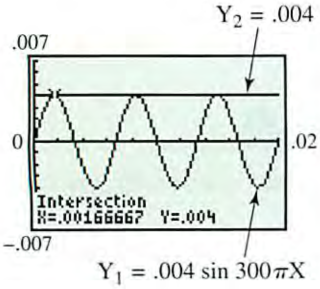


Figure 24

(c) If we reproduce the graph in Figure 23 on a calculator as Y_1 and also graph a second function as $Y_2 = .004$, we can determine that the approximate values of x at the points of intersection of the graphs over the interval $[0, .02]$ are

$$.0017, \quad .0083, \quad \text{and} \quad .015.$$

The first value is shown in Figure 24. These values represent time in seconds.

NOW TRY EXERCISE 53.

6.2 Exercises

Concept Check *Refer to the summary box on solving a trigonometric equation on page 277. Decide on the appropriate technique to begin the solution of each equation. Do not solve the equation.*

1. $2 \cot x + 1 = -1$

2. $\sin x + 2 = 3$

3. $5 \sec^2 x = 6 \sec x$

4. $2 \cos^2 x - \cos x = 1$

5. $9 \sin^2 x - 5 \sin x = 1$

6. $\tan^2 x - 4 \tan x + 2 = 0$

7. $\tan x - \cot x = 0$

8. $\cos^2 x = \sin^2 x + 1$

9. Suppose in solving an equation over the interval $[0°, 360°)$, you reach the step $\sin \theta = -\frac{1}{2}$. Why is $-30°$ not a correct answer?

10. Lindsay solved the equation $\sin x = 1 - \cos x$ by squaring both sides to get $\sin^2 x = 1 - 2 \cos x + \cos^2 x$. Several steps later, using correct algebra, she determined that the solution set for solutions over the interval $[0, 2\pi)$ is $\{0, \frac{\pi}{2}, \frac{3\pi}{2}\}$. Explain why this is not the correct solution set.

Solve each equation for exact solutions over the interval $[0, 2\pi)$. *See Examples 1–3.*

11. $2 \cot x + 1 = -1$

12. $\sin x + 2 = 3$

13. $2 \sin x + 3 = 4$

14. $2 \sec x + 1 = \sec x + 3$

15. $\tan^2 x + 3 = 0$

16. $\sec^2 x + 2 = -1$

17. $(\cot x - 1)(\sqrt{3} \cot x + 1) = 0$

18. $(\csc x + 2)(\csc x - \sqrt{2}) = 0$

19. $\cos^2 x + 2 \cos x + 1 = 0$

20. $2 \cos^2 x - \sqrt{3} \cos x = 0$

21. $-2 \sin^2 x = 3 \sin x + 1$

22. $2 \cos^2 x - \cos x = 1$

Solve each equation for solutions over the interval $[0°, 360°)$. *Give solutions to the nearest tenth as appropriate. See Examples 2–5.*

23. $(\cot \theta - \sqrt{3})(2 \sin \theta + \sqrt{3}) = 0$

24. $(\tan \theta - 1)(\cos \theta - 1) = 0$

25. $2 \sin \theta - 1 = \csc \theta$

26. $\tan \theta + 1 = \sqrt{3} + \sqrt{3} \cot \theta$

27. $\tan \theta - \cot \theta = 0$

28. $\cos^2 \theta = \sin^2 \theta + 1$

29. $\csc^2 \theta - 2 \cot \theta = 0$

30. $\sin^2 \theta \cos \theta = \cos \theta$

31. $2 \tan^2 \theta \sin \theta - \tan^2 \theta = 0$

32. $\sin^2 \theta \cos^2 \theta = 0$

33. $\sec^2 \theta \tan \theta = 2 \tan \theta$

34. $\cos^2 \theta - \sin^2 \theta = 0$

35. $9 \sin^2 \theta - 6 \sin \theta = 1$

36. $4 \cos^2 \theta + 4 \cos \theta = 1$

37. $\tan^2 \theta + 4 \tan \theta + 2 = 0$

38. $3 \cot^2 \theta - 3 \cot \theta - 1 = 0$

39. $\sin^2 \theta - 2 \sin \theta + 3 = 0$

40. $2 \cos^2 \theta + 2 \cos \theta - 1 = 0$

41. $\cot \theta + 2 \csc \theta = 3$

42. $2 \sin \theta = 1 - 2 \cos \theta$

Determine the solution set of each equation in radians (for x) to four decimal places or degrees (for θ) to the nearest tenth as appropriate. See Example 4.

43. $3\sin^2 x - \sin x - 1 = 0$

44. $2\cos^2 x + \cos x = 1$

45. $4\cos^2 x - 1 = 0$

46. $2\cos^2 x + 5\cos x + 2 = 0$

47. $5\sec^2\theta = 6\sec\theta$

48. $3\sin^2\theta - \sin\theta = 2$

49. $\dfrac{2\tan\theta}{3 - \tan^2\theta} = 1$

50. $\sec^2\theta = 2\tan\theta + 4$

The following equations cannot be solved by algebraic methods. Use a graphing calculator to find all solutions over the interval $[0, 2\pi)$. Express solutions to four decimal places.

51. $x^2 + \sin x - x^3 - \cos x = 0$

52. $x^3 - \cos^2 x = \dfrac{1}{2}x - 1$

(Modeling) *Solve each problem.*

53. *Pressure on the Eardrum* See Example 6. No musical instrument can generate a true pure tone. A pure tone has a unique, constant frequency and amplitude that sounds rather dull and uninteresting. The pressures caused by pure tones on the eardrum are sinusoidal. The change in pressure P in pounds per square foot on a person's eardrum from a pure tone at time t in seconds can be modeled using the equation

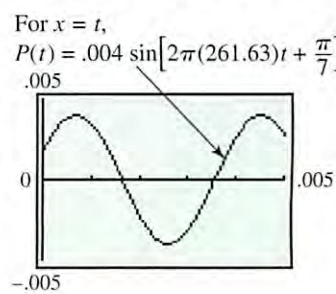
$$P = A\sin(2\pi ft + \phi),$$

where f is the frequency in cycles per second, and ϕ is the phase angle. When P is positive, there is an increase in pressure and the eardrum is pushed inward; when P is negative, there is a decrease in pressure and the eardrum is pushed outward. (*Source:* Roederer, J., *Introduction to the Physics and Psychophysics of Music,* Second Edition, Springer-Verlag, 1975.) A graph of the tone middle C is shown in the figure.

(a) Determine algebraically the values of t for which $P = 0$ over $[0, .005]$.

(b) From the graph and your answer in part (a), determine the interval for which $P \le 0$ over $[0, .005]$.

(c) Would an eardrum hearing this tone be vibrating outward or inward when $P < 0$?



54. *Accident Reconstruction* The model

$$.342D\cos\theta + h\cos^2\theta = \frac{16D^2}{V_0^2}$$

is used to reconstruct accidents in which a vehicle vaults into the air after hitting an obstruction. V_0 is velocity in feet per second of the vehicle when it hits, D is distance (in feet) from the obstruction to the landing point, and h is the difference in height (in feet) between landing point and takeoff point. Angle θ is the takeoff angle, the angle between the horizontal and the path of the vehicle. Find θ to the nearest degree if $V_0 = 60$, $D = 80$, and $h = 2$.

55. *Electromotive Force* In an electric circuit, let

$$V = \cos 2\pi t$$

model the electromotive force in volts at t seconds. Find the least value of t where $0 \le t \le \frac{1}{2}$ for each value of V.

(a) $V = 0$ **(b)** $V = .5$ **(c)** $V = .25$

56. ***Voltage Induced by a Coil of Wire*** A coil of wire rotating in a magnetic field induces a voltage modeled by

$$E = 20 \sin\left(\frac{\pi t}{4} - \frac{\pi}{2}\right),$$

where t is time in seconds. Find the least positive time to produce each voltage.

(a) 0 **(b)** $10\sqrt{3}$

57. ***Movement of a Particle*** A particle moves along a straight line. The distance of the particle from the origin at time t is modeled by

$$s(t) = \sin t + 2 \cos t.$$

Find a value of t that satisfies each equation.

(a) $s(t) = \frac{2 + \sqrt{3}}{2}$ **(b)** $s(t) = \frac{3\sqrt{2}}{2}$

58. Explain what is **WRONG** with the following solution for all x over the interval $[0, 2\pi)$ of the equation $\sin^2 x - \sin x = 0$.

$$\sin^2 x - \sin x = 0$$

$$\sin x - 1 = 0 \quad \text{Divide by } \sin x.$$

$$\sin x = 1 \quad \text{Add 1.}$$

$$x = \frac{\pi}{2}$$

The solution set is $\{\frac{\pi}{2}\}$.

6.3 Trigonometric Equations II

Equations with Half-Angles ▪ Equations with Multiple Angles

In this section, we discuss trigonometric equations that involve functions of half-angles and multiple angles. Solving these equations often requires adjusting solution intervals to fit given domains.

Equations with Half-Angles

▶ EXAMPLE 1 SOLVING AN EQUATION USING A HALF-ANGLE IDENTITY

Solve $2 \sin \frac{x}{2} = 1$

(a) over the interval $[0, 2\pi)$, and **(b)** give all solutions.

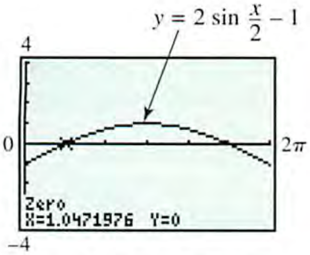
Solution

(a) Write the interval $[0, 2\pi)$ as the inequality

$$0 \le x < 2\pi.$$

The corresponding interval for $\frac{x}{2}$ is

$$0 \le \frac{x}{2} < \pi. \quad \text{Divide by 2. (Appendix A)}$$



The x-intercepts are the solutions found in Example 1. Using Xscl = $\frac{\pi}{3}$ makes it possible to support the exact solutions by counting the tick marks from 0 on the graph.

To find all values of $\frac{x}{2}$ over the interval $[0, \pi)$ that satisfy the given equation, first solve for $\sin \frac{x}{2}$.

$$2 \sin \frac{x}{2} = 1$$

$$\sin \frac{x}{2} = \frac{1}{2} \quad \text{Divide by 2.}$$

The two numbers over the interval $[0, \pi)$ with sine value $\frac{1}{2}$ are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$, so

$$\frac{x}{2} = \frac{\pi}{6} \quad \text{or} \quad \frac{x}{2} = \frac{5\pi}{6}$$

$$x = \frac{\pi}{3} \quad \text{or} \quad x = \frac{5\pi}{3}. \quad \text{Multiply by 2.}$$

The solution set over the given interval is $\left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$.

(b) Since this is a sine function with period 4π, the solution set is

$$\left\{\frac{\pi}{3} + 4n\pi, \frac{5\pi}{3} + 4n\pi, \text{ where } n \text{ is any integer}\right\}.$$

NOW TRY EXERCISE 15. ◀

Equations with Multiple Angles

▶ EXAMPLE 2 SOLVING AN EQUATION WITH A DOUBLE ANGLE

Solve $\cos 2x = \cos x$ over the interval $[0, 2\pi)$.

Solution First change $\cos 2x$ to a trigonometric function of x. Use the identity $\cos 2x = 2 \cos^2 x - 1$ so the equation involves only $\cos x$. Then factor.

$$\cos 2x = \cos x$$

$$2 \cos^2 x - 1 = \cos x \quad \text{Substitute; double-angle identity (Section 5.5)}$$

$$2 \cos^2 x - \cos x - 1 = 0 \quad \text{Subtract } \cos x.$$

$$(2 \cos x + 1)(\cos x - 1) = 0 \quad \text{Factor.}$$

$$2 \cos x + 1 = 0 \quad \text{or} \quad \cos x - 1 = 0 \quad \text{Zero-factor property (Appendix A)}$$

$$\cos x = -\frac{1}{2} \quad \text{or} \quad \cos x = 1 \quad \text{Solve each equation.}$$

Cosine is $-\frac{1}{2}$ in quadrants II and III with reference arc $\frac{\pi}{3}$, and has a value of 1 at 0 radians; thus, solutions over the required interval are

$$x = \frac{2\pi}{3} \quad \text{or} \quad x = \frac{4\pi}{3} \quad \text{or} \quad x = 0.$$

The solution set is $\left\{0, \frac{2\pi}{3}, \frac{4\pi}{3}\right\}$.

NOW TRY EXERCISE 17. ◀

> ▶ **Caution** In the solution of Example 2, $\cos 2x$ cannot be changed to $\cos x$ by dividing by 2 since 2 is not a factor of $\cos 2x$, that is, $\frac{\cos 2x}{2} \neq \cos x$. The only way to change $\cos 2x$ to a trigonometric function of x is by using one of the identities for $\cos 2x$.

▶ EXAMPLE 3 SOLVING AN EQUATION USING A MULTIPLE-ANGLE IDENTITY

Solve $4 \sin \theta \cos \theta = \sqrt{3}$ over the interval $[0°, 360°)$.

Solution The identity $2 \sin \theta \cos \theta = \sin 2\theta$ is useful here.

$$4 \sin \theta \cos \theta = \sqrt{3}$$

$$2(2 \sin \theta \cos \theta) = \sqrt{3} \qquad 4 = 2 \cdot 2$$

$$2 \sin 2\theta = \sqrt{3} \qquad 2 \sin \theta \cos \theta = \sin 2\theta \text{ (Section 5.5)}$$

$$\sin 2\theta = \frac{\sqrt{3}}{2} \qquad \text{Divide by 2.}$$

From the given interval $0° \leq \theta < 360°$, the interval for 2θ is $0° \leq 2\theta < 720°$. Since the sine is positive in quadrants I and II, solutions over this interval are

$$2\theta = 60°, 120°, 420°, 480°,$$

or

$$\theta = 30°, 60°, 210°, 240°. \qquad \text{Divide by 2.}$$

The final two solutions for 2θ were found by adding 360° to 60° and 120°, respectively, giving the solution set $\{30°, 60°, 210°, 240°\}$.

NOW TRY EXERCISE 37. ◀

▶ EXAMPLE 4 SOLVING AN EQUATION WITH A MULTIPLE ANGLE

Solve $\tan 3x + \sec 3x = 2$ over the interval $[0, 2\pi)$.

Solution Since the tangent and secant functions are related by the identity $1 + \tan^2 \theta = \sec^2 \theta$, one way to begin is to express everything in terms of secant.

$$\tan 3x + \sec 3x = 2$$

Don't forget the middle term.

$$\tan 3x = 2 - \sec 3x \qquad \text{Subtract } \sec 3x.$$

$$\tan^2 3x = 4 - 4 \sec 3x + \sec^2 3x \qquad \text{Square both sides; } (x - y)^2 = x^2 - 2xy + y^2.$$

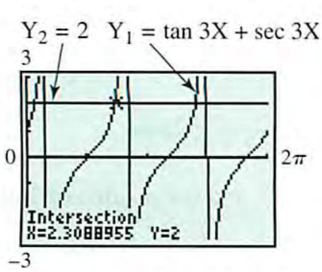
$$\sec^2 3x - 1 = 4 - 4 \sec 3x + \sec^2 3x \qquad \text{Replace } \tan^2 3x \text{ with } \sec^2 3x - 1. \text{ (Section 5.1)}$$

$$4 \sec 3x = 5 \qquad \text{Simplify.}$$

$$\sec 3x = \frac{5}{4} \qquad \text{Divide by 4.}$$

$$\frac{1}{\cos 3x} = \frac{5}{4} \qquad \sec \theta = \frac{1}{\cos \theta} \text{ (Section 5.1)}$$

$$\cos 3x = \frac{4}{5} \qquad \text{Use reciprocals.}$$



Connected mode; radian mode

The screen shows that one solution is approximately 2.3089. An advantage of using a graphing calculator is that extraneous values do not appear.

Multiply each term of the inequality $0 \leq x < 2\pi$ by 3 to find the interval for $3x$: $[0, 6\pi)$. Using a calculator and the fact that cosine is positive in quadrants I and IV,

$$3x \approx .6435, 5.6397, 6.9267, 11.9229, 13.2099, 18.2061$$

$$x \approx .2145, 1.8799, 2.3089, 3.9743, 4.4033, 6.0687. \quad \text{Divide by 3.}$$

Since both sides of the equation were squared, each proposed solution must be checked. Verify by substitution in the given equation that the solution set is $\{.2145, 2.3089, 4.4033\}$.

NOW TRY EXERCISE 33.

A piano string can vibrate at more than one frequency when it is struck. It produces a complex wave that can mathematically be modeled by a sum of several pure tones. If a piano key with a frequency of f_1 is played, then the corresponding string will not only vibrate at f_1 but it will also vibrate at the higher frequencies of $2f_1, 3f_1, 4f_1, \ldots, nf_1$. f_1 is called the **fundamental frequency** of the string, and higher frequencies are called the **upper harmonics.** The human ear will hear the sum of these frequencies as one complex tone. (*Source:* Roederer, J., *Introduction to the Physics and Psychophysics of Music,* Second Edition, Springer-Verlag, 1975.)

EXAMPLE 5 ANALYZING PRESSURES OF UPPER HARMONICS

Suppose that the A key above middle C is played. Its fundamental frequency is $f_1 = 440$ Hz, and its associated pressure is expressed as

$$P_1 = .002 \sin 880\pi t.$$

The string will also vibrate at

$$f_2 = 880, \quad f_3 = 1320, \quad f_4 = 1760, \quad f_5 = 2200, \ldots \text{ Hz}.$$

The corresponding pressures of these upper harmonics are

$$P_2 = \frac{.002}{2} \sin 1760\pi t, \qquad P_3 = \frac{.002}{3} \sin 2640\pi t,$$

$$P_4 = \frac{.002}{4} \sin 3520\pi t, \qquad \text{and} \qquad P_5 = \frac{.002}{5} \sin 4400\pi t.$$

The graph of

$$P = P_1 + P_2 + P_3 + P_4 + P_5,$$

shown in Figure 25, is "saw-toothed."

(a) What is the maximum value of P?

(b) At what values of $t = x$ does this maximum occur over the interval $[0, .01]$?

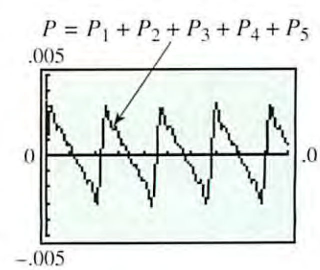


Figure 25

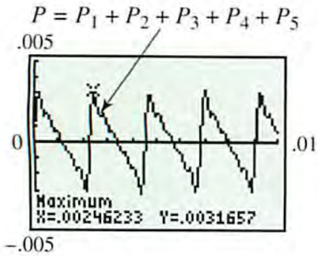


Figure 26

Solution

(a) A graphing calculator shows that the maximum value of P is approximately .00317. See Figure 26.

(b) The maximum occurs at $t = x \approx .000188, .00246, .00474, .00701$, and .00928. Figure 26 shows how the second value is found; the others are found similarly.

NOW TRY EXERCISE 43.

6.3 Exercises

Concept Check *Answer each question.*

1. Suppose you are solving a trigonometric equation for solutions over the interval $[0, 2\pi)$, and your work leads to $2x = \frac{2\pi}{3}, 2\pi, \frac{8\pi}{3}$. What are the corresponding values of x?

2. Suppose you are solving a trigonometric equation for solutions over the interval $[0, 2\pi)$, and your work leads to $\frac{1}{2}x = \frac{\pi}{16}, \frac{5\pi}{12}, \frac{5\pi}{8}$. What are the corresponding values of x?

3. Suppose you are solving a trigonometric equation for solutions over the interval $[0°, 360°)$, and your work leads to $3\theta = 180°, 630°, 720°, 930°$. What are the corresponding values of θ?

4. Suppose you are solving a trigonometric equation for solutions over the interval $[0°, 360°)$, and your work leads to $\frac{1}{3}\theta = 45°, 60°, 75°, 90°$. What are the corresponding values of θ?

5. Explain what is **WRONG** with the following solution.
Solve $\tan 2\theta = 2$ over the interval $[0, 2\pi)$.

$$\tan 2\theta = 2$$

$$\frac{\tan 2\theta}{2} = \frac{2}{2}$$

$$\tan \theta = 1$$

$$\theta = \frac{\pi}{4} \quad \text{or} \quad \theta = \frac{5\pi}{4}$$

The solution set is $\left\{\frac{\pi}{4}, \frac{5\pi}{4}\right\}$.

6. The equation $\cot \frac{x}{2} - \csc \frac{x}{2} - 1 = 0$ has no solution over the interval $[0, 2\pi)$. Using this information, what can be said about the graph of

$$y = \cot \frac{x}{2} - \csc \frac{x}{2} - 1$$

over this interval? Confirm your answer by actually graphing the function over the interval.

Solve each equation for exact solutions over the interval $[0, 2\pi)$. *See Examples 1–4.*

7. $\cos 2x = \frac{\sqrt{3}}{2}$

8. $\cos 2x = -\frac{1}{2}$

9. $\sin 3x = -1$

10. $\sin 3x = 0$

11. $3 \tan 3x = \sqrt{3}$

12. $\cot 3x = \sqrt{3}$

13. $\sqrt{2} \cos 2x = -1$

14. $2\sqrt{3} \sin 2x = \sqrt{3}$

15. $\sin \frac{x}{2} = \sqrt{2} - \sin \frac{x}{2}$

16. $\tan 4x = 0$

17. $\sin x = \sin 2x$

18. $\cos 2x - \cos x = 0$

19. $8 \sec^2 \frac{x}{2} = 4$
20. $\sin^2 \frac{x}{2} - 2 = 0$
21. $\sin \frac{x}{2} = \cos \frac{x}{2}$
22. $\sec \frac{x}{2} = \cos \frac{x}{2}$
23. $\cos 2x + \cos x = 0$
24. $\sin x \cos x = \frac{1}{4}$

Solve each equation in Exercises 25–32 for exact solutions over the interval $[0°, 360°)$. *In Exercises 33–40, give all solutions. If necessary, express solutions to the nearest tenth of a degree. See Examples 1–4.*

25. $\sqrt{2} \sin 3\theta - 1 = 0$
26. $-2 \cos 2\theta = \sqrt{3}$
27. $\cos \frac{\theta}{2} = 1$
28. $\sin \frac{\theta}{2} = 1$
29. $2\sqrt{3} \sin \frac{\theta}{2} = 3$
30. $2\sqrt{3} \cos \frac{\theta}{2} = -3$
31. $2 \sin \theta = 2 \cos 2\theta$
32. $\cos \theta - 1 = \cos 2\theta$
33. $1 - \sin \theta = \cos 2\theta$
34. $\sin 2\theta = 2 \cos^2 \theta$
35. $\csc^2 \frac{\theta}{2} = 2 \sec \theta$
36. $\cos \theta = \sin^2 \frac{\theta}{2}$
37. $2 - \sin 2\theta = 4 \sin 2\theta$
38. $4 \cos 2\theta = 8 \sin \theta \cos \theta$
39. $2 \cos^2 2\theta = 1 - \cos 2\theta$
40. $\sin \theta - \sin 2\theta = 0$

The following equations cannot be solved by algebraic methods. Use a graphing calculator to find all solutions over the interval $[0, 2\pi)$. *Express solutions to four decimal places.*

41. $2 \sin 2x - x^3 + 1 = 0$
42. $3 \cos \frac{x}{2} + \sqrt{x} - 2 = -\frac{1}{2}x + 2$

(Modeling) *Solve each problem. See Example 5.*

43. ***Pressure of a Plucked String*** If a string with a fundamental frequency of 110 Hz is plucked in the middle, it will vibrate at the odd harmonics of 110, 330, 550, . . . Hz but not at the even harmonics of 220, 440, 660, . . . Hz. The resulting pressure P caused by the string can be modeled by the equation

$$P = .003 \sin 220\pi t + \frac{.003}{3} \sin 660\pi t + \frac{.003}{5} \sin 1100\pi t + \frac{.003}{7} \sin 1540\pi t.$$

(*Source:* Benade, A., *Fundamentals of Musical Acoustics*, Dover Publications, 1990; Roederer, J., *Introduction to the Physics and Psychophysics of Music*, Second Edition, Springer-Verlag, 1975.)

(a) Graph P in the window $[0, .03]$ by $[-.005, .005]$.
(b) Use the graph to describe the shape of the sound wave that is produced.
(c) See **Section 6.2,** Exercise 53. At lower frequencies, the inner ear will hear a tone only when the eardrum is moving outward. Determine the times over the interval $[0, .03]$ when this will occur.

44. ***Hearing Beats in Music*** Musicians sometimes tune instruments by playing the same tone on two different instruments and listening for a phenomenon known as **beats.** Beats occur when two tones vary in frequency by only a few hertz. When the two instruments are in tune, the beats disappear. The ear hears beats because the pressure slowly rises and falls as a result of this slight variation in the frequency.

This phenomenon can be seen using a graphing calculator. (*Source:* Pierce, J., *The Science of Musical Sound,* Scientific American Books, 1992.)

(a) Consider two tones with frequencies of 220 and 223 Hz and pressures $P_1 = .005 \sin 440\pi t$ and $P_2 = .005 \sin 446\pi t$, respectively. Graph the pressure $P = P_1 + P_2$ felt by an eardrum over the 1-sec interval $[.15, 1.15]$. How many beats are there in 1 sec?

(b) Repeat part (a) with frequencies of 220 and 216 Hz.

(c) Determine a simple way to find the number of beats per second if the frequency of each tone is given.

45. ***Hearing Difference Tones*** Small speakers like those found in older radios and telephones often cannot vibrate slower than 200 Hz—yet 35 keys on a piano have frequencies below 200 Hz. When a musical instrument creates a tone of 110 Hz, it also creates tones at 220, 330, 440, 550, 660, . . . Hz. A small speaker cannot reproduce the 110-Hz vibration but it can reproduce the higher frequencies, which are called the upper harmonics. The low tones can still be heard because the speaker produces **difference tones** of the upper harmonics. The difference between consecutive frequencies is 110 Hz, and this difference tone will be heard by a listener. We can model this phenomenon using a graphing calculator. (*Source:* Benade, A., *Fundamentals of Musical Acoustics,* Dover Publications, 1990.)

(a) In the window $[0, .03]$ by $[-1, 1]$, graph the upper harmonics represented by the pressure

$$P = \frac{1}{2}\sin[2\pi(220)t] + \frac{1}{3}\sin[2\pi(330)t] + \frac{1}{4}\sin[2\pi(440)t].$$

(b) Estimate all t-coordinates where P is maximum.

(c) What does a person hear in addition to the frequencies of 220, 330, and 440 Hz?

(d) Graph the pressure produced by a speaker that can vibrate at 110 Hz and above.

46. ***Daylight Hours in New Orleans*** The seasonal variation in length of daylight can be modeled by a sine function. For example, the daily number of hours of daylight in New Orleans is given by

$$h = \frac{35}{3} + \frac{7}{3}\sin\frac{2\pi x}{365},$$

where x is the number of days after March 21 (disregarding leap year). (*Source:* Bushaw, D., et al., *A Sourcebook of Applications of School Mathematics.* Copyright © 1980 by The Mathematical Association of America.)

(a) On what date will there be about 14 hr of daylight?

(b) What date has the least number of hours of daylight?

(c) When will there be about 10 hr of daylight?

(Modeling) Alternating Electric Current *The study of alternating electric current requires the solutions of equations of the form*

$$i = I_{max} \sin 2\pi f t,$$

for time t in seconds, where i is instantaneous current in amperes, I_{max} is maximum current in amperes, and f is the number of cycles per second. (Source: Hannon, R. H., *Basic Technical Mathematics with Calculus,* W. B. Saunders Company, 1978.) *Find the least positive value of t, given the following data.*

47. $i = 40, I_{max} = 100, f = 60$

48. $i = 50, I_{max} = 100, f = 120$

49. $i = I_{max}, f = 60$

50. $i = \frac{1}{2}I_{max}, f = 60$

CHAPTER 6 ▶ Quiz (Sections 6.1–6.3)

1. Graph $y = \cos^{-1} x$, and indicate the coordinates of three points on the graph. Give the domain and range.

2. Find the exact value of each real number y.

(a) $y = \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ **(b)** $y = \tan^{-1}\sqrt{3}$ **(c)** $y = \sec^{-1}\left(-\frac{2\sqrt{3}}{3}\right)$

3. Use a calculator to give each value in decimal degrees.

(a) $\theta = \arccos .92341853$ **(b)** $\theta = \cot^{-1}(-1.0886767)$

4. Give the exact value of each expression without using a calculator.

(a) $\cos\left(\tan^{-1}\frac{4}{5}\right)$ **(b)** $\sin\left(\cos^{-1}\left(-\frac{1}{2}\right) + \tan^{-1}(-\sqrt{3})\right)$

Solve each equation for exact solutions over the interval $[0°, 360°)$.

5. $2\sin\theta - \sqrt{3} = 0$ **6.** $\cos\theta + 1 = 2\sin^2\theta$

Solve each equation for solutions over the interval $[0, 2\pi)$.

7. $\tan^2 x - 5\tan x + 3 = 0$ **8.** $3\cot 2x - \sqrt{3} = 0$

9. Solve $\cos\frac{x}{2} + \sqrt{3} = -\cos\frac{x}{2}$, giving all solutions in radians.

10. ***Electromotive Force*** In an electric circuit, let

$$V = \cos 2\pi t$$

model the electromotive force in volts at t seconds. Find the least value of t where $0 \le t \le \frac{1}{2}$ for each value of V.

(a) $V = 1$ **(b)** $V = .30$

6.4 Equations Involving Inverse Trigonometric Functions

Solving for x in Terms of y Using Inverse Functions ▪ Solving Inverse Trigonometric Equations

Until now, the equations in this chapter have involved trigonometric functions of angles or real numbers. Now we examine equations involving *inverse* trigonometric functions.

Solving for x in Terms of y Using Inverse Functions

▶ EXAMPLE 1 **SOLVING AN EQUATION FOR A VARIABLE USING INVERSE NOTATION**

Solve $y = 3\cos 2x$ for x.

Solution We want $\cos 2x$ alone on one side of the equation so we can solve for $2x$, and then for x.

$$y = 3\cos 2x$$ Our goal is to isolate x.

$$\frac{y}{3} = \cos 2x$$ Divide by 3. (Appendix A)

$$2x = \arccos \frac{y}{3}$$ Definition of arccosine (Section 6.1)

Do not multiply the argument by $\frac{1}{2}$.

$$x = \frac{1}{2}\arccos \frac{y}{3}$$ Multiply by $\frac{1}{2}$.

An equivalent form of this answer is $x = \frac{1}{2}\cos^{-1}\frac{y}{3}$.

NOW TRY EXERCISE 9.

Solving Inverse Trigonometric Equations

EXAMPLE 2 SOLVING AN EQUATION INVOLVING AN INVERSE TRIGONOMETRIC FUNCTION

Solve $2 \arcsin x = \pi$.

Solution First solve for $\arcsin x$, and then for x.

$$2 \arcsin x = \pi$$

$$\arcsin x = \frac{\pi}{2}$$ Divide by 2.

$$x = \sin \frac{\pi}{2}$$ Definition of arcsine (Section 6.1)

$$x = 1$$ (Section 3.3)

Verify that the solution satisfies the given equation. The solution set is $\{1\}$.

NOW TRY EXERCISE 25.

EXAMPLE 3 SOLVING AN EQUATION INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

Solve $\cos^{-1} x = \sin^{-1} \frac{1}{2}$.

Solution Let $\sin^{-1}\frac{1}{2} = u$. Then $\sin u = \frac{1}{2}$ and for u in quadrant I, we have

$$\cos^{-1} x = \sin^{-1}\frac{1}{2}$$

$$\cos^{-1} x = u$$ Substitute.

$$\cos u = x.$$ Alternative form (Section 6.1)

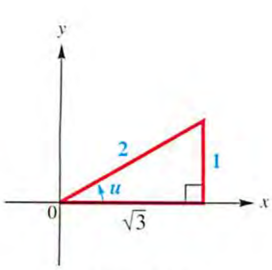


Figure 27

Sketch a triangle and label it using the facts that u is in quadrant I and $\sin u = \frac{1}{2}$. See Figure 27. Since $x = \cos u$, $x = \frac{\sqrt{3}}{2}$, and the solution set is $\left\{\frac{\sqrt{3}}{2}\right\}$. *Check.*

NOW TRY EXERCISE 33.

▶ EXAMPLE 4 **SOLVING AN INVERSE TRIGONOMETRIC EQUATION USING AN IDENTITY**

Solve $\arcsin x - \arccos x = \frac{\pi}{6}$.

Solution Isolate one inverse function on one side of the equation.

$$\arcsin x - \arccos x = \frac{\pi}{6}$$

$$\arcsin x = \arccos x + \frac{\pi}{6} \qquad \text{Add } \arccos x. \quad (1)$$

$$x = \sin\left(\arccos x + \frac{\pi}{6}\right) \qquad \text{Definition of arcsine}$$

Let $u = \arccos x$. The arccosine function yields angles in quadrants I and II, so $0 \le u \le \pi$ by definition.

$$x = \sin\left(u + \frac{\pi}{6}\right) \qquad \text{Substitute.}$$

$$x = \sin u \cos\frac{\pi}{6} + \cos u \sin\frac{\pi}{6} \qquad \text{Sine sum identity (Section 5.4)} \quad (2)$$

From equation (1) and by the definition of the arcsine function,

$$-\frac{\pi}{2} \le \arccos x + \frac{\pi}{6} \le \frac{\pi}{2} \qquad \text{Range of arcsine is } \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right].$$

$$-\frac{2\pi}{3} \le \arccos x \le \frac{\pi}{3}. \qquad \text{Subtract } \tfrac{\pi}{6} \text{ from each part. (Appendix A)}$$

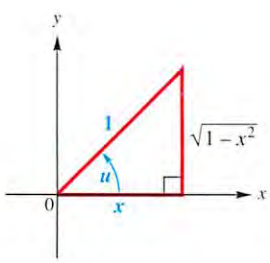


Figure 28

Since $0 \le \arccos x \le \pi$ **and** $-\frac{2\pi}{3} \le \arccos x \le \frac{\pi}{3}$, the intersection yields $0 \le \arccos x \le \frac{\pi}{3}$. This places u in quadrant I, and we can sketch the triangle in Figure 28. From this triangle we find that $\sin u = \sqrt{1 - x^2}$. Now substitute into equation (2) using $\sin u = \sqrt{1 - x^2}$, $\sin\frac{\pi}{6} = \frac{1}{2}$, $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, and $\cos u = x$.

$$x = \sin u \cos\frac{\pi}{6} + \cos u \sin\frac{\pi}{6} \qquad (2)$$

$$x = \left(\sqrt{1 - x^2}\right)\frac{\sqrt{3}}{2} + x \cdot \frac{1}{2} \qquad \text{Substitute.}$$

$$2x = \left(\sqrt{1 - x^2}\right)\sqrt{3} + x \qquad \text{Multiply by 2.}$$

$$x = \left(\sqrt{3}\right)\sqrt{1 - x^2} \qquad \text{Subtract } x.$$

Square *each* term.

$$x^2 = 3(1 - x^2) \qquad \text{Square both sides.}$$

$$x^2 = 3 - 3x^2 \qquad \text{Distributive property}$$

$$4x^2 = 3 \qquad \text{Add } 3x^2.$$

$$x^2 = \frac{3}{4} \qquad \text{Divide by 4.}$$

Choose the positive square root; $x > 0$.

$$x = \sqrt{\frac{3}{4}} \qquad \text{Take the square root of both sides. (Appendix A)}$$

$$x = \frac{\sqrt{3}}{2} \qquad \text{Quotient rule: } \sqrt[n]{\tfrac{a}{b}} = \tfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

To *check*, replace x with $\frac{\sqrt{3}}{2}$ in the original equation:

$$\arcsin \frac{\sqrt{3}}{2} - \arccos \frac{\sqrt{3}}{2} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6},$$

as required. The solution set is $\left\{\frac{\sqrt{3}}{2}\right\}$.

NOW TRY EXERCISE 35.

6.4 Exercises

Concept Check *Answer each question.*

1. Which one of the following equations has solution 0?

A. $\arctan 1 = x$ **B.** $\arccos 0 = x$ **C.** $\arcsin 0 = x$

2. Which one of the following equations has solution $\frac{\pi}{4}$?

A. $\arcsin \frac{\sqrt{2}}{2} = x$ **B.** $\arccos\left(-\frac{\sqrt{2}}{2}\right) = x$ **C.** $\arctan \frac{\sqrt{3}}{3} = x$

3. Which one of the following equations has solution $\frac{3\pi}{4}$?

A. $\arctan 1 = x$ **B.** $\arcsin \frac{\sqrt{2}}{2} = x$ **C.** $\arccos\left(-\frac{\sqrt{2}}{2}\right) = x$

4. Which one of the following equations has solution $-\frac{\pi}{6}$?

A. $\arctan \frac{\sqrt{3}}{3} = x$ **B.** $\arccos\left(-\frac{1}{2}\right) = x$ **C.** $\arcsin\left(-\frac{1}{2}\right) = x$

Solve each equation for x. See Example 1.

5. $y = 5 \cos x$

6. $4y = \sin x$

7. $2y = \cot 3x$

8. $6y = \frac{1}{2} \sec x$

9. $y = 3 \tan 2x$

10. $y = 3 \sin \frac{x}{2}$

11. $y = 6 \cos \frac{x}{4}$

12. $y = -\sin \frac{x}{3}$

13. $y = -2 \cos 5x$

14. $y = 3 \cot 5x$

15. $y = \cos(x + 3)$

16. $y = \tan(2x - 1)$

17. $y = \sin x - 2$

18. $y = \cot x + 1$

19. $y = 2 \sin x - 4$

20. $y = 4 + 3 \cos x$

21. $y = \sqrt{2} + 3 \sec 2x$

22. $y = 2 \csc \frac{x}{2} - \sqrt{3}$

23. Refer to Exercise 17. A student attempting to solve this equation wrote as the first step $y = \sin(x - 2)$, inserting parentheses as shown. Explain why this is incorrect.

24. Explain why the equation $\sin^{-1} x = \cos^{-1} 2$ cannot have a solution. (No work is required.)

Solve each equation for exact solutions. See Examples 2 and 3.

25. $-4 \arcsin x = \pi$

26. $6 \arccos x = 5\pi$

27. $\frac{4}{3} \cos^{-1} \frac{y}{4} = \pi$

28. $4\pi + 4 \tan^{-1} y = \pi$

29. $2 \arccos\left(\frac{y - \pi}{3}\right) = 2\pi$

30. $\arccos\left(y - \frac{\pi}{3}\right) = \frac{\pi}{6}$

31. $\arcsin x = \arctan \frac{3}{4}$

32. $\arctan x = \arccos \frac{5}{13}$

33. $\cos^{-1} x = \sin^{-1} \frac{3}{5}$

34. $\cot^{-1} x = \tan^{-1} \frac{4}{3}$

Solve each equation for exact solutions. See Example 4.

35. $\sin^{-1} x - \tan^{-1} 1 = -\frac{\pi}{4}$

36. $\sin^{-1} x + \tan^{-1} \sqrt{3} = \frac{2\pi}{3}$

37. $\arccos x + 2 \arcsin \frac{\sqrt{3}}{2} = \pi$

38. $\arccos x + 2 \arcsin \frac{\sqrt{3}}{2} = \frac{\pi}{3}$

39. $\arcsin 2x + \arccos x = \frac{\pi}{6}$

40. $\arcsin 2x + \arcsin x = \frac{\pi}{2}$

41. $\cos^{-1} x + \tan^{-1} x = \frac{\pi}{2}$

42. $\sin^{-1} x + \tan^{-1} x = 0$

43. Provide graphical support for the solution in Example 4 by showing that the graph of $y = \arcsin x - \arccos x - \frac{\pi}{6}$ has x-intercept $\frac{\sqrt{3}}{2} \approx .8660254$.

44. Provide graphical support for the solution in Example 4 by showing that the x-coordinate of the point of intersection of the graphs of $Y_1 = \arcsin X - \arccos X$ and $Y_2 = \frac{\pi}{6}$ is $\frac{\sqrt{3}}{2} \approx .8660254$.

The following equations cannot be solved by algebraic methods. Use a graphing calculator to find all solutions over the interval $[0, 6]$. *Express solutions to four decimal places.*

45. $(\arctan x)^3 - x + 2 = 0$

46. $\pi \sin^{-1}(.2x) - 3 = -\sqrt{x}$

(Modeling) *Solve each problem.*

47. ***Tone Heard by a Listener*** When two sources located at different positions produce the same pure tone, the human ear will often hear one sound that is equal to the sum of the individual tones. Since the sources are at different locations, they will have different phase angles ϕ. If two speakers located at different positions produce pure tones $P_1 = A_1 \sin(2\pi ft + \phi_1)$ and $P_2 = A_2 \sin(2\pi ft + \phi_2)$, where $-\frac{\pi}{4} \leq \phi_1, \phi_2 \leq \frac{\pi}{4}$, then the resulting tone heard by a listener can be written as $P = A \sin(2\pi ft + \phi)$, where

$$A = \sqrt{(A_1 \cos \phi_1 + A_2 \cos \phi_2)^2 + (A_1 \sin \phi_1 + A_2 \sin \phi_2)^2}$$

$$\text{and} \quad \phi = \arctan\left(\frac{A_1 \sin \phi_1 + A_2 \sin \phi_2}{A_1 \cos \phi_1 + A_2 \cos \phi_2}\right).$$

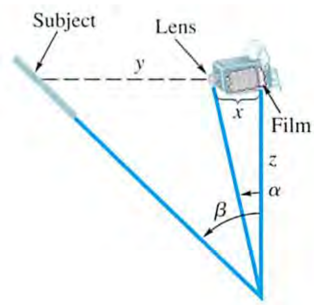
(*Source:* Fletcher, N. and T. Rossing, *The Physics of Musical Instruments,* Second Edition, Springer-Verlag, 1998.)

(a) Calculate A and ϕ if $A_1 = .0012$, $\phi_1 = .052$, $A_2 = .004$, and $\phi_2 = .61$. Also find an expression for $P = A \sin(2\pi ft + \phi)$ if $f = 220$.

(b) Graph $Y_1 = P$ and $Y_2 = P_1 + P_2$ on the same coordinate axes over the interval $[0, .01]$. Are the two graphs the same?

48. ***Tone Heard by a Listener*** Repeat Exercise 47, using $A_1 = .0025$, $\phi_1 = \frac{\pi}{7}$, $A_2 = .001$, $\phi_2 = \frac{\pi}{6}$, and $f = 300$.

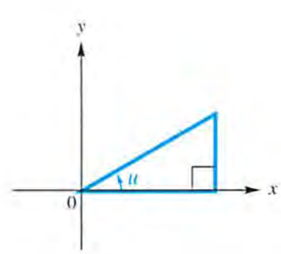
49. ***Depth of Field*** When a large-view camera is used to take a picture of an object that is not parallel to the film, the lens board should be tilted so that the planes containing the subject, the lens board, and the film intersect in a line. This gives the best "depth of field." See the figure. (*Source:* Bushaw, D., et al., *A Sourcebook of Applications of School Mathematics.* Copyright © 1980 by The Mathematical Association of America.)



(a) Write two equations, one relating α, x, and z, and the other relating β, x, y, and z.
(b) Eliminate z from the equations in part (a) to get one equation relating α, β, x, and y.
(c) Solve the equation from part (b) for α.
(d) Solve the equation from part (b) for β.

50. ***Programming Language for Inverse Functions*** In Visual Basic, a widely used programming language for PCs, the only inverse trigonometric function available is arctangent. The other inverse trigonometric functions can be expressed in terms of arctangent as follows.

(a) Let $u = \arcsin x$. Solve the equation for x in terms of u.
(b) Use the result of part (a) to label the three sides of the triangle in the figure in terms of x.



(c) Use the triangle from part (b) to write an equation for tan u in terms of x.
(d) Solve the equation from part (c) for u.

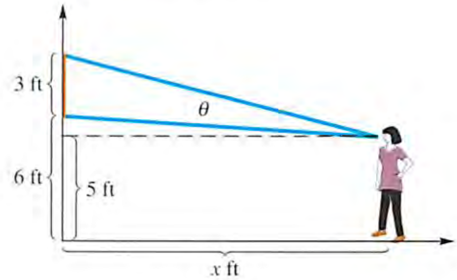
51. ***Alternating Electric Current*** In the study of alternating electric current, instantaneous voltage is modeled by

$$E = E_{\max} \sin 2\pi f t,$$

where f is the number of cycles per second, $E_{\max}$ is the maximum voltage, and t is time in seconds.

(a) Solve the equation for t.
(b) Find the least positive value of t if $E_{\max} = 12$, $E = 5$, and $f = 100$. Use a calculator.

52. ***Viewing Angle of an Observer*** While visiting a museum, Marsha Langlois views a painting that is 3 ft high and hangs 6 ft above the ground. See the figure. Assume her eyes are 5 ft above the ground, and let x be the distance from the spot where she is standing to the wall displaying the painting.



(a) Show that θ, the viewing angle subtended by the painting, is given by

$$\theta = \tan^{-1}\left(\frac{4}{x}\right) - \tan^{-1}\left(\frac{1}{x}\right).$$

(b) Find the value of x to the nearest hundredth for each value of θ.

(i) $\theta = \frac{\pi}{6}$ **(ii)** $\theta = \frac{\pi}{8}$

(c) Find the value of θ to the nearest hundredth for each value of x.

(i) $x = 4$ **(ii)** $x = 3$

53. ***Movement of an Arm*** In the **Section 4.1** Exercises we found the equation

$$y = \frac{1}{3}\sin\frac{4\pi t}{3},$$

where t is time (in seconds) and y is the angle formed by a rhythmically moving arm.

(a) Solve the equation for t.

(b) At what time(s) does the arm form an angle of .3 radian?

54. The function $y = \sec^{-1} x$ is not found on graphing calculators. However, with some models it can be graphed as

$$y = \frac{\pi}{2} - ((x > 0) - (x < 0))\left(\frac{\pi}{2} - \tan^{-1}\left(\sqrt{(x^2 - 1)}\right)\right).$$

(This formula appears as Y_1 in the TI-83/84 Plus screen here.) Use the formula to obtain the graph of $y = \sec^{-1} x$ in the window $[-4, 4]$ by $[0, \pi]$.

Chapter 6 Summary

KEY TERMS

6.1 one-to-one function
inverse function

NEW SYMBOLS

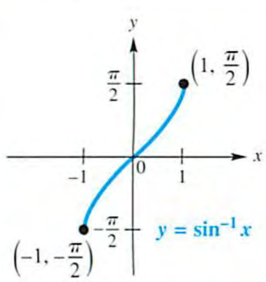
f^{-1}	inverse of function f
$\sin^{-1} x$ (arcsin x)	inverse sine of x
$\cos^{-1} x$ (arccos x)	inverse cosine of x
$\tan^{-1} x$ (arctan x)	inverse tangent of x
$\cot^{-1} x$ (arccot x)	inverse cotangent of x
$\sec^{-1} x$ (arcsec x)	inverse secant of x
$\csc^{-1} x$ (arccsc x)	inverse cosecant of x

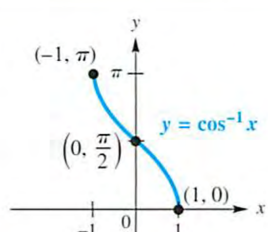
QUICK REVIEW

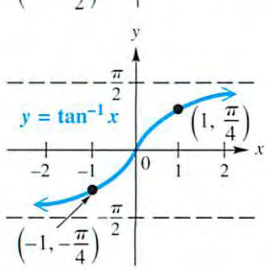
CONCEPTS **EXAMPLES**

6.1 Inverse Circular Functions

Inverse Function	Domain	Range	
		Interval	Quadrants of the Unit Circle
$y = \sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	I and IV
$y = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$	I and II
$y = \tan^{-1} x$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	I and IV
$y = \cot^{-1} x$	$(-\infty, \infty)$	$(0, \pi)$	I and II
$y = \sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi], y \neq \frac{\pi}{2}$	I and II
$y = \csc^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right], y \neq 0$	I and IV







See Figure 14 on page 264 for graphs of the other inverse circular (trigonometric) functions.

Evaluate $y = \cos^{-1} 0$.

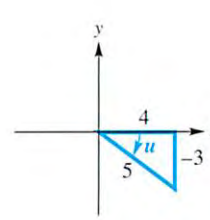
Write $y = \cos^{-1} 0$ as $\cos y = 0$. Then $y = \frac{\pi}{2}$, because $\cos \frac{\pi}{2} = 0$ and $\frac{\pi}{2}$ is in the range of $\cos^{-1} x$.

Use a calculator to find y in radians if $y = \sec^{-1}(-3)$.

With the calculator in radian mode, enter $\sec^{-1}(-3)$ as $\cos^{-1}\left(-\frac{1}{3}\right)$ to get $y \approx 1.9106332$.

Evaluate $\sin\left(\tan^{-1}\left(-\frac{3}{4}\right)\right)$.

Let $u = \tan^{-1}\left(-\frac{3}{4}\right)$. Then $\tan u = -\frac{3}{4}$. Since $\tan u$ is negative when u is in quadrant IV, sketch a triangle as shown.



We want $\sin\left(\tan^{-1}\left(-\frac{3}{4}\right)\right) = \sin u$. From the triangle,

$$\sin u = -\frac{3}{5}.$$

CONCEPTS	EXAMPLES

6.2 Trigonometric Equations I
6.3 Trigonometric Equations II

Solving a Trigonometric Equation

1. Decide whether the equation is linear or quadratic in form, so you can determine the solution method.
2. If only one trigonometric function is present, solve the equation for that function.
3. If more than one trigonometric function is present, rearrange the equation so that one side equals 0. Then try to factor and set each factor equal to 0 to solve.
4. If the equation is quadratic in form, but not factorable, use the quadratic formula. Check that solutions are in the desired interval.
5. Try using identities to change the form of the equation. It may be helpful to square both sides of the equation first. If this is done, check for extraneous solutions.

Solve $\tan\theta + \sqrt{3} = 2\sqrt{3}$ over the interval $[0°, 360°)$. Use a linear method.

$$\tan\theta + \sqrt{3} = 2\sqrt{3}$$
$$\tan\theta = \sqrt{3}$$
$$\theta = 60°$$

Another solution over $[0°, 360°)$ is

$$\theta = 60° + 180° = 240°.$$

The solution set is $\{60°, 240°\}$.

Solve $2\cos^2 x = 1$ over the interval $[0, 2\pi)$ using a double-angle identity.

$$2\cos^2 x = 1$$
$$2\cos^2 x - 1 = 0$$
$$\cos 2x = 0 \quad \text{Cosine double-angle identity}$$
$$2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \quad 0 \le x < 2\pi, \text{ so } 0 \le 2x < 4\pi.$$
$$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \quad \text{Divide by 2.}$$

The solution set is $\left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$.

6.4 Equations Involving Inverse Trigonometric Functions

We solve equations of the form $y = f(x)$ where $f(x)$ is a trigonometric function using inverse trigonometric functions.

Solve $y = 2\sin 3x$ for x.

$$y = 2\sin 3x$$
$$\frac{y}{2} = \sin 3x \quad \text{Divide by 2.}$$
$$3x = \arcsin\frac{y}{2} \quad \text{Definition of arcsine}$$
$$x = \frac{1}{3}\arcsin\frac{y}{2} \quad \text{Multiply by } \tfrac{1}{3}.$$

Techniques introduced in this section also show how to solve equations that involve inverse functions.

Solve $4\tan^{-1} x = \pi$.

$$\tan^{-1} x = \frac{\pi}{4} \quad \text{Divide by 4.}$$
$$x = \tan\frac{\pi}{4} = 1 \quad \text{Definition of arctangent}$$

The solution set is $\{1\}$.

CHAPTER 6 ▶ Review Exercises

1. Graph the inverse sine, cosine, and tangent functions, indicating three points on each graph. Give the domain and range for each.

Concept Check *Tell whether each statement is* true *or* false. *If false, tell why.*

2. The ranges of the inverse tangent and inverse cotangent functions are the same.

3. It is true that $\sin \frac{11\pi}{6} = -\frac{1}{2}$, and therefore $\arcsin\left(-\frac{1}{2}\right) = \frac{11\pi}{6}$.

4. For all x, $\tan(\tan^{-1} x) = x$.

Give the exact real number value of y. Do not use a calculator.

5. $y = \sin^{-1} \frac{\sqrt{2}}{2}$ **6.** $y = \arccos\left(-\frac{1}{2}\right)$ **7.** $y = \tan^{-1}(-\sqrt{3})$

8. $y = \arcsin(-1)$ **9.** $y = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ **10.** $y = \arctan \frac{\sqrt{3}}{3}$

11. $y = \sec^{-1}(-2)$ **12.** $y = \operatorname{arccsc} \frac{2\sqrt{3}}{3}$ **13.** $y = \operatorname{arccot}(-1)$

Give the degree measure of θ. Do not use a calculator.

14. $\theta = \arccos \frac{1}{2}$ **15.** $\theta = \arcsin\left(-\frac{\sqrt{3}}{2}\right)$ **16.** $\theta = \tan^{-1} 0$

Use a calculator to give the degree measure of θ.

17. $\theta = \arctan 1.7804675$ **18.** $\theta = \sin^{-1}(-.66045320)$ **19.** $\theta = \cos^{-1} .80396577$

20. $\theta = \cot^{-1} 4.5046388$ **21.** $\theta = \operatorname{arcsec} 3.4723155$ **22.** $\theta = \csc^{-1} 7.4890096$

Evaluate the following without using a calculator.

23. $\cos(\arccos(-1))$ **24.** $\sin\left(\arcsin\left(-\frac{\sqrt{3}}{2}\right)\right)$ **25.** $\arccos\left(\cos \frac{3\pi}{4}\right)$

26. $\operatorname{arcsec}(\sec \pi)$ **27.** $\tan^{-1}\left(\tan \frac{\pi}{4}\right)$ **28.** $\cos^{-1}(\cos 0)$

29. $\sin\left(\arccos \frac{3}{4}\right)$ **30.** $\cos(\arctan 3)$ **31.** $\cos(\csc^{-1}(-2))$

32. $\sec\left(2 \sin^{-1}\left(-\frac{1}{3}\right)\right)$ **33.** $\tan\left(\arcsin \frac{3}{5} + \arccos \frac{5}{7}\right)$

Write each of the following as an algebraic (nontrigonometric) expression in u, u > 0.

34. $\cos\left(\arctan \frac{u}{\sqrt{1 - u^2}}\right)$ **35.** $\tan\left(\operatorname{arcsec} \frac{\sqrt{u^2 + 1}}{u}\right)$

Solve each equation for solutions over the interval $[0, 2\pi)$. If necessary, express approximate solutions to four decimal places.

36. $\sin^2 x = 1$ **37.** $2 \tan x - 1 = 0$

38. $3 \sin^2 x - 5 \sin x + 2 = 0$ **39.** $\tan x = \cot x$

40. $\sec^4 2x = 4$

41. $\tan^2 2x - 1 = 0$

Give all solutions for each equation.

42. $\sec \frac{x}{2} = \cos \frac{x}{2}$

43. $\cos 2x + \cos x = 0$

44. $4 \sin x \cos x = \sqrt{3}$

Solve each equation for solutions over the interval $[0°, 360°)$. *If necessary, express solutions to the nearest tenth of a degree.*

45. $\sin^2 \theta + 3 \sin \theta + 2 = 0$

46. $2 \tan^2 \theta = \tan \theta + 1$

47. $\sin 2\theta = \cos 2\theta + 1$

48. $2 \sin 2\theta = 1$

49. $3 \cos^2 \theta + 2 \cos \theta - 1 = 0$

50. $5 \cot^2 \theta - \cot \theta - 2 = 0$

Solve each equation in Exercises 51–57 for x.

51. $4y = 2 \sin x$

52. $y = 3 \cos \frac{x}{2}$

53. $2y = \tan(3x + 2)$

54. $5y = 4 \sin x - 3$

55. $\frac{4}{3} \arctan \frac{x}{2} = \pi$

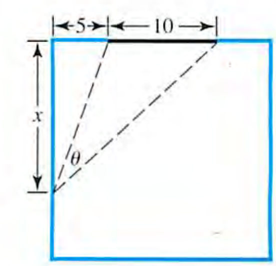
56. $\arccos x = \arcsin \frac{2}{7}$

57. $\arccos x + \arctan 1 = \frac{11\pi}{12}$

58. Solve $d = 550 + 450 \cos\left(\frac{\pi}{50}t\right)$ for t in terms of d.

(Modeling) *Solve each problem.*

59. *Viewing Angle of an Observer* A 10-ft-wide chalkboard is situated 5 ft from the left wall of a classroom. See the figure. A student sitting next to the wall x feet from the front of the classroom has a viewing angle of θ radians.



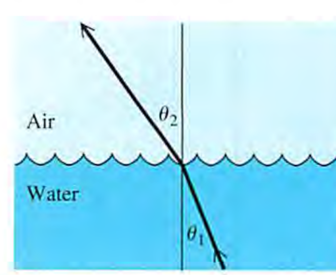
(a) Show that the value of θ is given by the function defined by

$$f(x) = \arctan\left(\frac{15}{x}\right) - \arctan\left(\frac{5}{x}\right).$$

(b) Graph $f(x)$ with a graphing calculator to estimate the value of x that maximizes the viewing angle.

60. *Snell's Law* Recall Snell's law from Exercises 65 and 66 of **Section 2.3:**

$$\frac{c_1}{c_2} = \frac{\sin \theta_1}{\sin \theta_2},$$



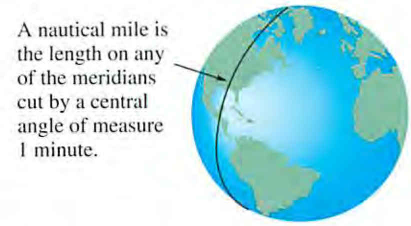
where c_1 is the speed of light in one medium, c_2 is the speed of light in a second medium, and θ_1 and θ_2 are the angles shown in the figure. Suppose a light is shining up through water into the air as in the figure. As θ_1 increases, θ_2 approaches 90°, at which point no light will emerge from the water. Assume the ratio $\frac{c_1}{c_2}$ in this case is .752. For what value of θ_1 does $\theta_2 = 90°$? This value of θ_1 is called the **critical angle** for water.

61. ***Snell's Law*** Refer to Exercise 60. What happens when θ_1 is greater than the critical angle?

62. ***British Nautical Mile*** The British nautical mile is defined as the length of a minute of arc of a meridian. Since Earth is flat at its poles, the nautical mile, in feet, is given by

$$L = 6077 - 31 \cos 2\theta,$$

where θ is the latitude in degrees. See the figure. (*Source:* Bushaw, D., et al., *A Sourcebook of Applications of School Mathematics.* Copyright © 1980 by The Mathematical Association of America.)



(a) Find the latitude between 0° and 90° at which the nautical mile is 6074 ft.

(b) At what latitude between 0° and 180° is the nautical mile 6108 ft?

(c) In the United States, the nautical mile is defined everywhere as 6080.2 ft. At what latitude between 0° and 90° does this agree with the British nautical mile?

63. The function $y = \csc^{-1} x$ is not found on graphing calculators. However, with some models it can be graphed as

$$y = ((x > 0) - (x < 0))\left(\frac{\pi}{2} - \tan^{-1}\left(\sqrt{(x^2 - 1)}\right)\right).$$

(This formula appears as Y_1 in the TI-83/84 Plus screen here.) Use the formula to obtain the graph of $y = \csc^{-1} x$ in the window $[-4, 4]$ by $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

64. **(a)** Use the graph of $y = \sin^{-1} x$ to approximate $\sin^{-1} .4$.

(b) Use the inverse sine key of a graphing calculator to approximate $\sin^{-1} .4$.

CHAPTER 6 ▶ Test

1. Graph $y = \sin^{-1} x$, and indicate the coordinates of three points on the graph. Give the domain and range.

2. Find the exact value of y for each equation.

(a) $y = \arccos\left(-\frac{1}{2}\right)$ **(b)** $y = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

(c) $y = \tan^{-1} 0$ **(d)** $y = \text{arcsec}(-2)$

3. Give the degree measure of θ.

(a) $\theta = \arccos \frac{\sqrt{3}}{2}$ **(b)** $\theta = \tan^{-1}(-1)$

(c) $\theta = \cot^{-1}(-1)$ **(d)** $\theta = \csc^{-1}\left(-\frac{2\sqrt{3}}{3}\right)$

4. Use a calculator to give each value in decimal degrees to the nearest hundredth.

(a) $\sin^{-1} .67610476$ (b) $\sec^{-1} 1.0840880$ (c) $\cot^{-1}(-.7125586)$

5. Find each exact value.

(a) $\cos\left(\arcsin \frac{2}{3}\right)$ (b) $\sin\left(2 \cos^{-1} \frac{1}{3}\right)$

6. Explain why $\sin^{-1} 3$ cannot be defined.

7. Explain why $\arcsin\left(\sin \frac{5\pi}{6}\right) \neq \frac{5\pi}{6}$.

8. Write $\tan(\arcsin u)$ as an algebraic (nontrigonometric) expression in u, $u > 0$.

Solve each equation for solutions over the interval $[0°, 360°)$. *Express approximate solutions to the nearest tenth of a degree.*

9. $-3 \sec \theta + 2\sqrt{3} = 0$ 10. $\sin^2 \theta = \cos^2 \theta + 1$ 11. $\csc^2 \theta - 2 \cot \theta = 4$

Solve each equation for solutions over the interval $[0, 2\pi)$. *Express approximate solutions to four decimal places.*

12. $\cos x = \cos 2x$ 13. $\sqrt{2} \cos 3x - 1 = 0$ 14. $\sin x \cos x = \frac{1}{3}$

15. Solve $\sin^2 \theta = -\cos 2\theta$, giving all solutions in degrees.

16. Solve $2\sqrt{3} \sin \frac{x}{2} = 3$, giving all solutions in radians.

17. Solve each equation for x.

(a) $y = \cos 3x$ (b) $\arcsin x = \arctan \frac{4}{3}$

18. ***(Modeling) Movement of a Runner's Arm*** A runner's arm swings rhythmically according to the model

$$y = \frac{\pi}{8} \cos\left[\pi\left(t - \frac{1}{3}\right)\right],$$

where y represents the angle between the actual position of the upper arm and the downward vertical position, and t represents time in seconds. At what times over the interval $[0, 3)$ is the angle y equal to 0?

CHAPTER 6 ▶ Quantitative Reasoning

How can we determine the amount of oil in a submerged storage tank?

The level of oil in a storage tank buried in the ground can be found in much the same way a dipstick is used to determine the oil level in an automobile crankcase. The person in the figure on the left has lowered a calibrated rod into an oil storage tank. When the rod is removed, the reading on the rod can be used with the dimensions of the storage tank to calculate the amount of oil in the tank.

Suppose the ends of the cylindrical storage tank in the figure are circles of radius 3 ft and the cylinder is 20 ft long. Determine the volume of oil in the tank to the nearest cubic foot if the rod shows a depth of 2 ft. (*Hint:* The volume will be 20 times the area of the shaded segment of the circle shown in the figure on the right.)