7 Applications of Trigonometry and Vectors

World technology took a giant step forward when the *Global Positioning System (GPS),* a network of more than 24 communications satellites, achieved full operational capability on July 17, 1995. GPS satellites transmit signals from 12,552 miles above Earth's surface.

The magnitude and direction of four arrows, or *vectors,* representing the signals from four GPS satellites to a receiver on Earth, are used to calculate location on the globe. Navigators use a similar calculation process with three vectors, called *triangulation,* to determine the location of a ship or an airplane. (*Source:* A. El-Rabbany, *Introduction to GPS, The Global Positioning System.*)

In this chapter, we use trigonometry and vectors to solve problems involving navigation, weight and force balancing, and distance and angle determination, among others.

7.1 Oblique Triangles and the Law of Sines

Congruency and Oblique Triangles ▪ Derivation of the Law of Sines ▪ Solving SAA and ASA Triangles (Case 1) ▪ Area of a Triangle

The concepts of solving triangles developed in **Chapter 2** can be extended to *all* triangles. If any three of the six side and angle measures of a triangle are known (provided at least one measure is the length of a side), then the other three measures can be found.

Congruency and Oblique Triangles The following axioms from geometry allow us to prove that two triangles are congruent (that is, their corresponding sides and angles are equal).

CONGRUENCE AXIOMS

Side-Angle-Side (SAS)	If two sides and the included angle of one triangle are equal, respectively, to two sides and the included angle of a second triangle, then the triangles are congruent.
Angle-Side-Angle (ASA)	If two angles and the included side of one triangle are equal, respectively, to two angles and the included side of a second triangle, then the triangles are congruent.
Side-Side-Side (SSS)	If three sides of one triangle are equal, respectively, to three sides of a second triangle, then the triangles are congruent.

If a side and *any* two angles are given (SAA), the third angle is easily determined by the angle sum formula ($A + B + C = 180°$), and then the ASA axiom can be applied. Keep in mind that whenever SAS, ASA, or SSS is given, the triangle is unique. We continue to label triangles as we did earlier with right triangles: side a opposite angle A, side b opposite angle B, and side c opposite angle C.

A triangle that is not a right triangle is called an **oblique triangle.** The measures of the three sides and the three angles of a triangle can be found if at least one side and any other two measures are known. There are four possible cases.

DATA REQUIRED FOR SOLVING OBLIQUE TRIANGLES

Case 1	One side and two angles are known (SAA or ASA).
Case 2	Two sides and one angle not included between the two sides are known (SSA). This case may lead to more than one triangle.
Case 3	Two sides and the angle included between the two sides are known (SAS).
Case 4	Three sides are known (SSS).

▶ **Note** ***If we know three angles of a triangle, we cannot find unique side lengths since* AAA *assures us only of similarity, not congruence.*** For example, there are infinitely many triangles ABC with $A = 35°$, $B = 65°$, and $C = 80°$.

Case 1, discussed in this section, and Case 2, discussed in **Section 7.2,** require the *law of sines.* Cases 3 and 4, discussed in **Section 7.3,** require the *law of cosines.*

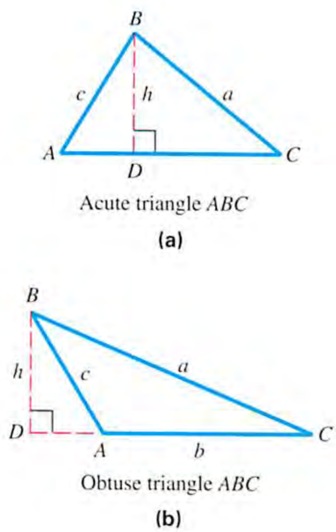


Figure 1

Derivation of the Law of Sines To derive the law of sines, we start with an oblique triangle, such as the *acute triangle* in Figure 1(a) or the *obtuse triangle* in Figure 1(b). This discussion applies to both triangles. First, construct the perpendicular from B to side AC (or its extension). Let h be the length of this perpendicular. Then c is the hypotenuse of right triangle ADB, and a is the hypotenuse of right triangle BDC.

In triangle ADB, $\sin A = \frac{h}{c}$, or $h = c \sin A$.

(Section 2.1)

In triangle BDC, $\sin C = \frac{h}{a}$, or $h = a \sin C$.

Since $h = c \sin A$ and $h = a \sin C$,

$$a \sin C = c \sin A$$

$$\frac{a}{\sin A} = \frac{c}{\sin C}.$$ Divide each side by $\sin A \sin C$.

In a similar way, by constructing perpendicular lines from the other vertices, it can be shown that

$$\frac{a}{\sin A} = \frac{b}{\sin B} \quad \text{and} \quad \frac{b}{\sin B} = \frac{c}{\sin C}.$$

This discussion proves the following theorem.

LAW OF SINES

In any triangle ABC, with sides a, b, and c,

$$\frac{a}{\sin A} = \frac{b}{\sin B}, \quad \frac{a}{\sin A} = \frac{c}{\sin C}, \quad \text{and} \quad \frac{b}{\sin B} = \frac{c}{\sin C}.$$

This can be written in compact form as

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

That is, according to the law of sines, the lengths of the sides in a triangle are proportional to the sines of the measures of the angles opposite them.

When solving for an angle, we use an alternative form of the law of sines,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$ Alternative form

▶ **Note** When using the law of sines, a good strategy is to select an equation so that the unknown variable is in the numerator and all other variables are known.

Solving SAA and ASA Triangles (Case 1) If two angles and one side of a triangle are known (Case 1, SAA or ASA), then the law of sines can be used to solve the triangle.

▶ EXAMPLE 1 USING THE LAW OF SINES TO SOLVE A TRIANGLE (SAA)

Solve triangle ABC if $A = 32.0°$, $B = 81.8°$, and $a = 42.9$ cm.

Solution Start by drawing a triangle, roughly to scale, and labeling the given parts as in Figure 2. Since the values of A, B, and a are known, use the form of the law of sines that involves these variables, and then solve for b.

Be sure to label a sketch carefully to help set up the correct equation.

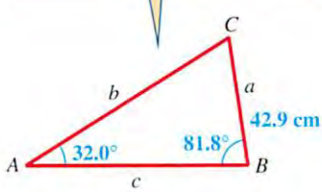


Figure 2

Choose a form that has the unknown variable in the numerator.

$$\frac{a}{\sin A} = \frac{b}{\sin B} \quad \text{Law of sines}$$

$$\frac{42.9}{\sin 32.0°} = \frac{b}{\sin 81.8°} \quad \text{Substitute the given values.}$$

$$b = \frac{42.9 \sin 81.8°}{\sin 32.0°} \quad \text{Multiply by } \sin 81.8°\text{; rewrite.}$$

$$b \approx 80.1 \text{ cm} \quad \text{Approximate with a calculator.}$$

To find C, use the fact that the sum of the angles of any triangle is 180°.

$$A + B + C = 180° \quad \text{(Section 1.2)}$$

$$C = 180° - A - B$$

$$C = 180° - 32.0° - 81.8°$$

$$C = 66.2°$$

Use the law of sines to find c. (Why does the Pythagorean theorem not apply?)

$$\frac{a}{\sin A} = \frac{c}{\sin C} \quad \text{Law of sines}$$

$$\frac{42.9}{\sin 32.0°} = \frac{c}{\sin 66.2°} \quad \text{Substitute.}$$

$$c = \frac{42.9 \sin 66.2°}{\sin 32.0°} \quad \text{Multiply by } \sin 66.2°\text{; rewrite.}$$

$$c \approx 74.1 \text{ cm} \quad \text{Approximate with a calculator.}$$

NOW TRY EXERCISE 7. ◀

▶ EXAMPLE 2 USING THE LAW OF SINES IN AN APPLICATION (ASA)

Jerry Keefe wishes to measure the distance across the Big Muddy River. See Figure 3. He determines that $C = 112.90°$, $A = 31.10°$, and $b = 347.6$ ft. Find the distance a across the river.

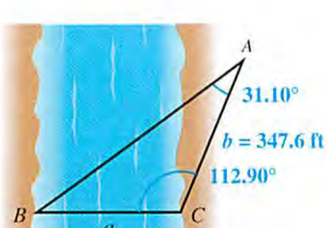


Figure 3

Solution To use the law of sines, one side and the angle opposite it must be known. Since b is the only side whose length is given, angle B must be found before the law of sines can be used.

$$B = 180° - A - C$$

$$B = 180° - 31.10° - 112.90° = 36.00°$$

Now use the form of the law of sines involving A, B, and b to find a.

Solve for a.

$$\frac{a}{\sin A} = \frac{b}{\sin B} \quad \text{Law of sines}$$

$$\frac{a}{\sin 31.10°} = \frac{347.6}{\sin 36.00°} \quad \text{Substitute known values.}$$

$$a = \frac{347.6 \sin 31.10°}{\sin 36.00°} \quad \text{Multiply by } \sin 31.10°.$$

$$a \approx 305.5 \text{ ft} \quad \text{Use a calculator.}$$

NOW TRY EXERCISE 25.

The next example involves the concept of bearing, first discussed in **Chapter 2.**

EXAMPLE 3 USING THE LAW OF SINES IN AN APPLICATION (ASA)

Two ranger stations are on an east-west line 110 mi apart. A forest fire is located on a bearing of N 42° E from the western station at A and a bearing of N 15° E from the eastern station at B. How far is the fire from the western station?

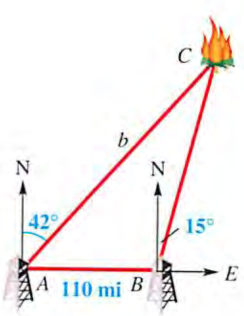


Figure 4

Solution Figure 4 shows the two stations at points A and B and the fire at point C. Angle $BAC = 90° - 42° = 48°$, the obtuse angle at B measures $90° + 15° = 105°$, and the third angle, C, measures $180° - 105° - 48° = 27°$. We use the law of sines to find side b.

Solve for b.

$$\frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{Law of sines}$$

$$\frac{b}{\sin 105°} = \frac{110}{\sin 27°} \quad \text{Substitute known values.}$$

$$b = \frac{110 \sin 105°}{\sin 27°} \quad \text{Multiply by } \sin 105°.$$

$$b \approx 234 \text{ mi} \quad \text{Use a calculator.}$$

NOW TRY EXERCISE 27.

Area of a Triangle

The method used to derive the law of sines can also be used to derive a formula to find the area of a triangle. A familiar formula for the area of a triangle is $\mathcal{A} = \frac{1}{2}bh$, where $\mathcal{A}$ represents area, b base, and h height. This formula cannot always be used easily since in practice, h is often unknown. To find another formula, refer to acute triangle ABC in Figure 5(a) or obtuse triangle ABC in Figure 5(b), shown on the next page.

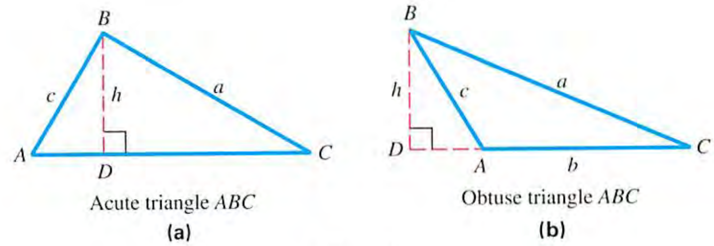


Figure 5

A perpendicular has been drawn from B to the base of the triangle (or the extension of the base). This perpendicular forms two right triangles. Using triangle ABD,

$$\sin A = \frac{h}{c}, \qquad \text{or} \qquad h = c \sin A.$$

Substituting into the formula for the area of a triangle,

$$\mathcal{A} = \frac{1}{2}bh = \frac{1}{2}bc \sin A.$$

Any other pair of sides and the angle between them could have been used.

AREA OF A TRIANGLE (SAS)

In any triangle ABC, the area $\mathcal{A}$ is given by the following formulas:

$$\mathcal{A} = \frac{1}{2}bc \sin A, \quad \mathcal{A} = \frac{1}{2}ab \sin C, \quad \text{and} \quad \mathcal{A} = \frac{1}{2}ac \sin B.$$

That is, the area is half the product of the lengths of two sides and the sine of the angle included between them.

▶ **Note** If the included angle measures 90°, its sine is 1, and the formula becomes the familiar $\mathcal{A} = \frac{1}{2}bh$.

▶ EXAMPLE 4 FINDING THE AREA OF A TRIANGLE (SAS)

Find the area of triangle ABC in Figure 6.

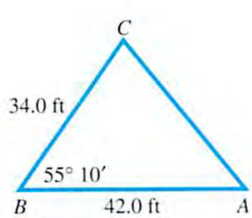


Figure 6

Solution We are given $B = 55° \, 10'$, $a = 34.0$ ft, and $c = 42.0$ ft, so

$$\mathcal{A} = \frac{1}{2}ac \sin B = \frac{1}{2}(34.0)(42.0) \sin 55° \, 10' \approx 586 \text{ ft}^2.$$

NOW TRY EXERCISE 43. ◀

▶ EXAMPLE 5 FINDING THE AREA OF A TRIANGLE (ASA)

Find the area of triangle ABC if $A = 24° \, 40'$, $b = 27.3$ cm, and $C = 52° \, 40'$.

Solution Before the area formula can be used, we must find either a or c. Since the sum of the measures of the angles of any triangle is 180°,

$$B = 180° - 24° \, 40' - 52° \, 40' = 102° \, 40'.$$

We can use the law of sines to find a.

Solve for a.

$$\frac{a}{\sin A} = \frac{b}{\sin B} \qquad \text{Law of sines}$$

$$\frac{a}{\sin 24^\circ 40'} = \frac{27.3}{\sin 102^\circ 40'} \qquad \text{Substitute known values.}$$

$$a = \frac{27.3 \sin 24^\circ 40'}{\sin 102^\circ 40'} \qquad \text{Multiply by } \sin 24^\circ 40'.$$

$$a \approx 11.7 \text{ cm} \qquad \text{Use a calculator.}$$

Now, we find the area.

$$\mathcal{A} = \frac{1}{2}ab \sin C = \frac{1}{2}(11.7)(27.3) \sin 52^\circ 40' \approx 127 \text{ cm}^2$$

NOW TRY EXERCISE 49.

▶ **Caution** Whenever possible, use given values in solving triangles or finding areas rather than values obtained in intermediate steps to avoid possible rounding errors.

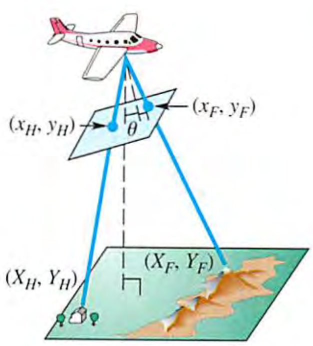


Figure 7

CONNECTIONS

Aerial photographs can be used to provide coordinates of ordered pairs to determine distances on the ground. Suppose we assign coordinates as shown in Figure 7. If an object's photographic coordinates are (x, y), then its ground coordinates (X, Y) in feet can be computed using the following formulas.

$$X = \frac{(a-h)x}{f \sec \theta - y \sin \theta}, \qquad Y = \frac{(a-h)y \cos \theta}{f \sec \theta - y \sin \theta}$$

Here, f is focal length of the camera in inches, a is altitude in feet of the airplane, and h is elevation in feet of the object. Suppose that a house has photographic coordinates $(x_H, y_H) = (.9, 3.5)$ with elevation 150 ft, while a nearby forest fire has photographic coordinates $(x_F, y_F) = (2.1, -2.4)$ and is at elevation 690 ft. If the photograph was taken at 7400 ft by a camera with focal length 6 in. and tilt angle $\theta = 4.1^\circ$, we can use these formulas to find the distance on the ground between the house and the fire. (*Source:* Moffitt, F. and E. Mikhail, *Photogrammetry*, Third Edition, Harper & Row, 1980.)

FOR DISCUSSION OR WRITING

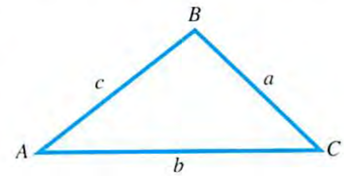
1. Use the formulas to find the ground coordinates of the house and the fire to the nearest tenth of a foot.
2. Use the distance formula given in **Appendix B** to find the required distance on the ground to the nearest tenth of a foot.

7.1 Exercises

1. *Concept Check* Consider the oblique triangle ABC. Which one of the following proportions is *not* valid?

A. $\frac{a}{b} = \frac{\sin A}{\sin B}$ B. $\frac{a}{\sin A} = \frac{b}{\sin B}$

C. $\frac{\sin A}{a} = \frac{b}{\sin B}$ D. $\frac{\sin A}{a} = \frac{\sin B}{b}$

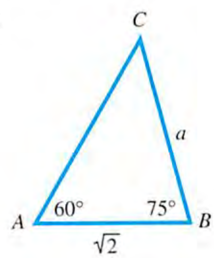


2. *Concept Check* Which two of the following situations do not provide sufficient information for solving a triangle by the law of sines?

A. You are given two angles and the side included between them.
B. You are given two angles and a side opposite one of them.
C. You are given two sides and the angle included between them.
D. You are given three sides.

Find the length of each side a. Do not use a calculator.

3.



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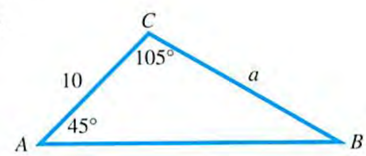
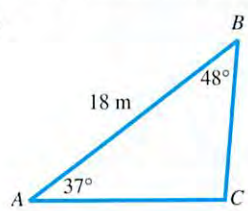


Determine the remaining sides and angles of each triangle ABC. See Example 1.

5.



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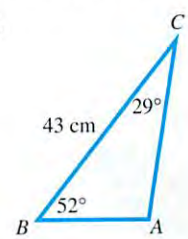

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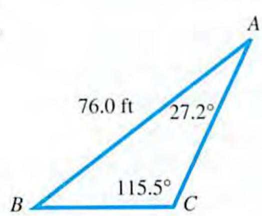


8.

9. $A = 68.41°, B = 54.23°, a = 12.75$ ft

10. $C = 74.08°, B = 69.38°, c = 45.38$ m

11. $A = 87.2°, b = 75.9$ yd, $C = 74.3°$

12. $B = 38° 40', a = 19.7$ cm, $C = 91° 40'$

13. $B = 20° 50'$, $C = 103° 10'$, $AC = 132$ ft

14. $A = 35.3°$, $B = 52.8°$, $AC = 675$ ft

15. $A = 39.70°$, $C = 30.35°$, $b = 39.74$ m

16. $C = 71.83°$, $B = 42.57°$, $a = 2.614$ cm

17. $B = 42.88°$, $C = 102.40°$, $b = 3974$ ft

18. $A = 18.75°$, $B = 51.53°$, $c = 2798$ yd

19. $A = 39° 54'$, $a = 268.7$ m, $B = 42° 32'$

20. $C = 79° 18'$, $c = 39.81$ mm, $A = 32° 57'$

21. Explain why the law of sines cannot be used to solve a triangle if we are given the lengths of the three sides of the triangle.

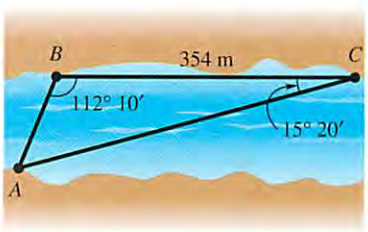
22. *Concept Check* In Example 1, we asked the question, "Why does the Pythagorean theorem not apply?" Answer this question.

23. Terry Harris, a perceptive trigonometry student, makes the statement, "If we know *any* two angles and one side of a triangle, then the triangle is uniquely determined." Is this a valid statement? Explain, referring to the congruence axioms given in this section.

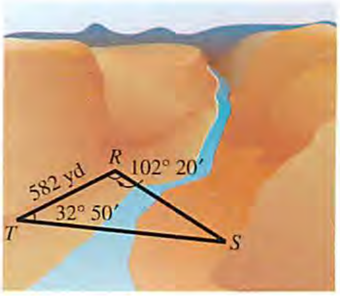
24. *Concept Check* If a is twice as long as b, is A necessarily twice as large as B?

Solve each problem. See Examples 2 and 3.

25. *Distance Across a River* To find the distance AB across a river, a surveyor laid off a distance $BC = 354$ m on one side of the river. It is found that $B = 112° 10'$ and $C = 15° 20'$. Find AB. See the figure.



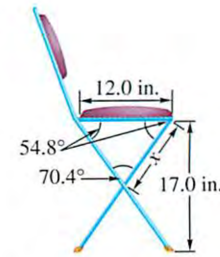
26. *Distance Across a Canyon* To determine the distance RS across a deep canyon, Rhonda lays off a distance $TR = 582$ yd. She then finds that $T = 32° 50'$ and $R = 102° 20'$. Find RS.



27. *Distance a Ship Travels* A ship is sailing due north. At a certain point the bearing of a lighthouse 12.5 km away is N 38.8° E. Later on, the captain notices that the bearing of the lighthouse has become S 44.2° E. How far did the ship travel between the two observations of the lighthouse?

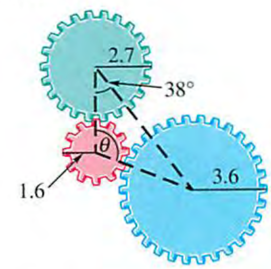
28. *Distance Between Radio Direction Finders* Radio direction finders are placed at points A and B, which are 3.46 mi apart on an east-west line, with A west of B. From A the bearing of a certain radio transmitter is 47.7°, and from B the bearing is 302.5°. Find the distance of the transmitter from A.

29. ***Measurement of a Folding Chair*** A folding chair is to have a seat 12.0 in. deep with angles as shown in the figure. How far down from the seat should the crossing legs be joined? (Find x in the figure.)

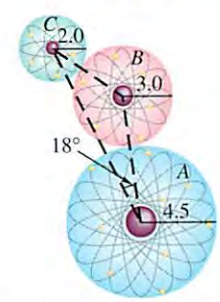


30. ***Distance Across a River*** Standing on one bank of a river flowing north, Mark notices a tree on the opposite bank at a bearing of 115.45°. Lisa is on the same bank as Mark, but 428.3 m away. She notices that the bearing of the tree is 45.47°. The two banks are parallel. What is the distance across the river?

31. ***Angle Formed by Radii of Gears*** Three gears are arranged as shown in the figure. Find angle θ.

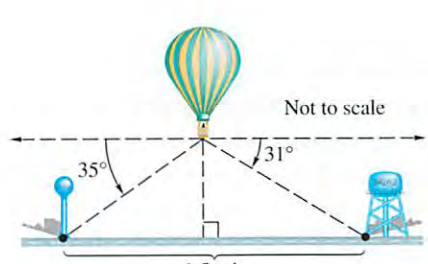


32. ***Distance Between Atoms*** Three atoms with atomic radii of 2.0, 3.0, and 4.5 are arranged as in the figure. Find the distance between the centers of atoms A and C.

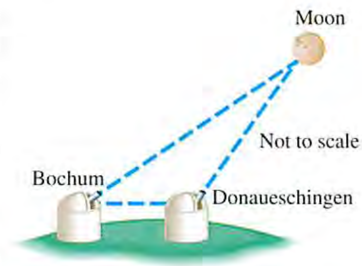


33. ***Distance Between a Ship and a Lighthouse*** The bearing of a lighthouse from a ship was found to be N 37° E. After the ship sailed 2.5 mi due south, the new bearing was N 25° E. Find the distance between the ship and the lighthouse at each location.

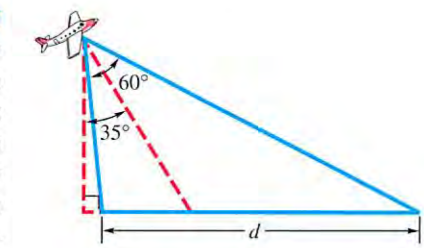
34. ***Height of a Balloon*** A balloonist is directly above a straight road 1.5 mi long that joins two villages. She finds that the town closer to her is at an angle of depression of 35°, and the farther town is at an angle of depression of 31°. How high above the ground is the balloon?



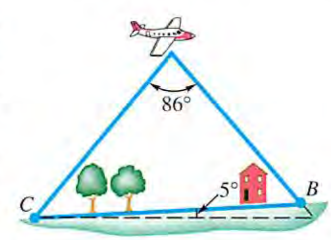
35. ***Distance to the Moon*** Since the moon is a relatively close celestial object, its distance can be measured directly by taking two different photographs at precisely the same time from two different locations. The moon will have a different angle of elevation at each location. On April 29, 1976, at 11:35 A.M., the lunar angles of elevation during a partial solar eclipse at Bochum in upper Germany and at Donaueschingen in lower Germany were measured as 52.6997° and 52.7430°, respectively. The two cities are 398 km apart. Calculate the distance to the moon from Bochum on this day, and compare it with the actual value of 406,000 km. Disregard the curvature of Earth in this calculation. (*Source:* Scholosser, W., T. Schmidt-Kaler, and E. Milone, *Challenges of Astronomy,* Springer-Verlag, 1991.)



36. ***Ground Distances Measured by Aerial Photography*** The distance covered by an aerial photograph is determined by both the focal length of the camera and the tilt of the camera from the perpendicular to the ground. Although the tilt is usually small, both archaeological and Canadian photographs often use larger tilts. A camera lens with a 12-in. focal length will have an angular coverage of 60°. If an aerial photograph is taken with this camera tilted $\theta = 35°$ at an altitude of 5000 ft, calculate the ground distance d to the nearest foot that will be shown in this photograph. (*Source:* Brooks, R. and D. Johannes, *Phytoarchaeology,* Dioscorides Press, 1990; Moffitt, F. and E. Mikhail, *Photogrammetry,* Third Edition, Harper & Row, 1980.)



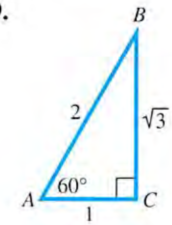
37. ***Ground Distances Measured by Aerial Photography*** Refer to Exercise 36. A camera lens with a 6-in. focal length has an angular coverage of 86°. Suppose an aerial photograph is taken vertically with no tilt at an altitude of 3500 ft over ground with an increasing slope of 5°, as shown in the figure. Calculate the ground distance CB that would appear in the resulting photograph. (*Source:* Moffitt, F. and E. Mikhail, *Photogrammetry,* Third Edition, Harper & Row, 1980.)



38. ***Ground Distances Measured by Aerial Photography*** Repeat Exercise 37 if the camera lens has an 8.25-in. focal length with an angular coverage of 72°.

Find the area of each triangle using the formula $\mathcal{A} = \frac{1}{2}bh$, *and then verify that the formula* $\mathcal{A} = \frac{1}{2}ab \sin C$ *gives the same result.*

39.



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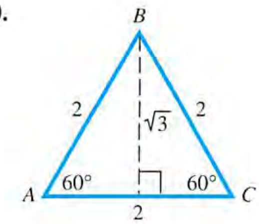

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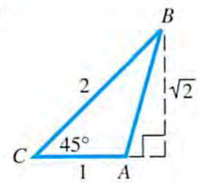

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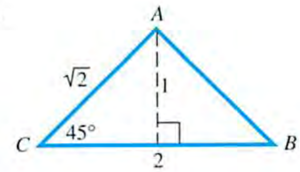


Find the area of each triangle ABC. See Examples 4 and 5.

43. $A = 42.5°, b = 13.6$ m, $c = 10.1$ m

44. $C = 72.2°, b = 43.8$ ft, $a = 35.1$ ft

45. $B = 124.5°, a = 30.4$ cm, $c = 28.4$ cm

46. $C = 142.7°, a = 21.9$ km, $b = 24.6$ km

47. $A = 56.80°, b = 32.67$ in., $c = 52.89$ in.

48. $A = 34.97°, b = 35.29$ m, $c = 28.67$ m

49. $A = 30.50°, b = 13.00$ cm, $C = 112.60°$

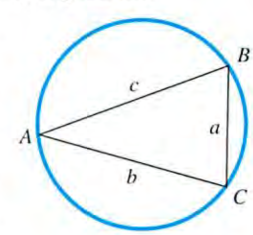
50. $A = 59.80°, b = 15.00$ m, $C = 53.10°$

Solve each problem.

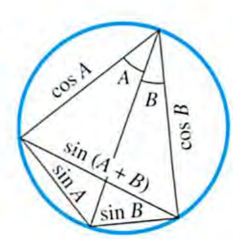
51. ***Area of a Metal Plate*** A painter is going to apply a special coating to a triangular metal plate on a new building. Two sides measure 16.1 m and 15.2 m. She knows that the angle between these sides is 125°. What is the area of the surface she plans to cover with the coating?

52. ***Area of a Triangular Lot*** A real estate agent wants to find the area of a triangular lot. A surveyor takes measurements and finds that two sides are 52.1 m and 21.3 m, and the angle between them is 42.2°. What is the area of the triangular lot?

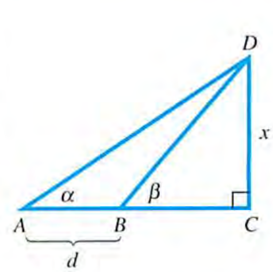
53. ***Triangle Inscribed in a Circle*** For a triangle inscribed in a circle of radius r, each law of sines ratio $\frac{a}{\sin A}$, $\frac{b}{\sin B}$, and $\frac{c}{\sin C}$ has value $2r$. The circle in the figure has diameter 1. What are the values of a, b, and c? (*Note:* This result provides an alternative way to define the sine function for angles between 0° and 180°. It was used nearly 2000 yr ago by the mathematician Ptolemy to construct one of the earliest trigonometric tables.)



54. ***Theorem of Ptolemy*** The following theorem is also attributed to Ptolemy: *In a quadrilateral inscribed in a circle, the product of the diagonals is equal to the sum of the products of the opposite sides.* (*Source:* Eves, H., *An Introduction to the History of Mathematics,* Sixth Edition, Saunders College Publishing, 1990.) The circle in the figure has diameter 1. Explain why the lengths of the line segments are as shown, and then apply Ptolemy's theorem to derive the formula for the sine of the sum of two angles.

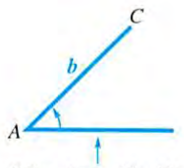


55. Several of the exercises on right triangle applications involved a figure similar to the one shown here, in which angles α and β and the length of line segment AB are known, and the length of side CD is to be determined. Use the law of sines to obtain x in terms of α, β, and d.



7.2 The Ambiguous Case of the Law of Sines

Description of the Ambiguous Case ▪ Solving SSA Triangles (Case 2) ▪ Analyzing Data for Possible Number of Triangles



B lies along this side if a triangle exists.

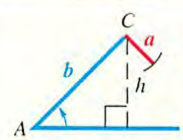
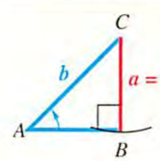
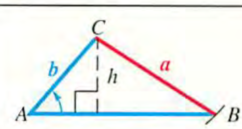
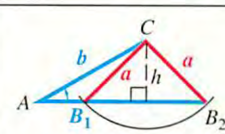
Figure 8

Description of the Ambiguous Case

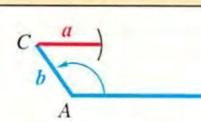
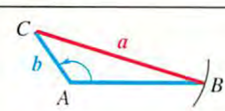
We used the law of sines to solve triangles involving Case 1, SAA or ASA, in **Section 7.1.** If we are given the lengths of two sides and the angle opposite one of them (Case 2, SSA), then zero, one, or two such triangles may exist. (There is no SSA axiom.)

Suppose we know the measure of acute angle A of triangle ABC, the length of side a, and the length of side b, as shown in Figure 8. Now we must draw the side of length a opposite angle A. The table shows possible outcomes. This situation (SSA) is called the **ambiguous case** of the law of sines.

If angle A is acute, there are four possible outcomes.

Number of Triangles	Sketch	Applying Law of Sines Leads to
0		$\sin B > 1$, $a < h < b$
1		$\sin B = 1$, $a = h$ and $h < b$
1		$0 < \sin B < 1$, $a \geq b$
2		$0 < \sin B_2 < 1$, $h < a < b$

If angle A is obtuse, there are two possible outcomes.

Number of Triangles	Sketch	Applying Law of Sines Leads to
0		$\sin B \geq 1$, $a \leq b$
1		$0 < \sin B < 1$, $a > b$

The following basic facts should be kept in mind to determine which situation applies.

APPLYING THE LAW OF SINES

1. For any angle θ of a triangle, $0 < \sin\theta \leq 1$. If $\sin\theta = 1$, then $\theta = 90°$ and the triangle is a right triangle.
2. $\sin\theta = \sin(180° - \theta)$ (Supplementary angles have the same sine value.)
3. The smallest angle is opposite the shortest side, the largest angle is opposite the longest side, and the middle-valued angle is opposite the intermediate side (assuming the triangle has sides that are all of different lengths).

Solving SSA Triangles (Case 2)

EXAMPLE 1 SOLVING THE AMBIGUOUS CASE (NO SUCH TRIANGLE)

Solve triangle ABC if $B = 55° 40'$, $b = 8.94$ m, and $a = 25.1$ m.

Solution Since we are given B, b, and a, we can use the law of sines to find A.

Choose a form that has the unknown variable in the numerator.

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$ Law of sines (alternative form) (Section 7.1)

$$\frac{\sin A}{25.1} = \frac{\sin 55° 40'}{8.94}$$ Substitute the given values.

$$\sin A = \frac{25.1 \sin 55° 40'}{8.94}$$ Multiply by 25.1.

$$\sin A \approx 2.3184379$$ Use a calculator.

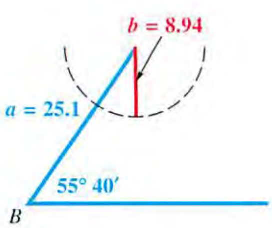


Figure 9

Since $\sin A$ cannot be greater than 1, there can be no such angle A and thus, no triangle with the given information. An attempt to sketch such a triangle leads to the situation shown in Figure 9.

NOW TRY EXERCISE 17.

Note In the ambiguous case, we are given two sides and an angle opposite one of the sides (SSA). For example, suppose b, c, and angle C are given. This situation represents the ambiguous case because angle C is opposite side c.

EXAMPLE 2 SOLVING THE AMBIGUOUS CASE (TWO TRIANGLES)

Solve triangle ABC if $A = 55.3°$, $a = 22.8$ ft, and $b = 24.9$ ft.

Solution To begin, use the law of sines to find angle B.

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$ Solve for sin B.

$$\frac{\sin 55.3°}{22.8} = \frac{\sin B}{24.9}$$ Substitute known values.

$$\sin B = \frac{24.9 \sin 55.3°}{22.8}$$ Multiply by 24.9; rewrite.

$$\sin B \approx .8978678$$ Use a calculator.

There are two angles B between 0° and 180° that satisfy this condition. Since $\sin B \approx .8978678$, to the nearest tenth one value of B is

$$B_1 = 63.9°. \quad \text{Use the inverse sine function. (Section 6.1)}$$

Supplementary angles have the same sine value, so another *possible* value of B is

$$B_2 = 180° - 63.9° = 116.1°. \quad \text{(Section 1.1)}$$

To see if $B_2 = 116.1°$ is a valid possibility, add 116.1° to the measure of A, 55.3°. Since $116.1° + 55.3° = 171.4°$, and this sum is less than 180°, it is a valid angle measure for this triangle.

Now separately solve triangles AB_1C_1 and AB_2C_2 shown in Figure 10. Begin with AB_1C_1. Find C_1 first.

$$C_1 = 180° - A - B_1 \quad \text{(Section 1.2)}$$
$$C_1 = 180° - 55.3° - 63.9°$$
$$C_1 = 60.8°$$

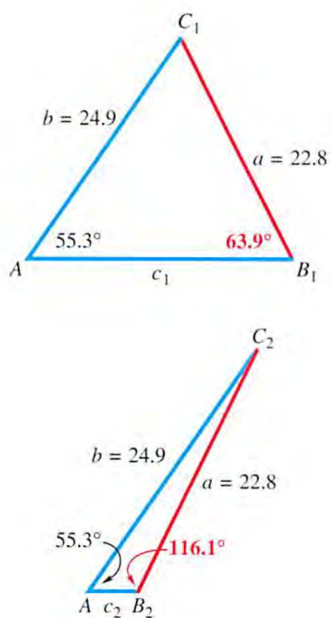


Figure 10

Now, use the law of sines to find c_1.

$$\frac{a}{\sin A} = \frac{c_1}{\sin C_1} \quad \text{Solve for } c_1.$$
$$\frac{22.8}{\sin 55.3°} = \frac{c_1}{\sin 60.8°}$$
$$c_1 = \frac{22.8 \sin 60.8°}{\sin 55.3°}$$
$$c_1 \approx 24.2 \text{ ft}$$

To solve triangle AB_2C_2, first find C_2.

$$C_2 = 180° - A - B_2$$
$$C_2 = 180° - 55.3° - 116.1°$$
$$C_2 = 8.6°$$

Use the law of sines to find c_2.

$$\frac{a}{\sin A} = \frac{c_2}{\sin C_2} \quad \text{Solve for } c_2.$$
$$\frac{22.8}{\sin 55.3°} = \frac{c_2}{\sin 8.6°}$$
$$c_2 = \frac{22.8 \sin 8.6°}{\sin 55.3°}$$
$$c_2 \approx 4.15 \text{ ft}$$

NOW TRY EXERCISE 25.

The ambiguous case results in zero, one, or two triangles. The following guidelines can be used to determine how many triangles there are.

NUMBER OF TRIANGLES SATISFYING THE AMBIGUOUS CASE (SSA)

Let sides a and b and angle A be given in triangle ABC. (The law of sines can be used to calculate the value of $\sin B$.)

1. If applying the law of sines results in an equation having $\sin B > 1$, then *no triangle* satisfies the given conditions.
2. If $\sin B = 1$, then *one triangle* satisfies the given conditions and $B = 90°$.
3. If $0 < \sin B < 1$, then either *one or two triangles* satisfy the given conditions.
   (a) If $\sin B = k$, then let $B_1 = \sin^{-1} k$ and use B_1 for B in the first triangle.
   (b) Let $B_2 = 180° - B_1$. If $A + B_2 < 180°$, then a second triangle exists. In this case, use B_2 for B in the second triangle.

▶ EXAMPLE 3 SOLVING THE AMBIGUOUS CASE (ONE TRIANGLE)

Solve triangle ABC, given $A = 43.5°$, $a = 10.7$ in., and $c = 7.2$ in.

Solution To find angle C, use an alternative form of the law of sines.

$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

$$\frac{\sin C}{7.2} = \frac{\sin 43.5°}{10.7}$$

$$\sin C = \frac{7.2 \sin 43.5°}{10.7} \approx .46319186$$

$$C \approx 27.6° \quad \text{Use the inverse sine function.}$$

There is another angle C that has sine value .46319186; it is $C = 180° - 27.6° = 152.4°$. However, notice in the given information that $c < a$, meaning that in the triangle, angle C must have measure *less than* angle A. Notice also that when we add this obtuse value to the given angle $A = 43.5°$, we obtain

$$152.4° + 43.5° = 195.9°,$$

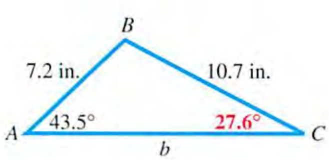


Figure 11

which is greater than 180°. So either of these approaches shows that there can be only one triangle. See Figure 11. Then

$$B = 180° - 27.6° - 43.5° = 108.9°,$$

and we can find side b with the law of sines.

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

$$\frac{b}{\sin 108.9°} = \frac{10.7}{\sin 43.5°}$$

$$b = \frac{10.7 \sin 108.9°}{\sin 43.5°}$$

$$b \approx 14.7 \text{ in.}$$

NOW TRY EXERCISE 21. ◀

Analyzing Data for Possible Number of Triangles

▶ EXAMPLE 4 ANALYZING DATA INVOLVING AN OBTUSE ANGLE

Without using the law of sines, explain why $A = 104°$, $a = 26.8$ m, and $b = 31.3$ m cannot be valid for a triangle ABC.

Solution Since A is an obtuse angle, it is the largest angle and so the longest side of the triangle must be a. However, we are given $b > a$; thus, $B > A$, which is impossible if A is obtuse. Therefore, no such triangle ABC exists.

NOW TRY EXERCISE 33. ◀

7.2 Exercises

1. *Concept Check* Which one of the following sets of data does not determine a unique triangle?

A. $A = 40°, B = 60°, C = 80°$ **B.** $a = 5, b = 12, c = 13$
C. $a = 3, b = 7, C = 50°$ **D.** $a = 2, b = 2, c = 2$

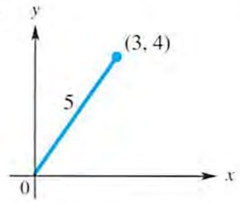
2. *Concept Check* Which one of the following sets of data determines a unique triangle?

A. $A = 50°, B = 50°, C = 80°$ **B.** $a = 3, b = 5, c = 20$
C. $A = 40°, B = 20°, C = 30°$ **D.** $a = 7, b = 24, c = 25$

Concept Check *In each figure, a line of length h is to be drawn from the given point to the positive x-axis in order to form a triangle. For what value(s) of h can you draw the following?*

(a) *two triangles* **(b)** *exactly one triangle* **(c)** *no triangle*

3.



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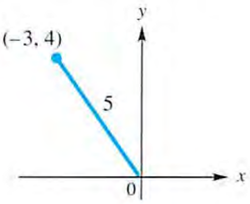
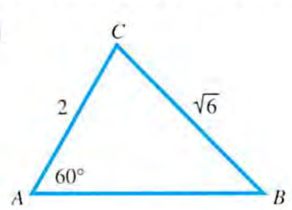


Determine the number of triangles ABC possible with the given parts. See Examples 1–4.

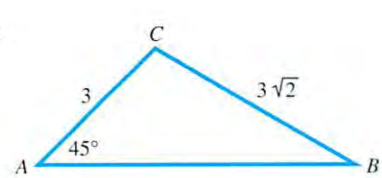
5. $a = 50, b = 26, A = 95°$ **6.** $b = 60, a = 82, B = 100°$
7. $a = 31, b = 26, B = 48°$ **8.** $a = 35, b = 30, A = 40°$
9. $a = 50, b = 61, A = 58°$ **10.** $B = 54°, c = 28, b = 23$

Find each angle B. Do not use a calculator.

11.



12.



Find the unknown angles in triangle ABC for each triangle that exists. See Examples 1–3.

13. $A = 29.7°$, $b = 41.5$ ft, $a = 27.2$ ft

14. $B = 48.2°$, $a = 890$ cm, $b = 697$ cm

15. $C = 41° 20'$, $b = 25.9$ m, $c = 38.4$ m

16. $B = 48° 50'$, $a = 3850$ in., $b = 4730$ in.

17. $B = 74.3°$, $a = 859$ m, $b = 783$ m

18. $C = 82.2°$, $a = 10.9$ km, $c = 7.62$ km

19. $A = 142.13°$, $b = 5.432$ ft, $a = 7.297$ ft

20. $B = 113.72°$, $a = 189.6$ yd, $b = 243.8$ yd

Solve each triangle ABC that exists. See Examples 1–3.

21. $A = 42.5°$, $a = 15.6$ ft, $b = 8.14$ ft

22. $C = 52.3°$, $a = 32.5$ yd, $c = 59.8$ yd

23. $B = 72.2°$, $b = 78.3$ m, $c = 145$ m

24. $C = 68.5°$, $c = 258$ cm, $b = 386$ cm

25. $A = 38° 40'$, $a = 9.72$ km, $b = 11.8$ km

26. $C = 29° 50'$, $a = 8.61$ m, $c = 5.21$ m

27. $A = 96.80°$, $b = 3.589$ ft, $a = 5.818$ ft

28. $C = 88.70°$, $b = 56.87$ yd, $c = 112.4$ yd

29. $B = 39.68°$, $a = 29.81$ m, $b = 23.76$ m

30. $A = 51.20°$, $c = 7986$ cm, $a = 7208$ cm

31. Apply the law of sines to the following: $a = \sqrt{5}$, $c = 2\sqrt{5}$, $A = 30°$. What is the value of sin C? What is the measure of C? Based on its angle measures, what kind of triangle is triangle ABC?

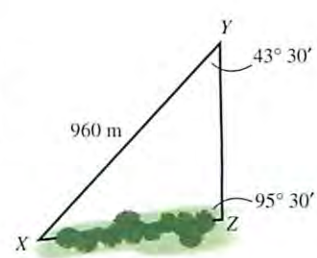
32. Explain the condition that must exist to determine that there is no triangle satisfying the given values of a, b, and B, once the value of sin B is found.

33. Without using the law of sines, explain why no triangle ABC exists satisfying $A = 103° 20'$, $a = 14.6$ ft, $b = 20.4$ ft.

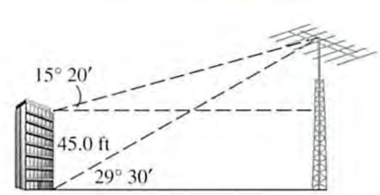
34. Apply the law of sines to the data given in Example 4. Describe in your own words what happens when you try to find the measure of angle B using a calculator.

Use the law of sines to solve each problem.

35. ***Distance Between Inaccessible Points*** To find the distance between a point X and an inaccessible point Z, a line segment XY is constructed. It is found that $XY = 960$ m, angle $XYZ = 43° 30'$, and angle $YZX = 95° 30'$. Find the distance between X and Z to the nearest meter.



36. ***Height of an Antenna Tower*** The angle of elevation from the top of a building 45.0 ft high to the top of a nearby antenna tower is 15° 20′. From the base of the building, the angle of elevation of the tower is 29° 30′. Find the height of the tower.



37. ***Height of a Building*** A flagpole 95.0 ft tall is on the top of a building. From a point on level ground, the angle of elevation of the top of the flagpole is 35.0°, while the angle of elevation of the bottom of the flagpole is 26.0°. Find the height of the building.

38. ***Flight Path of a Plane*** A pilot flies her plane on a heading of 35° 00′ from point X to point Y, 400 mi from X. Then she turns and flies on a heading of 145° 00′ to point Z, which is 400 mi from her starting point X. What is the heading of Z from X, and what is the distance YZ?

Use the law of sines to prove that each statement is true for any triangle ABC, with corresponding sides a, b, and c.

39. $\dfrac{a+b}{b} = \dfrac{\sin A + \sin B}{\sin B}$

40. $\dfrac{a-b}{a+b} = \dfrac{\sin A - \sin B}{\sin A + \sin B}$

7.3 The Law of Cosines

Derivation of the Law of Cosines ▪ Solving SAS and SSS Triangles (Cases 3 and 4) ▪ Heron's Formula for the Area of a Triangle

As mentioned in **Section 7.1,** if we are given two sides and the included angle (Case 3) or three sides (Case 4) of a triangle, then a unique triangle is determined. These are the SAS and SSS cases, respectively. Both cases require using the *law of cosines*. Remember the following property of triangles when applying the law of cosines to solve a triangle.

TRIANGLE SIDE LENGTH RESTRICTION

In any triangle, the sum of the lengths of any two sides must be greater than the length of the remaining side.

No triangle is formed.

Figure 12

For example, it would be impossible to construct a triangle with sides of lengths 3, 4, and 10. See Figure 12.

Derivation of the Law of Cosines

To derive the law of cosines, let ABC be any oblique triangle. Choose a coordinate system so that vertex B is at the origin and side BC is along the positive x-axis. See Figure 13.

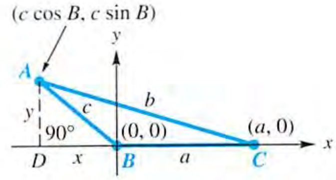


Figure 13

Let (x, y) be the coordinates of vertex A of the triangle. Verify that for angle B, whether obtuse or acute,

$$\sin B = \frac{y}{c} \quad \text{and} \quad \cos B = \frac{x}{c}. \quad \text{(Section 2.2)}$$

$$y = c \sin B \quad \text{and} \quad x = c \cos B. $$

Here x is negative when B is obtuse.

Thus, the coordinates of point A become $(c \cos B, c \sin B)$.

Point C in Figure 13 has coordinates $(a, 0)$, and AC has length b. Since point A has coordinates $(c \cos B, c \sin B)$, by the distance formula,

$$b = \sqrt{(c \cos B - a)^2 + (c \sin B - 0)^2} \quad \text{(Appendix B)}$$

$$b^2 = (c \cos B - a)^2 + (c \sin B)^2 \quad \text{Square both sides.}$$

$$= (c^2 \cos^2 B - 2ac \cos B + a^2) + c^2 \sin^2 B \quad \text{Multiply: } (x-y)^2 = x^2 - 2xy + y^2$$

Remember the middle term.

$$= a^2 + c^2(\cos^2 B + \sin^2 B) - 2ac \cos B \quad \text{Properties of real numbers}$$

$$= a^2 + c^2(1) - 2ac \cos B \quad \text{Fundamental identity (Section 5.1)}$$

$$b^2 = a^2 + c^2 - 2ac \cos B.$$

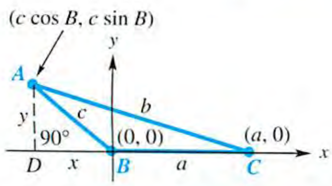


Figure 13 (repeated)

This result is one form of the law of cosines. In our work, we could just as easily have placed A or C at the origin. This would have given the same result, but with the variables rearranged.

LAW OF COSINES

In any triangle ABC, with sides a, b, and c,

$$\mathbf{a^2 = b^2 + c^2 - 2bc \cos A,}$$
$$\mathbf{b^2 = a^2 + c^2 - 2ac \cos B,}$$
$$\mathbf{c^2 = a^2 + b^2 - 2ab \cos C.}$$

That is, according to the law of cosines, the square of a side of a triangle is equal to the sum of the squares of the other two sides, minus twice the product of those two sides and the cosine of the angle included between them.

▶ **Note** If we let $C = 90°$ in the third form of the law of cosines, then $\cos C = \cos 90° = 0$, and the formula becomes $c^2 = a^2 + b^2$, the Pythagorean theorem (Appendix B). The Pythagorean theorem is a special case of the law of cosines.

Solving SAS and SSS Triangles (Cases 3 and 4)

▶ EXAMPLE 1 USING THE LAW OF COSINES IN AN APPLICATION (SAS)

A surveyor wishes to find the distance between two inaccessible points A and B on opposite sides of a lake. While standing at point C, she finds that $AC = 259$ m, $BC = 423$ m, and angle ACB measures 132° 40′. Find the distance AB. See Figure 14.

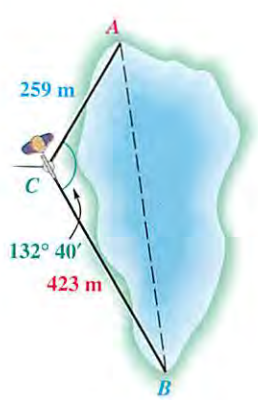


Figure 14

Solution The law of cosines can be used here since we know the lengths of two sides of the triangle and the measure of the included angle.

$$AB^2 = 259^2 + 423^2 - 2(259)(423) \cos 132° 40'$$

$$AB^2 \approx 394{,}510.6 \quad \text{Use a calculator.}$$

$$AB \approx 628 \quad \text{Take the square root of each side. (Appendix A)}$$

The distance between the points is approximately 628 m.

NOW TRY EXERCISE 39. ◀

▶ EXAMPLE 2 USING THE LAW OF COSINES TO SOLVE A TRIANGLE (SAS)

Solve triangle ABC if $A = 42.3°$, $b = 12.9$ m, and $c = 15.4$ m.

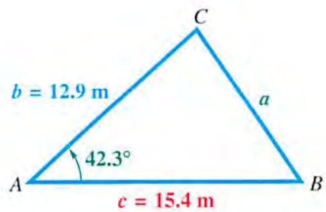


Figure 15

Solution See Figure 15. We start by finding a with the law of cosines.

$$a^2 = b^2 + c^2 - 2bc \cos A \qquad \text{Law of cosines}$$
$$a^2 = 12.9^2 + 15.4^2 - 2(12.9)(15.4) \cos 42.3° \qquad \text{Substitute.}$$
$$a^2 \approx 109.7 \qquad \text{Use a calculator.}$$
$$a \approx 10.47 \text{ m} \qquad \text{Take square roots.}$$

Of the two remaining angles B and C, B must be the smaller since it is opposite the shorter of the two sides b and c. Therefore, B cannot be obtuse.

$$\frac{\sin A}{a} = \frac{\sin B}{b} \qquad \text{Law of sines (Section 7.1)}$$
$$\frac{\sin 42.3°}{10.47} = \frac{\sin B}{12.9} \qquad \text{Substitute.}$$
$$\sin B = \frac{12.9 \sin 42.3°}{10.47} \qquad \text{Multiply by 12.9; rewrite.}$$
$$B \approx 56.0° \qquad \text{Use the inverse sine function. (Section 6.1)}$$

The easiest way to find C is to subtract the measures of A and B from 180°.

$$C = 180° - 42.3° - 56.0° = 81.7° \qquad \text{(Section 1.2)}$$

NOW TRY EXERCISE 19. ◀

▶ **Caution** Had we used the law of sines to find C rather than B in Example 2, we would not have known whether C is equal to 81.7° or its supplement, 98.3°.

▶ EXAMPLE 3 USING THE LAW OF COSINES TO SOLVE A TRIANGLE (SSS)

Solve triangle ABC if $a = 9.47$ ft, $b = 15.9$ ft, and $c = 21.1$ ft.

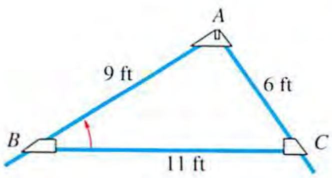
Solution We can use the law of cosines to solve for any angle of the triangle. We solve for C, the largest angle. We will know that C is obtuse if $\cos C < 0$.

$$c^2 = a^2 + b^2 - 2ab \cos C \qquad \text{Law of cosines}$$
$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \qquad \text{Solve for } \cos C.$$
$$\cos C = \frac{9.47^2 + 15.9^2 - 21.1^2}{2(9.47)(15.9)} \qquad \text{Substitute.}$$
$$\cos C \approx -.34109402 \qquad \text{Use a calculator.}$$
$$C \approx 109.9° \qquad \text{Use the inverse cosine function. (Section 6.1)}$$

We can use either the law of sines or the law of cosines to find $B \approx 45.1°$. (Verify this.) Since $A = 180° - B - C$, we obtain $A \approx 25.0°$.

NOW TRY EXERCISE 23. ◀

(a)



(b)

Figure 16

Trusses are frequently used to support roofs on buildings, as illustrated in Figure 16(a). The simplest type of roof truss is a triangle, as shown in Figure 16(b). One basic task when constructing a roof truss is to cut the ends of the rafters so that the roof has the correct slope. (*Source:* Riley, W., L. Sturges, and D. Morris, *Statics and Mechanics of Materials*, John Wiley and Sons, 1995.)

▶ EXAMPLE 4 DESIGNING A ROOF TRUSS (SSS)

Find angle B for the truss shown in Figure 16(b).

Solution

$$b^2 = a^2 + c^2 - 2ac\cos B \quad \text{Law of cosines}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} \quad \text{Solve for } \cos B.$$

$$\cos B = \frac{11^2 + 9^2 - 6^2}{2(11)(9)} \quad \text{Let } a = 11,\ b = 6, \text{ and } c = 9.$$

$$\cos B \approx .83838384 \quad \text{Use a calculator.}$$

$$B \approx 33° \quad \text{Use the inverse cosine function.}$$

NOW TRY EXERCISE 45. ◀

Four possible cases can occur when solving an oblique triangle, as summarized in the following table. In all four cases, it is assumed that the given information actually produces a triangle.

Oblique Triangle	Suggested Procedure for Solving
Case 1: One side and two angles are known. **(SAA or ASA)**	*Step 1* Find the remaining angle using the angle sum formula ($A + B + C = 180°$). *Step 2* Find the remaining sides using the law of sines.
Case 2: Two sides and one angle (not included between the two sides) are known. **(SSA)**	*This is the ambiguous case; there may be no triangle, one triangle, or two triangles.* *Step 1* Find an angle using the law of sines. *Step 2* Find the remaining angle using the angle sum formula. *Step 3* Find the remaining side using the law of sines. *If two triangles exist, repeat Steps 2 and 3.*
Case 3: Two sides and the included angle are known. **(SAS)**	*Step 1* Find the third side using the law of cosines. *Step 2* Find the smaller of the two remaining angles using the law of sines. *Step 3* Find the remaining angle using the angle sum formula.
Case 4: Three sides are known. **(SSS)**	*Step 1* Find the largest angle using the law of cosines. *Step 2* Find either remaining angle using the law of sines. *Step 3* Find the remaining angle using the angle sum formula.

Heron of Alexandria

Heron's Formula for the Area of a Triangle

The law of cosines can be used to derive a formula for the area of a triangle given the lengths of the three sides. This formula is known as **Heron's formula,** named after the Greek mathematician Heron of Alexandria, who lived around A.D. 75. It is found in his work *Metrica*. Heron's formula can be used for the case SSS.

HERON'S AREA FORMULA (SSS)

If a triangle has sides of lengths a, b, and c, with **semiperimeter**

$$s = \frac{1}{2}(a + b + c),$$

then the area of the triangle is

$$\mathcal{A} = \sqrt{s(s-a)(s-b)(s-c)}.$$

That is, according to Heron's formula, the area of a triangle is the square root of the product of four factors: (1) the semiperimeter, (2) the semiperimeter minus the first side, (3) the semiperimeter minus the second side, and (4) the semiperimeter minus the third side. A derivation of Heron's formula is given in the Connections at the end of this section.

▶ EXAMPLE 5 USING HERON'S FORMULA TO FIND AN AREA (SSS)

The distance "as the crow flies" from Los Angeles to New York is 2451 mi, from New York to Montreal is 331 mi, and from Montreal to Los Angeles is 2427 mi. What is the area of the triangular region having these three cities as vertices? (Ignore the curvature of Earth.)

Solution In Figure 17 we let $a = 2451$, $b = 331$, and $c = 2427$.

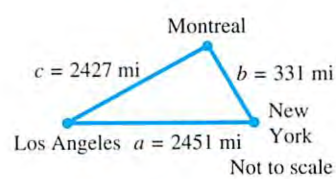


Figure 17

The semiperimeter s is

$$s = \frac{1}{2}(2451 + 331 + 2427) = 2604.5.$$

Using Heron's formula, the area $\mathcal{A}$ is

$$\mathcal{A} = \sqrt{s(s-a)(s-b)(s-c)}$$

Don't forget the factor s.

$$\mathcal{A} = \sqrt{2604.5(2604.5 - 2451)(2604.5 - 331)(2604.5 - 2427)}$$

$$\mathcal{A} \approx 401{,}700 \text{ mi}^2.$$

NOW TRY EXERCISE 75. ◀

CONNECTIONS A trigonometric derivation of Heron's formula illustrates some ingenious manipulation involving the law of cosines, algebraic techniques, double-angle identities, and the area formula $\mathcal{A} = \frac{1}{2}bc \sin A$.

Let triangle ABC have sides of lengths a, b, and c. By the law of cosines,

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}. \qquad \text{Solve for } \cos A. \quad (1)$$

The perimeter of the triangle is $a + b + c$, so half of the perimeter (the semiperimeter) is given by

$$s = \frac{1}{2}(a + b + c). \quad (2)$$

$$2s = a + b + c \qquad \text{Multiply by 2.} \quad (3)$$

$$b + c - a = 2s - 2a \qquad \text{Subtract } 2a \text{ from both sides.}$$

$$b + c - a = 2(s - a) \qquad \text{Factor.} \quad (4)$$

Subtracting $2b$ and $2c$ in a similar way in equation (3) yields

$$a - b + c = 2(s - b) \quad (5)$$

and

$$a + b - c = 2(s - c). \quad (6)$$

Now we obtain an expression for $1 - \cos A$.

$$1 - \cos A = 1 - \underbrace{\frac{b^2 + c^2 - a^2}{2bc}}_{\cos A, \text{ from (1)}}$$

$$= \frac{2bc + a^2 - b^2 - c^2}{2bc} \qquad \text{Find a common denominator; distribute the } - \text{ sign.}$$

$$= \frac{a^2 - (b^2 - 2bc + c^2)}{2bc} \qquad \text{Regroup.}$$

Pay attention to signs.

$$= \frac{a^2 - (b - c)^2}{2bc} \qquad \text{Factor the perfect square trinomial.}$$

$$= \frac{[a - (b - c)]\,[a + (b - c)]}{2bc} \qquad \text{Factor the difference of squares.}$$

$$= \frac{(a - b + c)(a + b - c)}{2bc}$$

$$= \frac{2(s - b) \cdot 2(s - c)}{2bc} \qquad \text{From (5) and (6)}$$

$$1 - \cos A = \frac{2(s - b)(s - c)}{bc} \qquad \text{Lowest terms} \quad (7)$$

Similarly, it can be shown that

$$1 + \cos A = \frac{2s(s - a)}{bc}. \quad (8)$$

(continued)

Recall the double-angle identities for $\cos 2\theta$ from **Section 5.5.**

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\cos A = 2\cos^2\left(\frac{A}{2}\right) - 1 \quad \text{Let } \theta = \tfrac{A}{2}.$$

$$1 + \cos A = 2\cos^2\left(\frac{A}{2}\right)$$

$$\underbrace{\frac{2s(s-a)}{bc}}_{\text{From (8)}} = 2\cos^2\left(\frac{A}{2}\right)$$

$$\frac{s(s-a)}{bc} = \cos^2\left(\frac{A}{2}\right)$$

$$\cos\left(\frac{A}{2}\right) = \sqrt{\frac{s(s-a)}{bc}} \quad \textbf{(9)}$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$\cos A = 1 - 2\sin^2\left(\frac{A}{2}\right) \quad \text{Let } \theta = \tfrac{A}{2}.$$

$$1 - \cos A = 2\sin^2\left(\frac{A}{2}\right)$$

$$\underbrace{\frac{2(s-b)(s-c)}{bc}}_{\text{From (7)}} = 2\sin^2\left(\frac{A}{2}\right)$$

$$\frac{(s-b)(s-c)}{bc} = \sin^2\left(\frac{A}{2}\right)$$

$$\sin\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{bc}} \quad \textbf{(10)}$$

From **Section 7.2,** we know that the area of triangle ABC can be expressed as

$$\mathcal{A} = \frac{1}{2}bc\sin A,$$

so

$$2\mathcal{A} = bc\sin A \quad \text{Multiply by 2.}$$

$$\frac{2\mathcal{A}}{bc} = \sin A. \quad \text{Divide by } bc. \quad \textbf{(11)}$$

Recall the double-angle identity for $\sin 2\theta$ from **Section 5.5.**

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\sin A = 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right) \quad \text{Let } \theta = \tfrac{A}{2}.$$

$$\frac{2\mathcal{A}}{bc} = 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right) \quad \text{Use equation (11).} \quad \textbf{(12)}$$

$$\frac{2\mathcal{A}}{bc} = 2\sqrt{\frac{(s-b)(s-c)}{bc}} \cdot \sqrt{\frac{s(s-a)}{bc}} \quad \text{Use equations (9) and (10).} \quad \textbf{(13)}$$

$$\frac{2\mathcal{A}}{bc} = 2\sqrt{\frac{s(s-a)(s-b)(s-c)}{b^2c^2}} \quad \text{Multiply.}$$

$$\frac{2\mathcal{A}}{bc} = \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{bc} \quad \text{Simplify the denominator.}$$

$$\mathcal{A} = \sqrt{s(s-a)(s-b)(s-c)} \quad \text{Heron's formula}$$

FOR WRITING OR DISCUSSION

1. Provide the details to justify equations (5), (6), and (8).
2. Explain how equations (12) and (13) are justified by using equations (9), (10), and (11).

7.3 Exercises

Concept Check *Assume triangle ABC has standard labeling and complete the following.*

(a) *Determine whether* SAA, ASA, SSA, SAS, *or* SSS *is given.*

(b) *Decide whether the law of sines or the law of cosines should be used to begin solving the triangle.*

1. a, b, and C

2. A, C, and c

3. a, b, and A

4. a, b, and c

5. A, B, and c

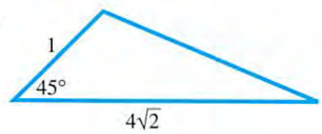
6. a, c, and A

7. a, B, and C

8. b, c, and A

Find the length of the remaining side of each triangle. Do not use a calculator.

9.



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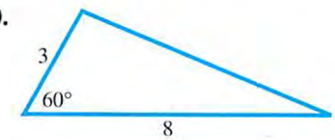
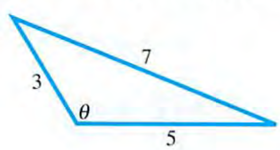


Find the value of θ in each triangle. Do not use a calculator.

11.



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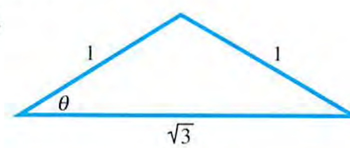
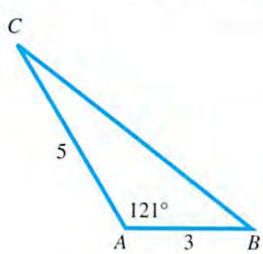


Solve each triangle. Approximate values to the nearest tenth.

13.



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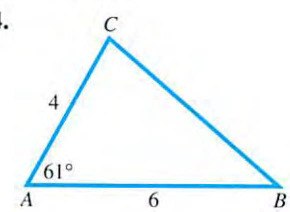

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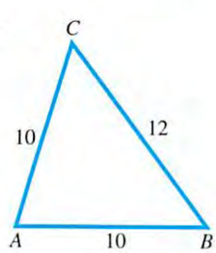

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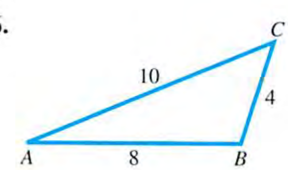

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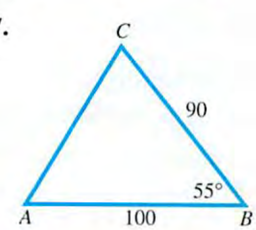

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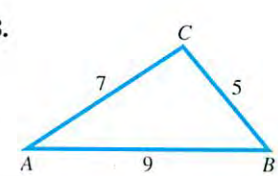


Solve each triangle. See Examples 2 and 3.

19. $A = 41.4°$, $b = 2.78$ yd, $c = 3.92$ yd

20. $C = 28.3°$, $b = 5.71$ in., $a = 4.21$ in.

21. $C = 45.6°$, $b = 8.94$ m, $a = 7.23$ m

22. $A = 67.3°$, $b = 37.9$ km, $c = 40.8$ km

23. $a = 9.3$ cm, $b = 5.7$ cm, $c = 8.2$ cm

24. $a = 28$ ft, $b = 47$ ft, $c = 58$ ft

25. $a = 42.9$ m, $b = 37.6$ m, $c = 62.7$ m

26. $a = 189$ yd, $b = 214$ yd, $c = 325$ yd

27. $AB = 1240$ ft, $AC = 876$ ft, $BC = 965$ ft

28. $AB = 298$ m, $AC = 421$ m, $BC = 324$ m

29. $A = 80° 40'$, $b = 143$ cm, $c = 89.6$ cm

30. $C = 72° 40'$, $a = 327$ ft, $b = 251$ ft

31. $B = 74.80°$, $a = 8.919$ in., $c = 6.427$ in.

32. $C = 59.70°$, $a = 3.725$ mi, $b = 4.698$ mi

33. $A = 112.8°$, $b = 6.28$ m, $c = 12.2$ m

34. $B = 168.2°$, $a = 15.1$ cm, $c = 19.2$ cm

35. $a = 3.0$ ft, $b = 5.0$ ft, $c = 6.0$ ft

36. $a = 4.0$ ft, $b = 5.0$ ft, $c = 8.0$ ft

37. Refer to Figure 12. If you attempt to find any angle of a triangle with the values $a = 3$, $b = 4$, and $c = 10$ with the law of cosines, what happens?

38. "The shortest distance between two points is a straight line." Explain how this relates to the geometric property that states that the sum of the lengths of any two sides of a triangle must be greater than the length of the remaining side.

Solve each problem. See Examples 1–4.

39. ***Distance Across a Lake*** Points A and B are on opposite sides of Lake Yankee. From a third point, C, the angle between the lines of sight to A and B is 46.3°. If AC is 350 m long and BC is 286 m long, find AB.

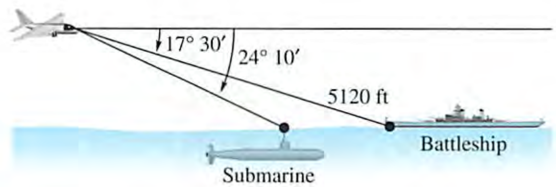
40. ***Diagonals of a Parallelogram*** The sides of a parallelogram are 4.0 cm and 6.0 cm. One angle is 58° while another is 122°. Find the lengths of the diagonals of the parallelogram.

41. ***Flight Distance*** Airports A and B are 450 km apart, on an east-west line. Tom flies in a northeast direction from A to airport C. From C he flies 359 km on a bearing of 128° 40′ to B. How far is C from A?

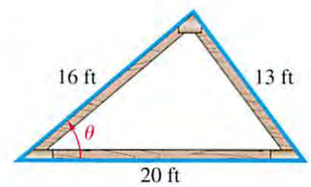
42. ***Distance Between Two Ships*** Two ships leave a harbor together, traveling on courses that have an angle of 135° 40′ between them. If they each travel 402 mi, how far apart are they?

43. ***Distance Between a Ship and a Rock*** A ship is sailing east. At one point, the bearing of a submerged rock is 45° 20′. After the ship has sailed 15.2 mi, the bearing of the rock has become 308° 40′. Find the distance of the ship from the rock at the latter point.

44. ***Distance Between a Ship and a Submarine*** From an airplane flying over the ocean, the angle of depression to a submarine lying under the surface is $24° 10'$. At the same moment, the angle of depression from the airplane to a battleship is $17° 30'$. (See the figure.) The distance from the airplane to the battleship is 5120 ft. Find the distance between the battleship and the submarine. (Assume the airplane, submarine, and battleship are in a vertical plane.)

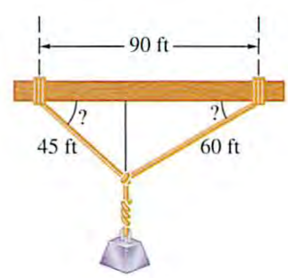


45. ***Truss Construction*** A triangular truss is shown in the figure. Find angle θ.

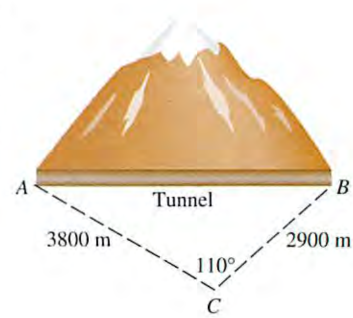


46. ***Distance Between Points on a Crane*** A crane with a counterweight is shown in the figure. Find the horizontal distance between points A and B to the nearest foot.

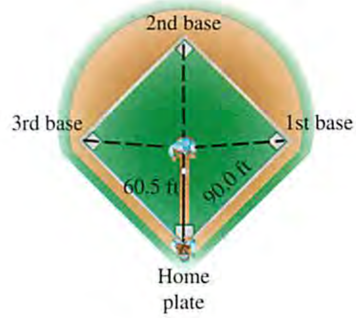
47. ***Distance Between a Beam and Cables*** A weight is supported by cables attached to both ends of a balance beam, as shown in the figure. What angles are formed between the beam and the cables?



48. ***Length of a Tunnel*** To measure the distance through a mountain for a proposed tunnel, a point C is chosen that can be reached from each end of the tunnel. (See the figure.) If $AC = 3800$ m, $BC = 2900$ m, and angle $C = 110°$, find the length of the tunnel.



49. ***Distance on a Baseball Diamond*** A baseball diamond is a square, 90.0 ft on a side, with home plate and the three bases as vertices. The pitcher's position is 60.5 ft from home plate. Find the distance from the pitcher's position to each of the bases.



50. ***Distance Between Ends of the Vietnam Memorial*** The Vietnam Veterans' Memorial in Washington, D.C., is V-shaped with equal sides of length 246.75 ft. The angle between these sides measures 125° 12′. Find the distance between the ends of the two sides. (*Source:* Pamphlet obtained at Vietnam Veterans' Memorial.)

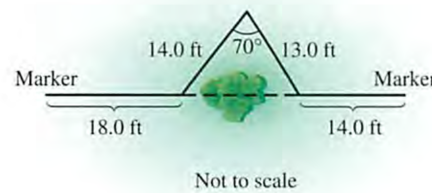
51. ***Distance Between a Ship and a Point*** Starting at point *A*, a ship sails 18.5 km on a bearing of 189°, then turns and sails 47.8 km on a bearing of 317°. Find the distance of the ship from point *A*.

52. ***Distance Between Two Factories*** Two factories blow their whistles at exactly 5:00. A man hears the two blasts at 3 sec and 6 sec after 5:00, respectively. The angle between his lines of sight to the two factories is 42.2°. If sound travels 344 m per sec, how far apart are the factories?

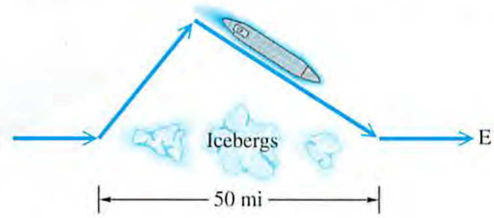
53. ***Bearing of One Town to Another*** Two towns 21 mi apart are separated by a dense forest. (See the figure.) To travel from town *A* to town *B*, a person must go 17 mi on a bearing of 325°, then turn and continue for 9 mi to reach town *B*. Find the bearing of *B* from *A*.

54. ***Distance Traveled by a Plane*** An airplane flies 180 mi from point *X* at a bearing of 125°, and then turns and flies at a bearing of 230° for 100 mi. How far is the plane from point *X*?

55. ***Measurement Using Triangulation*** Surveyors are often confronted with obstacles, such as trees, when measuring the boundary of a lot. One technique used to obtain an accurate measurement is the so-called **triangulation method.** In this technique, a triangle is constructed around the obstacle and one angle and two sides of the triangle are measured. Use this technique to find the length of the property line (the straight line between the two markers) in the figure. (*Source:* Kavanagh, B., *Surveying Principles and Applications,* Sixth Edition, Prentice-Hall, 2003.)



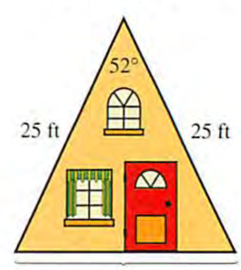
56. ***Path of a Ship*** A ship sailing due east in the North Atlantic has been warned to change course to avoid icebergs. The captain turns and sails on a bearing of 62°, then changes course again to a bearing of 115° until the ship reaches its original course. (See the figure.) How much farther did the ship have to travel to avoid the icebergs?



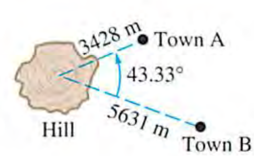
57. ***Angle in a Parallelogram*** A parallelogram has sides of length 25.9 cm and 32.5 cm. The longer diagonal has length 57.8 cm. Find the angle opposite the longer diagonal.

58. ***Distance Between an Airplane and a Mountain*** A person in a plane flying straight north observes a mountain at a bearing of 24.1°. At that time, the plane is 7.92 km from the mountain. A short time later, the bearing to the mountain becomes 32.7°. How far is the airplane from the mountain when the second bearing is taken?

59. ***Layout for a Playhouse*** The layout for a child's playhouse has the dimensions given in the figure. Find x.

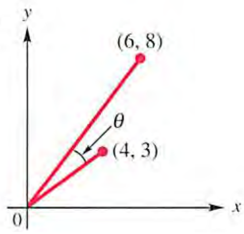


60. ***Distance Between Two Towns*** To find the distance between two small towns, an electronic distance measuring (EDM) instrument is placed on a hill from which both towns are visible. The distance to each town from the EDM and the angle between the two lines of sight are measured. (See the figure.) Find the distance between the towns.

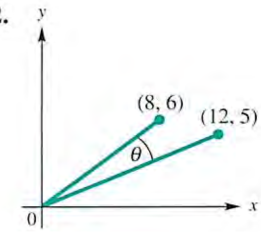


Find the measure of each angle θ to two decimal places.

61.

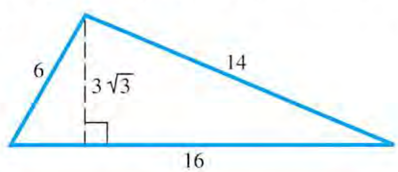


62.

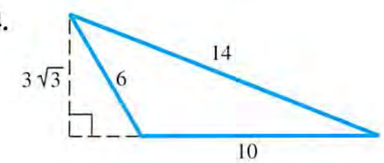


Find the exact area of each triangle using the formula $\mathscr{A} = \frac{1}{2}bh$, and then verify that Heron's formula gives the same result.

63.



64.



Find the area of each triangle ABC. See Example 5.

65. $a = 12$ m, $b = 16$ m, $c = 25$ m

66. $a = 22$ in., $b = 45$ in., $c = 31$ in.

67. $a = 154$ cm, $b = 179$ cm, $c = 183$ cm

68. $a = 25.4$ yd, $b = 38.2$ yd, $c = 19.8$ yd

69. $a = 76.3$ ft, $b = 109$ ft, $c = 98.8$ ft

70. $a = 15.89$ in., $b = 21.74$ in., $c = 10.92$ in.

Volcano Movement *To help predict eruptions from the volcano Mauna Loa on the island of Hawaii, scientists keep track of the volcano's movement by using a "super triangle" with vertices on the three volcanoes shown on the map at the right. (For example, in one year, Mauna Loa moved 6 in., a result of increasing internal pressure.) Refer to the map to work Exercises 71 and 72.*

71. $AB = 22.47928$ mi, $AC = 28.14276$ mi, $A = 58.56989°$; find BC

72. $AB = 22.47928$ mi, $BC = 25.24983$ mi, $A = 58.56989°$; find B

Solve each problem. See Example 5.

73. ***Perfect Triangles*** A **perfect triangle** is a triangle whose sides have whole number lengths and whose area is numerically equal to its perimeter. Show that the triangle with sides of length 9, 10, and 17 is perfect.

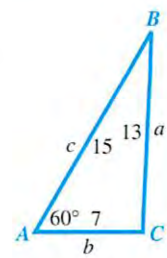
74. ***Heron Triangles*** A **Heron triangle** is a triangle having integer sides and area. Show that each of the following is a Heron triangle.

(a) $a = 11, b = 13, c = 20$ **(b)** $a = 13, b = 14, c = 15$
(c) $a = 7, b = 15, c = 20$ **(d)** $a = 9, b = 10, c = 17$

75. ***Area of the Bermuda Triangle*** Find the area of the Bermuda Triangle if the sides of the triangle have approximate lengths 850 mi, 925 mi, and 1300 mi.

76. ***Required Amount of Paint*** A painter needs to cover a triangular region 75 m by 68 m by 85 m. A can of paint covers 75 m^2 of area. How many cans (to the next higher number of cans) will be needed?

77. Consider triangle ABC shown here.

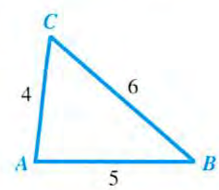


(a) Use the law of sines to find candidates for the value of angle C. Round angle measures to the nearest tenth of a degree.

(b) Rework part (a) using the law of cosines.

(c) Why is the law of cosines a better method in this case?

78. Show that the measure of angle A is twice the measure of angle B. (*Hint:* Use the law of cosines to find cos A and cos B, and then show that $\cos A = 2\cos^2 B - 1$.)



79. Let point D on side AB of triangle ABC be such that CD bisects angle C. Show that

$$\frac{AD}{DB} = \frac{b}{a}.$$

80. In addition to the law of sines and the law of cosines, there is a **law of tangents.** In any triangle ABC,

$$\frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)} = \frac{a - b}{a + b}.$$

Verify this law for the triangle ABC with $a = 2$, $b = 2\sqrt{3}$, $A = 30°$, and $B = 60°$.

RELATING CONCEPTS

For individual or collaborative investigation
(Exercises 81–85)

We have introduced two new formulas for the area of a triangle in this chapter. You should now be able to find the area $\mathcal{A}$ of a triangle using one of three formulas:

(a) $\mathcal{A} = \frac{1}{2}bh$

(b) $\mathcal{A} = \frac{1}{2}ab\sin C$ $\left(\text{or } \mathcal{A} = \frac{1}{2}ac\sin B \text{ or } \mathcal{A} = \frac{1}{2}bc\sin A\right)$

(c) $\mathcal{A} = \sqrt{s(s-a)(s-b)(s-c)}$ (Heron's formula).

(continued)

Another area formula can be used when the coordinates of the vertices of a triangle are given. If the vertices are the ordered pairs (x_1, y_1), (x_2, y_2), and (x_3, y_3), then the following is valid:

$$\text{(d)}\quad \mathcal{A} = \frac{1}{2}\left|(x_1y_2 - y_1x_2 + x_2y_3 - y_2x_3 + x_3y_1 - y_3x_1)\right|.$$

Work Exercises 81–85 in order, *showing that the various formulas all lead to the same area.*

81. Draw a triangle with vertices $A(2, 5)$, $B(-1, 3)$, and $C(4, 0)$.

82. Use the distance formula to find the lengths of the sides a, b, and c.

83. Find the area of triangle ABC using formula (b). (First use the law of cosines to find the measure of an angle.)

84. Find the area of triangle ABC using formula (c) (that is, Heron's formula).

85. Find the area of triangle ABC using new formula (d).

CHAPTER 7 ▶ Quiz (Sections 7.1–7.3)

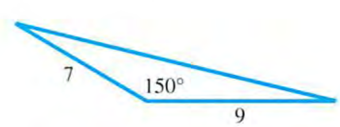
Find the indicated part of each triangle ABC.

1. Find A if $B = 30.6°$, $b = 7.42$ in., and $c = 4.54$ in.

2. Find a if $A = 144°$, $c = 135$ m, and $b = 75.0$ m.

3. Find C if $a = 28.4$ ft, $b = 16.9$ ft, and $c = 21.2$ ft.

4. Find the area of the triangle shown here.

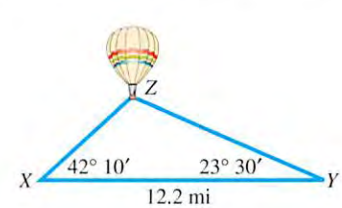


5. Find the area of triangle ABC if $a = 19.5$ km, $b = 21.0$ km, and $c = 22.5$ km.

6. For triangle ABC with $c = 345$, $a = 534$, and $C = 25.4°$, there are two possible values for angle A. What are they?

7. Solve triangle ABC if $c = 326$, $A = 111°$, and $B = 41°$.

8. ***Height of a Balloon*** The angles of elevation of a hot air balloon from two observation points X and Y on level ground are $42°\ 10'$ and $23°\ 30'$, respectively. As shown in the figure, points X, Y, and Z are in the same vertical plane and points X and Y are 12.2 mi apart. Approximate the height of the balloon to the nearest tenth of a mile.



7.4 Vectors, Operations, and the Dot Product

Basic Terminology ▪ Algebraic Interpretation of Vectors ▪ Operations with Vectors ▪ Dot Product and the Angle Between Vectors

Basic Terminology

Many quantities involve magnitudes, such as 45 lb or 60 mph. These quantities are called **scalars** and can be represented by real numbers. Other quantities, called **vector quantities,** involve both magnitude and direction. Typical vector quantities are velocity, acceleration, and force. For example, traveling 50 mph *east* represents a vector quantity.

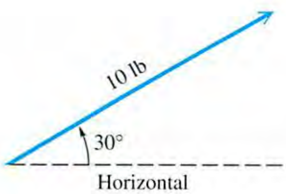


Figure 18

A vector quantity is often represented with a directed line segment (a segment that uses an arrowhead to indicate direction), called a **vector.** The length of the vector represents the **magnitude** of the vector quantity. The direction of the vector, indicated by the arrowhead, represents the direction of the quantity. For example, the vector in Figure 18 represents a force of 10 lb applied at an angle of 30° from the horizontal.

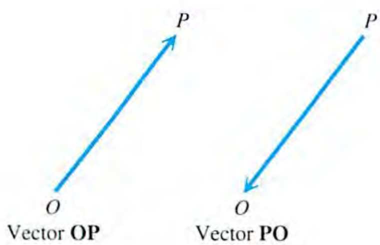


Figure 19

The symbol for a vector is often printed in boldface type. When writing vectors by hand, it is customary to use an arrow over the letter or letters. Thus **OP** and $\overrightarrow{OP}$ both represent the vector **OP**. Vectors may be named with either one lowercase or uppercase letter, or two uppercase letters. When two letters are used, the first indicates the **initial point** and the second indicates the **terminal point** of the vector. Knowing these points gives the direction of the vector. For example, vectors **OP** and **PO** in Figure 19 are not the same vector. They have the same magnitude, but *opposite* directions. The magnitude of vector **OP** is written $|\mathbf{OP}|$.

Two vectors are equal if and only if they have the same direction and the same magnitude. In Figure 20, vectors **A** and **B** are equal, as are vectors **C** and **D**. As Figure 20 shows, equal vectors need not coincide, but they must be parallel. Vectors **A** and **E** are unequal because they do not have the same direction, while $\mathbf{A} \neq \mathbf{F}$ because they have different magnitudes.

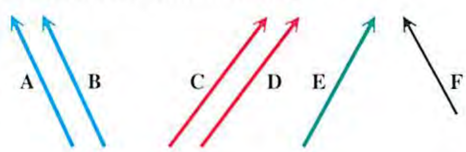
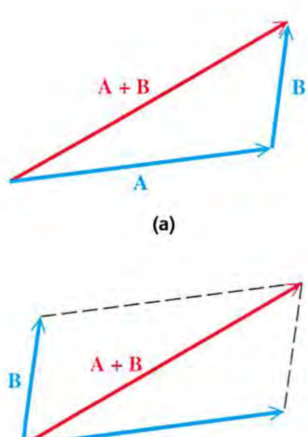


Figure 20

NOW TRY EXERCISE 1. ◀



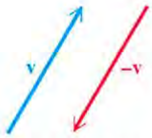
(a)

(b)

Figure 21

To find the sum of two vectors **A** and **B**, we place the initial point of vector **B** at the terminal point of vector **A**, as shown in Figure 21(a). The vector with the same initial point as **A** and the same terminal point as **B** is the sum $\mathbf{A} + \mathbf{B}$. The sum of two vectors is also a vector.

Another way to find the sum of two vectors is to use the **parallelogram rule.** Place vectors **A** and **B** so that their initial points coincide, as in Figure 21(b). Then, complete a parallelogram that has **A** and **B** as two sides. The diagonal of the parallelogram with the same initial point as **A** and **B** is the sum $\mathbf{A} + \mathbf{B}$ found by the definition. Compare Figures 21(a) and (b). Parallelograms can be used to show that vector $\mathbf{B} + \mathbf{A}$ is the same as vector $\mathbf{A} + \mathbf{B}$, or that $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$, so vector addition is commutative. The vector sum $\mathbf{A} + \mathbf{B}$ is called the **resultant** of vectors **A** and **B**.



Vectors **v** and **−v** are opposites.

Figure 22

For every vector **v** there is a vector −**v** that has the same magnitude as **v** but opposite direction. Vector −**v** is called the **opposite** of **v**. (See Figure 22.) The sum of **v** and −**v** has magnitude 0 and is called the **zero vector.** As with real numbers, to subtract vector **B** from vector **A**, find the vector sum **A** + (−**B**). (See Figure 23.)

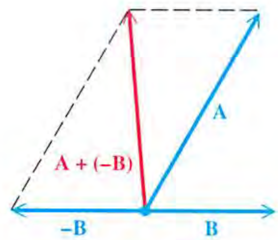


Figure 23

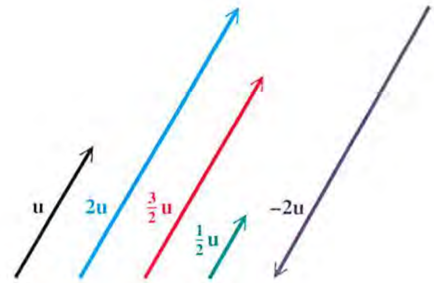


Figure 24

The **scalar product** of a real number (or scalar) k and a vector **u** is the vector $k \cdot \mathbf{u}$, which has magnitude $|k|$ times the magnitude of **u**. As suggested by Figure 24, the vector $k \cdot \mathbf{u}$ has the same direction as **u** if $k > 0$, and opposite direction if $k < 0$.

NOW TRY EXERCISES 5, 7, 9, AND 11. ◀

▼ LOOKING AHEAD TO CALCULUS

In addition to two-dimensional vectors in a plane, calculus courses introduce three-dimensional vectors in space. The magnitude of the two-dimensional vector $\langle a, b \rangle$ is given by $\sqrt{a^2 + b^2}$. If we extend this to the three-dimensional vector $\langle a, b, c \rangle$, the expression becomes $\sqrt{a^2 + b^2 + c^2}$. Similar extensions are made for other concepts.

Algebraic Interpretation of Vectors

We now consider vectors in a rectangular coordinate system. A vector with its initial point at the origin is called a **position vector.** A position vector **u** with its endpoint at the point (a, b) is written $\langle a, b \rangle$, so

$$\mathbf{u} = \langle a, b \rangle.$$

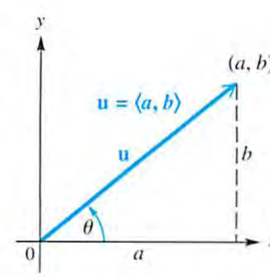


Figure 25

This means that every vector in the real plane corresponds to an ordered pair of real numbers. Thus, geometrically a vector is a directed line segment; algebraically, it is an ordered pair. The numbers a and b are the **horizontal component** and **vertical component,** respectively, of vector **u**. Figure 25 shows the vector $\mathbf{u} = \langle a, b \rangle$. The positive angle between the x-axis and a position vector is the **direction angle** for the vector. In Figure 25, θ is the direction angle for vector **u**.

From Figure 25, we can see that the magnitude and direction of a vector are related to its horizontal and vertical components.

MAGNITUDE AND DIRECTION ANGLE OF A VECTOR $\langle a, b \rangle$

The magnitude (length) of vector $\mathbf{u} = \langle a, b \rangle$ is given by

$$|\mathbf{u}| = \sqrt{a^2 + b^2}.$$

The direction angle θ satisfies $\tan \theta = \frac{b}{a}$, where $a \neq 0$.

▶ EXAMPLE 1 FINDING MAGNITUDE AND DIRECTION ANGLE

Find the magnitude and direction angle for $\mathbf{u} = \langle 3, -2 \rangle$.

Algebraic Solution

The magnitude is $|\mathbf{u}| = \sqrt{3^2 + (-2)^2} = \sqrt{13}$. To find the direction angle θ, start with $\tan\theta = \frac{b}{a} = \frac{-2}{3} = -\frac{2}{3}$. Vector $\mathbf{u}$ has a positive horizontal component and a negative vertical component, placing the position vector in quadrant IV. A calculator gives $\tan^{-1}\left(-\frac{2}{3}\right) \approx -33.7°$. Adding 360° yields the direction angle $\theta \approx 326.3°$. See Figure 26.

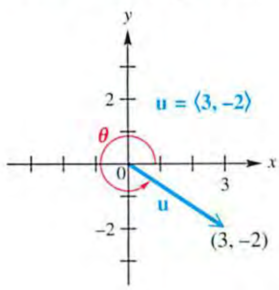


Figure 26

Graphing Calculator Solution

A calculator returns the magnitude and direction angle, given the horizontal and vertical components. An approximation for $\sqrt{13}$ is given, and the direction angle has a measure with least possible absolute value. We must add 360° to the value of θ to obtain the positive direction angle. See Figure 27.

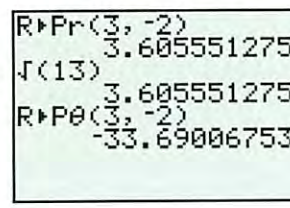


Figure 27

For more information, see your owner's manual or the graphing calculator manual that accompanies this text.

NOW TRY EXERCISE 33. ◀

HORIZONTAL AND VERTICAL COMPONENTS

The horizontal and vertical components, respectively, of a vector $\mathbf{u}$ having magnitude $|\mathbf{u}|$ and direction angle θ are given by

$$a = |\mathbf{u}| \cos\theta \qquad \text{and} \qquad b = |\mathbf{u}| \sin\theta.$$

That is, $\mathbf{u} = \langle a, b \rangle = \langle |\mathbf{u}| \cos\theta, |\mathbf{u}| \sin\theta \rangle$.

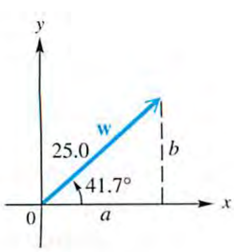


Figure 28

▶ EXAMPLE 2 FINDING HORIZONTAL AND VERTICAL COMPONENTS

Vector $\mathbf{w}$ in Figure 28 has magnitude 25.0 and direction angle 41.7°. Find the horizontal and vertical components.

Algebraic Solution

Use the two formulas in the box, with $|\mathbf{w}| = 25.0$ and $\theta = 41.7°$.

$$a = 25.0 \cos 41.7° \qquad b = 25.0 \sin 41.7°$$
$$a \approx 18.7 \qquad b \approx 16.6$$

Therefore, $\mathbf{w} = \langle 18.7, 16.6 \rangle$. The horizontal component is 18.7, and the vertical component is 16.6 (rounded to the nearest tenth).

Graphing Calculator Solution

See Figure 29. The results support the algebraic solution.

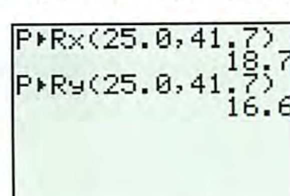


Figure 29

NOW TRY EXERCISE 37. ◀

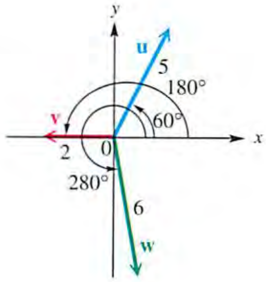


Figure 30

▶ EXAMPLE 3 WRITING VECTORS IN THE FORM ⟨*a*, *b*⟩

Write each vector in Figure 30 in the form $\langle a, b\rangle$.

Solution

$$\mathbf{u} = \langle 5\cos 60°, 5\sin 60°\rangle = \left\langle 5\cdot\frac{1}{2}, 5\cdot\frac{\sqrt{3}}{2}\right\rangle = \left\langle \frac{5}{2}, \frac{5\sqrt{3}}{2}\right\rangle$$

$$\mathbf{v} = \langle 2\cos 180°, 2\sin 180°\rangle = \langle 2(-1), 2(0)\rangle = \langle -2, 0\rangle$$

$$\mathbf{w} = \langle 6\cos 280°, 6\sin 280°\rangle \approx \langle 1.0419, -5.9088\rangle \quad \text{Use a calculator.}$$

NOW TRY EXERCISE 43. ◀

The following properties of parallelograms are helpful when studying vectors.

PROPERTIES OF PARALLELOGRAMS

1. A parallelogram is a quadrilateral whose opposite sides are parallel.
2. The opposite sides and opposite angles of a parallelogram are equal, and adjacent angles of a parallelogram are supplementary.
3. The diagonals of a parallelogram bisect each other, but do not necessarily bisect the angles of the parallelogram.

▶ EXAMPLE 4 FINDING THE MAGNITUDE OF A RESULTANT

Two forces of 15 and 22 newtons act on a point in the plane. (A **newton** is a unit of force that equals .225 lb.) If the angle between the forces is 100°, find the magnitude of the resultant force.

Solution As shown in Figure 31, a parallelogram that has the forces as adjacent sides can be formed. The angles of the parallelogram adjacent to angle P measure 80°, since adjacent angles of a parallelogram are supplementary. Opposite sides of the parallelogram are equal in length. The resultant force divides the parallelogram into two triangles. Use the law of cosines with either triangle.

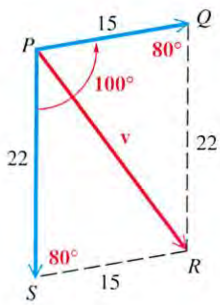


Figure 31

$$|\mathbf{v}|^2 = 15^2 + 22^2 - 2(15)(22)\cos 80° \quad \text{Law of cosines (Section 7.3)}$$

$$\approx 225 + 484 - 115$$

$$|\mathbf{v}|^2 \approx 594$$

$$|\mathbf{v}| \approx 24 \quad \text{Take square roots. (Appendix A)}$$

To the nearest unit, the magnitude of the resultant force is 24 newtons.

NOW TRY EXERCISE 49. ◀

Operations with Vectors In Figure 32, $\mathbf{m} = \langle a, b\rangle$, $\mathbf{n} = \langle c, d\rangle$, and $\mathbf{p} = \langle a + c, b + d\rangle$. Using geometry, we can show that the endpoints of the three vectors and the origin form a parallelogram. Since a diagonal of this parallelogram gives the resultant of **m** and **n**, we have $\mathbf{p} = \mathbf{m} + \mathbf{n}$ or

$$\langle a + c, b + d\rangle = \langle a, b\rangle + \langle c, d\rangle.$$

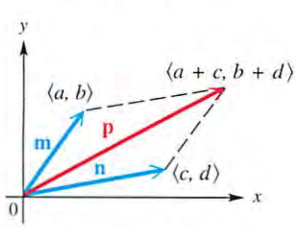


Figure 32

Similarly, we could verify the following vector operations.

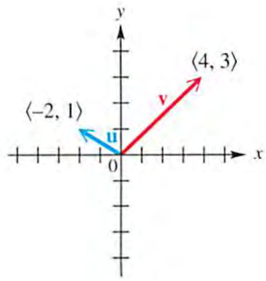


Figure 33

VECTOR OPERATIONS

For any real numbers a, b, c, d, and k,

$$\langle a, b\rangle + \langle c, d\rangle = \langle a + c, b + d\rangle$$
$$k \cdot \langle a, b\rangle = \langle ka, kb\rangle.$$
$$\text{If } \mathbf{a} = \langle a_1, a_2\rangle, \text{ then } -\mathbf{a} = \langle -a_1, -a_2\rangle.$$
$$\langle a, b\rangle - \langle c, d\rangle = \langle a, b\rangle + (-\langle c, d\rangle) = \langle a - c, b - d\rangle$$

▶ EXAMPLE 5 PERFORMING VECTOR OPERATIONS

Let $\mathbf{u} = \langle -2, 1\rangle$ and $\mathbf{v} = \langle 4, 3\rangle$. (See Figure 33.) Find the following: **(a)** $\mathbf{u} + \mathbf{v}$, **(b)** $-2\mathbf{u}$, **(c)** $4\mathbf{u} - 3\mathbf{v}$.

Algebraic Solution

(a) $\mathbf{u} + \mathbf{v} = \langle -2, 1\rangle + \langle 4, 3\rangle$
$= \langle -2 + 4, 1 + 3\rangle$
$= \langle 2, 4\rangle$

(b) $-2\mathbf{u} = -2 \cdot \langle -2, 1\rangle$
$= \langle -2(-2), -2(1)\rangle$
$= \langle 4, -2\rangle$

(c) $4\mathbf{u} - 3\mathbf{v} = 4 \cdot \langle -2, 1\rangle - 3 \cdot \langle 4, 3\rangle$
$= \langle -8, 4\rangle - \langle 12, 9\rangle$
$= \langle -8 - 12, 4 - 9\rangle$
$= \langle -20, -5\rangle$

Graphing Calculator Solution

Vector arithmetic can be performed with a graphing calculator, as shown in Figure 34.

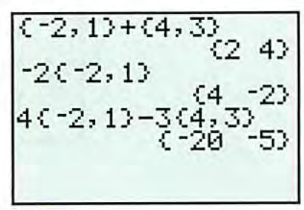


Figure 34

NOW TRY EXERCISES 59, 61, AND 63. ◀

A **unit vector** is a vector that has magnitude 1. Two very useful unit vectors are defined as follows and shown in Figure 35(a).

$$\mathbf{i} = \langle 1, 0\rangle \qquad \mathbf{j} = \langle 0, 1\rangle$$

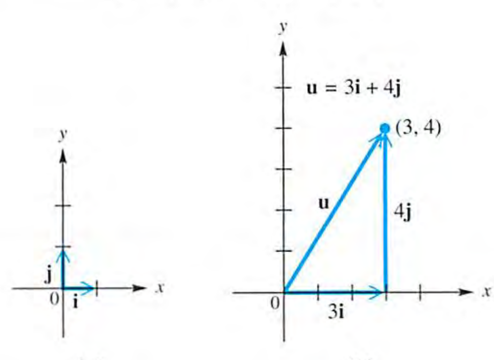


Figure 35

With the unit vectors **i** and **j**, we can express any other vector $\langle a, b\rangle$ in the form $a\mathbf{i} + b\mathbf{j}$, as shown in Figure 35(b), where $\langle 3, 4\rangle = 3\mathbf{i} + 4\mathbf{j}$. The vector operations previously given can be restated, using $a\mathbf{i} + b\mathbf{j}$ notation.

i, j FORM FOR VECTORS

If $\mathbf{v} = \langle a, b \rangle$, then $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$, where $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$.

Dot Product and the Angle Between Vectors The *dot product* of two vectors is a real number, not a vector. It is also known as the *inner product*. Dot products are used to determine the angle between two vectors, derive geometric theorems, and solve physics problems.

DOT PRODUCT

The **dot product** of the two vectors $\mathbf{u} = \langle a, b \rangle$ and $\mathbf{v} = \langle c, d \rangle$ is denoted $\mathbf{u} \cdot \mathbf{v}$, read "**u** dot **v**," and given by

$$\mathbf{u} \cdot \mathbf{v} = ac + bd.$$

That is, the dot product of two vectors is the sum of the product of their first components and the product of their second components.

▶ EXAMPLE 6 FINDING DOT PRODUCTS

Find each dot product.

(a) $\langle 2, 3 \rangle \cdot \langle 4, -1 \rangle$

(b) $\langle 6, 4 \rangle \cdot \langle -2, 3 \rangle$

Solution

(a) $\langle 2, 3 \rangle \cdot \langle 4, -1 \rangle = 2(4) + 3(-1) = 5$

(b) $\langle 6, 4 \rangle \cdot \langle -2, 3 \rangle = 6(-2) + 4(3) = 0$

NOW TRY EXERCISE 71. ◀

The following properties of dot products are easily verified.

PROPERTIES OF THE DOT PRODUCT

For all vectors **u**, **v**, and **w** and real numbers k,

(a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

(b) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

(c) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

(d) $(k\mathbf{u}) \cdot \mathbf{v} = k(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (k\mathbf{v})$

(e) $\mathbf{0} \cdot \mathbf{u} = 0$

(f) $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$.

For example, to prove the first part of (d), we let $\mathbf{u} = \langle a, b \rangle$ and $\mathbf{v} = \langle c, d \rangle$.

$$\begin{aligned}(k\mathbf{u}) \cdot \mathbf{v} &= (k\langle a, b \rangle) \cdot \langle c, d \rangle = \langle ka, kb \rangle \cdot \langle c, d \rangle \\ &= kac + kbd = k(ac + bd) \\ &= k(\langle a, b \rangle \cdot \langle c, d \rangle) = k(\mathbf{u} \cdot \mathbf{v})\end{aligned}$$

The proofs of the remaining properties are similar.

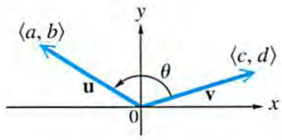


Figure 36

The dot product of two vectors can be positive, 0, or negative. A geometric interpretation of the dot product explains when each of these cases occurs. This interpretation involves the angle between the two vectors. Consider the vectors $\mathbf{u} = \langle a, b \rangle$ and $\mathbf{v} = \langle c, d \rangle$, as shown in Figure 36. The **angle θ between u and v** is defined to be the angle having the two vectors as its sides for which $0° \le \theta \le 180°$. The following theorem relates the dot product to the angle between the vectors. Its proof is outlined in Exercise 32 in **Section 7.5.**

GEOMETRIC INTERPRETATION OF DOT PRODUCT

If θ is the angle between the two nonzero vectors **u** and **v**, where $0° \le \theta \le 180°$, then

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta \quad \text{or, equivalently,} \quad \cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}.$$

▶ EXAMPLE 7 FINDING THE ANGLE BETWEEN TWO VECTORS

Find the angle θ between the two vectors $\mathbf{u} = \langle 3, 4 \rangle$ and $\mathbf{v} = \langle 2, 1 \rangle$.

Solution By the geometric interpretation,

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{\langle 3,4 \rangle \cdot \langle 2,1 \rangle}{|\langle 3,4 \rangle||\langle 2,1 \rangle|}$$

$$= \frac{3(2) + 4(1)}{\sqrt{9+16} \cdot \sqrt{4+1}}$$

$$= \frac{10}{5\sqrt{5}} \approx .894427191.$$

Therefore, $\theta \approx \cos^{-1} .894427191 \approx 26.57°$. (Section 6.1)

NOW TRY EXERCISE 77. ◀

For angles θ between 0° and 180°, $\cos\theta$ is positive, 0, or negative when θ is less than, equal to, or greater than 90°, respectively. Therefore, the dot product is positive, 0, or negative according to this table.

Dot Product	Angle Between Vectors
Positive	Acute
0	Right
Negative	Obtuse

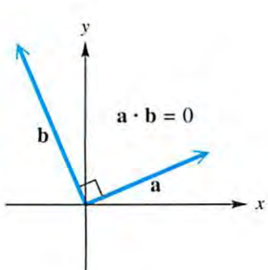


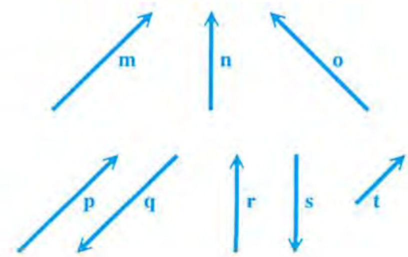
Figure 37

▶ **Note** If $\mathbf{a} \cdot \mathbf{b} = 0$ for two nonzero vectors **a** and **b**, then $\cos\theta = 0$ and $\theta = 90°$. Thus, **a** and **b** are perpendicular or **orthogonal vectors.** See Figure 37.

NOW TRY EXERCISES 87 AND 89. ◀

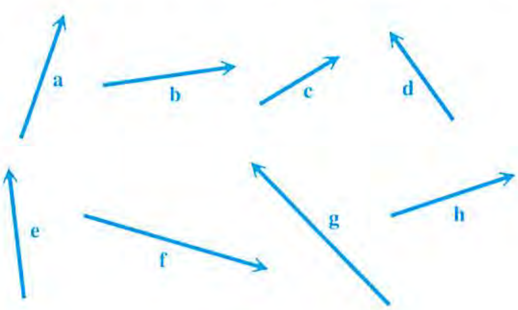
7.4 Exercises

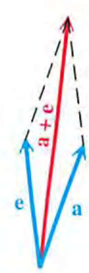
Concept Check *Exercises 1–4 refer to the vectors* **m**-**t** *at the right.*



1. Name all pairs of vectors that appear to be equal.
2. Name all pairs of vectors that are opposites.
3. Name all pairs of vectors where the first is a scalar multiple of the other, with the scalar positive.
4. Name all pairs of vectors where the first is a scalar multiple of the other, with the scalar negative.

Concept Check *Refer to vectors* **a**-**h** *below. Make a copy or a sketch of each vector, and then draw a sketch to represent each vector in Exercises 5–16. For example, find* **a** + **e** *by placing* **a** *and* **e** *so that their initial points coincide. Then use the parallelogram rule to find the resultant, as shown in the figure on the right.*





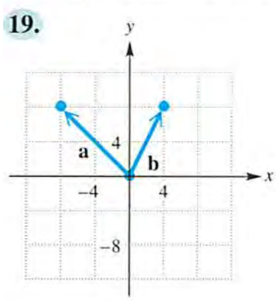
5. −**b**	6. −**g**	7. 3**a**	8. 2**h**
9. **a** + **b**	10. **h** + **g**	11. **a** − **c**	12. **d** − **e**
13. **a** + (**b** + **c**)	14. (**a** + **b**) + **c**	15. **c** + **d**	16. **d** + **c**

17. From the results of Exercises 13 and 14, does it appear that vector addition is associative?

18. From the results of Exercises 15 and 16, does it appear that vector addition is commutative?

In Exercises 19–24, use the figure to find each vector: **(a)** **a** + **b** **(b)** **a** − **b** **(c)** −**a**. *Use* $\langle x, y\rangle$ *notation as in Example 3.*

19.



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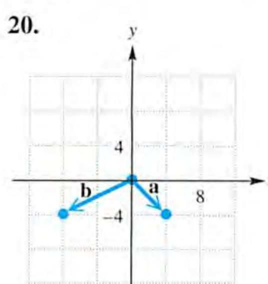

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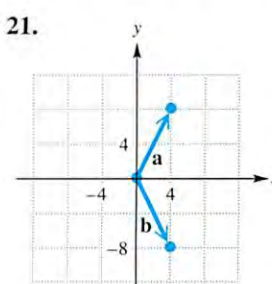

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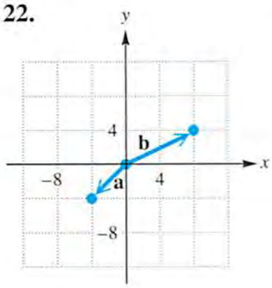

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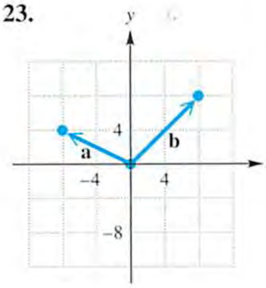
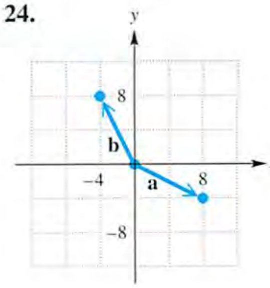


24.



Given vectors **a** *and* **b**, *find:* **(a)** 2**a** **(b)** 2**a** + 3**b** **(c)** **b** − 3**a**.

25. $\mathbf{a} = 2\mathbf{i}, \mathbf{b} = \mathbf{i} + \mathbf{j}$

26. $\mathbf{a} = -\mathbf{i} + 2\mathbf{j}, \mathbf{b} = \mathbf{i} - \mathbf{j}$

27. $\mathbf{a} = \langle -1, 2 \rangle, \mathbf{b} = \langle 3, 0 \rangle$

28. $\mathbf{a} = \langle -2, -1 \rangle, \mathbf{b} = \langle -3, 2 \rangle$

For each pair of vectors **u** *and* **w** *with angle θ between them, sketch the resultant.*

29. $|\mathbf{u}| = 12, |\mathbf{w}| = 20, \theta = 27°$

30. $|\mathbf{u}| = 8, |\mathbf{w}| = 12, \theta = 20°$

31. $|\mathbf{u}| = 20, |\mathbf{w}| = 30, \theta = 30°$

32. $|\mathbf{u}| = 50, |\mathbf{w}| = 70, \theta = 40°$

Find the magnitude and direction angle for **u**. *See Example 1.*

33. $\langle 15, -8 \rangle$

34. $\langle -7, 24 \rangle$

35. $\langle -4, 4\sqrt{3} \rangle$

36. $\langle 8\sqrt{2}, -8\sqrt{2} \rangle$

For each of the following, vector **v** *has the given magnitude and direction. Find the magnitudes of the horizontal and vertical components of* **v**, *if α is the direction angle of* **v** *from the horizontal. See Example 2.*

37. $\alpha = 20°, |\mathbf{v}| = 50$

38. $\alpha = 50°, |\mathbf{v}| = 26$

39. $\alpha = 35° 50', |\mathbf{v}| = 47.8$

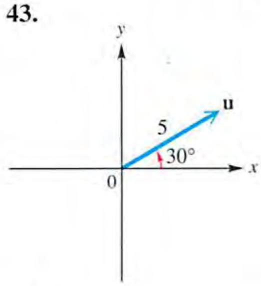
40. $\alpha = 27° 30', |\mathbf{v}| = 15.4$

41. $\alpha = 128.5°, |\mathbf{v}| = 198$

42. $\alpha = 146.3°, |\mathbf{v}| = 238$

Write each vector in the form ⟨a, b⟩. See Example 3.

43.



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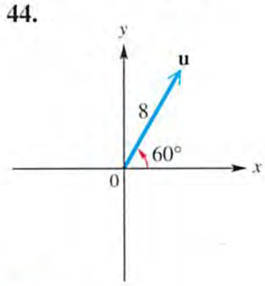
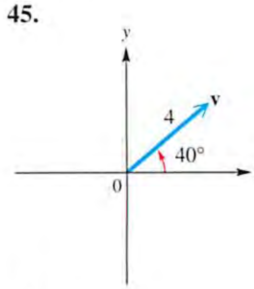
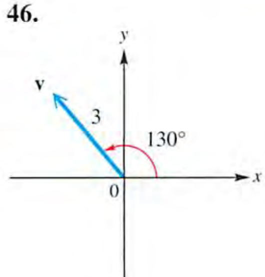


45.



46.



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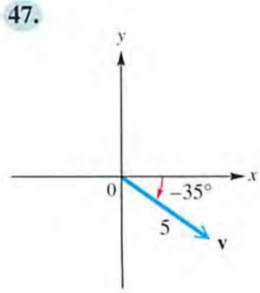
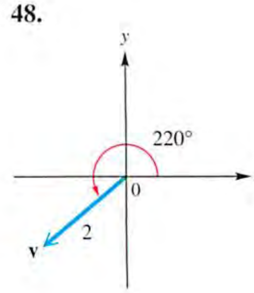


48.



Two forces act at a point in the plane. The angle between the two forces is given. Find the magnitude of the resultant force. See Example 4.

49. forces of 250 and 450 newtons, forming an angle of 85°

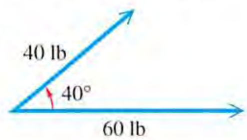
50. forces of 19 and 32 newtons, forming an angle of 118°

51. forces of 116 and 139 lb, forming an angle of 140° 50′

52. forces of 37.8 and 53.7 lb, forming an angle of 68.5°

Use the parallelogram rule to find the magnitude of the resultant force for the two forces shown in each figure. Round answers to the nearest tenth.

53.



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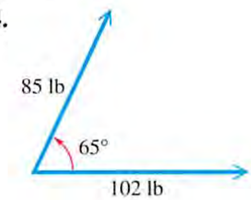

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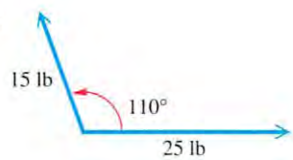
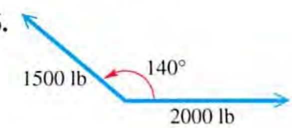


56.



57. ***Concept Check*** If $\mathbf{u} = \langle a, b\rangle$ and $\mathbf{v} = \langle c, d\rangle$, what is $\mathbf{u} + \mathbf{v}$?

58. Explain how addition of vectors is similar to addition of complex numbers.

Given $\mathbf{u} = \langle -2, 5\rangle$ *and* $\mathbf{v} = \langle 4, 3\rangle$, *find the following. See Example 5.*

59. $\mathbf{u} + \mathbf{v}$ **60.** $\mathbf{u} - \mathbf{v}$ **61.** $-4\mathbf{u}$ **62.** $-5\mathbf{v}$

63. $3\mathbf{u} - 6\mathbf{v}$ **64.** $-2\mathbf{u} + 4\mathbf{v}$ **65.** $\mathbf{u} + \mathbf{v} - 3\mathbf{u}$ **66.** $2\mathbf{u} + \mathbf{v} - 6\mathbf{v}$

Write each vector in the form $a\mathbf{i} + b\mathbf{j}$. *See Figure 35(b).*

67. $\langle -5, 8\rangle$ **68.** $\langle 6, -3\rangle$ **69.** $\langle 2, 0\rangle$ **70.** $\langle 0, -4\rangle$

Find the dot product for each pair of vectors. See Example 6.

71. $\langle 6, -1\rangle, \langle 2, 5\rangle$ **72.** $\langle -3, 8\rangle, \langle 7, -5\rangle$ **73.** $\langle 2, -3\rangle, \langle 6, 5\rangle$

74. $\langle 1, 2\rangle, \langle 3, -1\rangle$ **75.** $4\mathbf{i}, 5\mathbf{i} - 9\mathbf{j}$ **76.** $2\mathbf{i} + 4\mathbf{j}, -\mathbf{j}$

Find the angle between each pair of vectors. See Example 7.

77. $\langle 2, 1\rangle, \langle -3, 1\rangle$ **78.** $\langle 1, 7\rangle, \langle 1, 1\rangle$ **79.** $\langle 1, 2\rangle, \langle -6, 3\rangle$

80. $\langle 4, 0\rangle, \langle 2, 2\rangle$ **81.** $3\mathbf{i} + 4\mathbf{j}, \mathbf{j}$ **82.** $-5\mathbf{i} + 12\mathbf{j}, 3\mathbf{i} + 2\mathbf{j}$

Let $\mathbf{u} = \langle -2, 1\rangle$, $\mathbf{v} = \langle 3, 4\rangle$, *and* $\mathbf{w} = \langle -5, 12\rangle$. *Evaluate each expression.*

83. $(3\mathbf{u}) \cdot \mathbf{v}$ **84.** $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w})$ **85.** $\mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w}$ **86.** $\mathbf{u} \cdot (3\mathbf{v})$

Determine whether each pair of vectors is orthogonal. See Figure 37.

87. $\langle 1, 2\rangle, \langle -6, 3\rangle$ **88.** $\langle 3, 4\rangle, \langle 6, 8\rangle$

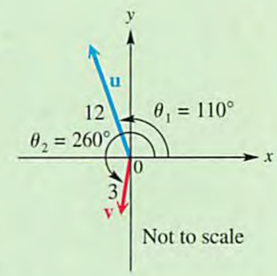
89. $\langle 1, 0\rangle, \langle \sqrt{2}, 0\rangle$ **90.** $\langle 1, 1\rangle, \langle 1, -1\rangle$

91. $\sqrt{5}\mathbf{i} - 2\mathbf{j}, -5\mathbf{i} + 2\sqrt{5}\mathbf{j}$ **92.** $-4\mathbf{i} + 3\mathbf{j}, 8\mathbf{i} - 6\mathbf{j}$

RELATING CONCEPTS

For individual or collaborative investigation
(Exercises 93–98)

Consider the two vectors **u** *and* **v** *shown. Assume all values are exact.* ***Work Exercises 93–98 in order.***



93. Use trigonometry alone (without using vector notation) to find the magnitude and direction angle of **u** + **v**. Use the law of cosines and the law of sines in your work.

94. Find the horizontal and vertical components of **u**, using your calculator.

95. Find the horizontal and vertical components of **v**, using your calculator.

96. Find the horizontal and vertical components of **u** + **v** by adding the results you obtained in Exercises 94 and 95.

97. Use your calculator to find the magnitude and direction angle of the vector **u** + **v**.

98. Compare your answers in Exercises 93 and 97. What do you notice? Which method of solution do you prefer?

7.5 Applications of Vectors

The Equilibrant ▪ Incline Applications ▪ Navigation Applications

The Equilibrant The previous section covered methods for finding the resultant of two vectors. Sometimes it is necessary to find a vector that will counterbalance the resultant. This opposite vector is called the **equilibrant;** that is, the equilibrant of vector **u** is the vector −**u**.

▶ EXAMPLE 1 FINDING THE MAGNITUDE AND DIRECTION OF AN EQUILIBRANT

Find the magnitude of the equilibrant of forces of 48 newtons and 60 newtons acting on a point A, if the angle between the forces is 50°. Then find the angle between the equilibrant and the 48-newton force.

Solution

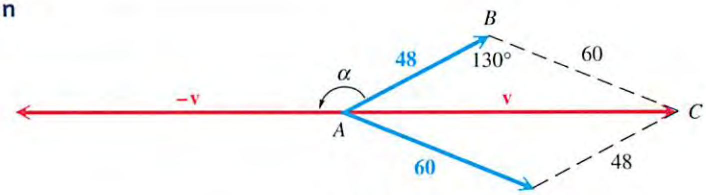


Figure 38

In Figure 38, the equilibrant is −**v**. The magnitude of **v**, and hence of −**v**, is found by using triangle ABC and the law of cosines.

$|\mathbf{v}|^2 = 48^2 + 60^2 - 2(48)(60)\cos 130°$ Law of cosines (Section 7.3)

$|\mathbf{v}|^2 \approx 9606.5$ Use a calculator.

$|\mathbf{v}| \approx 98$ newtons Two significant digits (Section 2.4)

The required angle, labeled α in Figure 38, can be found by subtracting angle CAB from 180°. Use the law of sines to find angle CAB.

$$\frac{98}{\sin 130°} = \frac{60}{\sin CAB} \quad \text{Law of sines (Section 7.1)}$$

$$\sin CAB \approx .46900680$$

$$CAB \approx 28° \quad \text{Use the inverse sine function. (Section 6.1)}$$

Finally, $\alpha \approx 180° - 28° = 152°$.

NOW TRY EXERCISE 1.

Incline Applications

We can use vectors to solve incline problems.

EXAMPLE 2 FINDING A REQUIRED FORCE

Find the force required to keep a 50-lb wagon from sliding down a ramp inclined at 20° to the horizontal. (Assume there is no friction.)

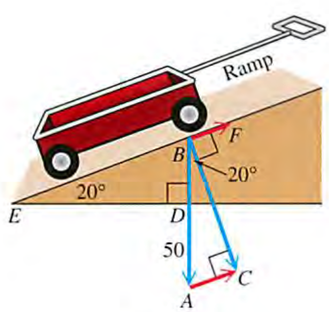


Figure 39

Solution In Figure 39, the vertical 50-lb force **BA** represents the force of gravity. It is the sum of vectors **BC** and $-$**AC**. The vector **BC** represents the force with which the weight pushes against the ramp. Vector **BF** represents the force that would pull the weight up the ramp. Since vectors **BF** and **AC** are equal, $|\mathbf{AC}|$ gives the magnitude of the required force.

Vectors **BF** and **AC** are parallel, so angle EBD equals angle A. Since angle BDE and angle C are right angles, triangles CBA and DEB have two corresponding angles equal and, thus, are similar triangles. Therefore, angle ABC equals angle E, which is 20°. From right triangle ABC,

$$\sin 20° = \frac{|\mathbf{AC}|}{50} \quad \text{(Section 2.1)}$$

$$|\mathbf{AC}| = 50 \sin 20° \approx 17.$$

Approximately a 17-lb force will keep the wagon from sliding down the ramp.

NOW TRY EXERCISE 9.

EXAMPLE 3 FINDING AN INCLINE ANGLE

A force of 16.0 lb is required to hold a 40.0 lb lawn mower on an incline. What angle does the incline make with the horizontal?

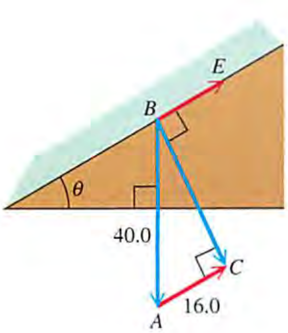


Figure 40

Solution Figure 40 illustrates the situation. Consider right triangle ABC. Angle B equals angle θ, the magnitude of vector **BA** represents the weight of the mower, and vector **AC** equals vector **BE**, which represents the force required to hold the mower on the incline. From the figure,

$$\sin B = \frac{16}{40} = .4$$

$$B \approx 23.5782°. \quad \text{Use the inverse sine function.}$$

Therefore, the hill makes an angle of about 23.6° with the horizontal.

NOW TRY EXERCISE 11.

Navigation Applications Problems involving bearing (defined in **Section 2.5**) can also be worked with vectors.

▶ EXAMPLE 4 APPLYING VECTORS TO A NAVIGATION PROBLEM

A ship leaves port on a bearing of 28.0° and travels 8.20 mi. The ship then turns due east and travels 4.30 mi. How far is the ship from port? What is its bearing from port?

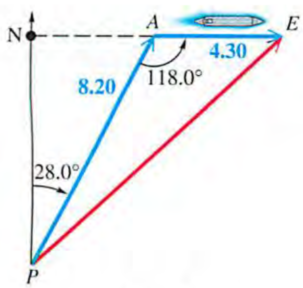


Figure 41

Solution In Figure 41, vectors **PA** and **AE** represent the ship's path. The magnitude and bearing of the resultant **PE** can be found as follows. Triangle *PNA* is a right triangle, so angle $NAP = 90° - 28.0° = 62.0°$. Then angle $PAE = 180° - 62.0° = 118.0°$. Use the law of cosines to find $|\mathbf{PE}|$, the magnitude of vector **PE**.

$$|\mathbf{PE}|^2 = 8.20^2 + 4.30^2 - 2(8.20)(4.30)\cos 118.0° \quad \text{Law of cosines}$$

$$|\mathbf{PE}|^2 \approx 118.84 \quad \text{Approximate.}$$

$$|\mathbf{PE}| \approx 10.9 \quad \text{Square root property (Appendix A)}$$

The ship is about 10.9 mi from port.

To find the bearing of the ship from port, first find angle *APE*. Use the law of sines.

$$\frac{\sin APE}{4.30} = \frac{\sin 118.0°}{10.9} \quad \text{Law of sines}$$

$$\sin APE = \frac{4.30 \sin 118.0°}{10.9} \quad \text{Multiply by 4.30.}$$

$$APE \approx 20.4° \quad \text{Use the inverse sine function.}$$

Now add 20.4° to 28.0° to find that the bearing is 48.4°.

NOW TRY EXERCISE 15. ◀

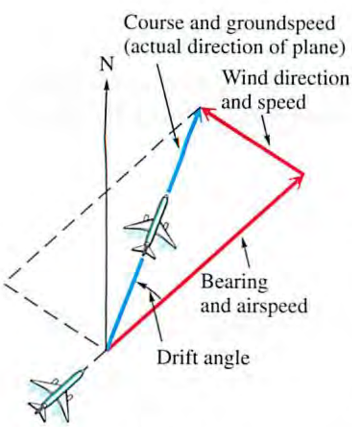


Figure 42

In air navigation, the **airspeed** of a plane is its speed relative to the air, while the **groundspeed** is its speed relative to the ground. Because of wind, these two speeds are usually different. The groundspeed of the plane is represented by the vector sum of the airspeed and windspeed vectors. See Figure 42.

▶ EXAMPLE 5 APPLYING VECTORS TO A NAVIGATION PROBLEM

A plane with an airspeed of 192 mph is headed on a bearing of 121°. A north wind is blowing (from north to south) at 15.9 mph. Find the groundspeed and the actual bearing of the plane.

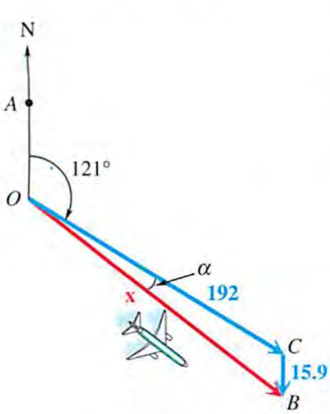


Figure 43

Solution In Figure 43, the groundspeed is represented by $|\mathbf{x}|$. We must find angle α to determine the bearing, which will be $121° + \alpha$. From Figure 43, angle BCO = angle AOC, which measures 121°. Find $|\mathbf{x}|$ by the law of cosines.

$$|\mathbf{x}|^2 = 192^2 + 15.9^2 - 2(192)(15.9)\cos 121°$$

$$|\mathbf{x}|^2 \approx 40{,}261$$

$$|\mathbf{x}| \approx 200.7, \quad \text{or} \quad \text{about 201 mph}$$

Now find α by using the law of sines. Use the value of $|\mathbf{x}|$ before rounding.

$$\frac{\sin \alpha}{15.9} = \frac{\sin 121^\circ}{200.7}$$

$$\sin \alpha \approx .0679$$

$$\alpha \approx 3.89^\circ$$

To the nearest degree, α is 4°. The groundspeed is about 201 mph on a bearing of $121^\circ + 4^\circ = 125^\circ$.

NOW TRY EXERCISE 25.

7.5 Exercises

Solve each problem. See Examples 1–3.

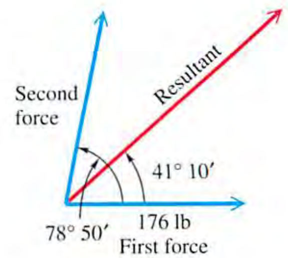
1. ***Direction and Magnitude of an Equilibrant*** Two tugboats are pulling a disabled speedboat into port with forces of 1240 lb and 1480 lb. The angle between these forces is 28.2°. Find the direction and magnitude of the equilibrant.

2. ***Direction and Magnitude of an Equilibrant*** Two rescue vessels are pulling a broken-down motorboat toward a boathouse with forces of 840 lb and 960 lb. The angle between these forces is 24.5°. Find the direction and magnitude of the equilibrant.

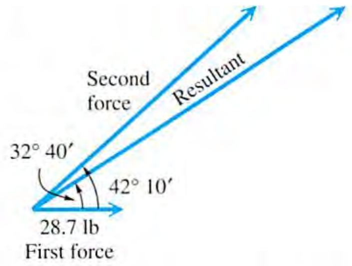
3. ***Angle Between Forces*** Two forces of 692 newtons and 423 newtons act at a point. The resultant force is 786 newtons. Find the angle between the forces.

4. ***Angle Between Forces*** Two forces of 128 lb and 253 lb act at a point. The equilibrant force is 320 lb. Find the angle between the forces.

5. ***Magnitudes of Forces*** A force of 176 lb makes an angle of 78° 50′ with a second force. The resultant of the two forces makes an angle of 41° 10′ with the first force. Find the magnitudes of the second force and of the resultant.



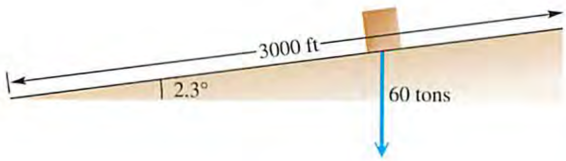
6. ***Magnitudes of Forces*** A force of 28.7 lb makes an angle of 42° 10′ with a second force. The resultant of the two forces makes an angle of 32° 40′ with the first force. Find the magnitudes of the second force and of the resultant.



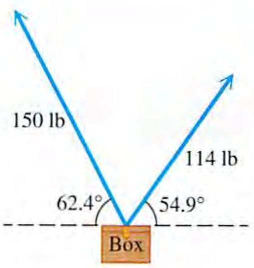
7. ***Angle of a Hill Slope*** A force of 25 lb is required to hold an 80-lb crate on a hill. What angle does the hill make with the horizontal?

8. ***Force Needed to Keep a Car Parked*** Find the force required to keep a 3000-lb car parked on a hill that makes an angle of 15° with the horizontal.

9. ***Force Needed for a Monolith*** To build the pyramids in Egypt, it is believed that giant causeways were constructed to transport the building materials to the site. One such causeway is said to have been 3000 ft long, with a slope of about 2.3°. How much force would be required to hold a 60-ton monolith on this causeway?



10. ***Weight of a Box*** Two people are carrying a box. One person exerts a force of 150 lb at an angle of 62.4° with the horizontal. The other person exerts a force of 114 lb at an angle of 54.9°. Find the weight of the box.



11. ***Incline Angle*** A force of 18.0 lb is required to hold a 60.0-lb stump grinder on an incline. What angle does the incline make with the horizontal?

12. ***Incline Angle*** A force of 30.0 lb is required to hold an 80.0-lb pressure washer on an incline. What angle does the incline make with the horizontal?

13. ***Weight of a Crate and Tension of a Rope*** A crate is supported by two ropes. One rope makes an angle of 46° 20′ with the horizontal and has a tension of 89.6 lb on it. The other rope is horizontal. Find the weight of the crate and the tension in the horizontal rope.

14. ***Angles Between Forces*** Three forces acting at a point are in equilibrium. The forces are 980 lb, 760 lb, and 1220 lb. Find the angles between the directions of the forces to the nearest tenth of a degree. (*Hint:* Arrange the forces to form the sides of a triangle.)

Solve each problem. See Examples 4 and 5.

15. ***Distance and Bearing of a Ship*** A ship leaves port on a bearing of 34.0° and travels 10.4 mi. The ship then turns due east and travels 4.6 mi. How far is the ship from port, and what is its bearing from port?

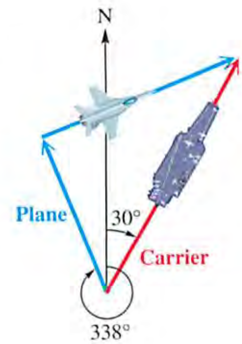
16. ***Distance and Bearing of a Luxury Liner*** A luxury liner leaves port on a bearing of 110.0° and travels 8.8 mi. It then turns due west and travels 2.4 mi. How far is the liner from port, and what is its bearing from port?

17. ***Distance of a Ship from Its Starting Point*** Starting at point A, a ship sails 18.5 km on a bearing of 189°, then turns and sails 47.8 km on a bearing of 317°. Find the distance of the ship from point A.

18. ***Distance of a Ship from Its Starting Point*** Starting at point X, a ship sails 15.5 km on a bearing of 200°, then turns and sails 2.4 km on a bearing of 320°. Find the distance of the ship from point X.

19. ***Distance and Direction of a Motorboat*** A motorboat sets out in the direction N 80° 00′ E. The speed of the boat in still water is 20.0 mph. If the current is flowing directly south, and the actual direction of the motorboat is due east, find the speed of the current and the actual speed of the motorboat.

20. ***Path Traveled by a Plane*** The aircraft carrier *Tallahassee* is traveling at sea on a steady course with a bearing of 30° at 32 mph. Patrol planes on the carrier have enough fuel for 2.6 hr of flight when traveling at a speed of 520 mph. One of the pilots takes off on a bearing of 338° and then turns and heads in a straight line, so as to be able to catch the carrier and land on the deck at the exact instant that his fuel runs out. If the pilot left at 2 P.M., at what time did he turn to head for the carrier?



21. ***Bearing and Groundspeed of a Plane*** An airline route from San Francisco to Honolulu is on a bearing of 233.0°. A jet flying at 450 mph on that bearing runs into a wind blowing at 39.0 mph from a direction of 114.0°. Find the resulting bearing and groundspeed of the plane.

22. ***Movement of a Motorboat*** Suppose you would like to cross a 132-ft-wide river in a motorboat. Assume that the motorboat can travel at 7.0 mph relative to the water and that the current is flowing west at the rate of 3.0 mph. The bearing θ is chosen so that the motorboat will land at a point exactly across from the starting point.

(a) At what speed will the motorboat be traveling relative to the banks?

(b) How long will it take for the motorboat to make the crossing?

(c) What is the measure of angle θ?

23. ***Airspeed and Groundspeed*** A pilot wants to fly on a bearing of 74.9°. By flying due east, he finds that a 42.0-mph wind, blowing from the south, puts him on course. Find the airspeed and the groundspeed.

24. ***Bearing of a Plane*** A plane flies 650 mph on a bearing of 175.3°. A 25-mph wind, from a direction of 266.6°, blows against the plane. Find the resulting bearing of the plane.

25. ***Bearing and Groundspeed of a Plane*** A pilot is flying at 190.0 mph. He wants his flight path to be on a bearing of 64° 30′. A wind is blowing from the south at 35.0 mph. Find the bearing he should fly, and find the plane's groundspeed.

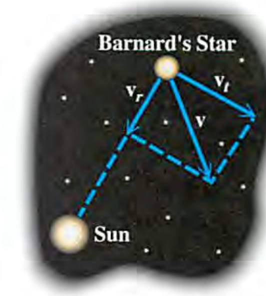
26. ***Bearing and Groundspeed of a Plane*** A pilot is flying at 168 mph. She wants her flight path to be on a bearing of 57° 40′. A wind is blowing from the south at 27.1 mph. Find the bearing the pilot should fly, and find the plane's groundspeed.

27. ***Bearing and Airspeed of a Plane*** What bearing and airspeed are required for a plane to fly 400 mi due north in 2.5 hr if the wind is blowing from a direction of 328° at 11 mph?

28. ***Groundspeed and Bearing of a Plane*** A plane is headed due south with an airspeed of 192 mph. A wind from a direction of 78.0° is blowing at 23.0 mph. Find the groundspeed and resulting bearing of the plane.

29. ***Groundspeed and Bearing of a Plane*** An airplane is headed on a bearing of 174° at an airspeed of 240 km per hr. A 30 km per hr wind is blowing from a direction of 245°. Find the groundspeed and resulting bearing of the plane.

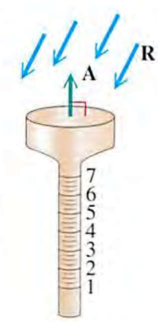
30. ***Velocity of a Star*** The space velocity **v** of a star relative to the sun can be expressed as the resultant vector of two perpendicular vectors—the radial velocity $\mathbf{v}_r$ and the tangential velocity $\mathbf{v}_t$, where $\mathbf{v} = \mathbf{v}_r + \mathbf{v}_t$. If a star is located near the sun and its space velocity is large, then its motion across the sky will also be large. Barnard's Star is a relatively close star with a distance of 35 trillion mi from the sun. It moves across the sky through an angle of 10.34″ per year, which is the largest motion of any known star. Its radial velocity is $\mathbf{v}_r = 67$ mi per sec toward the sun. (*Source:* Zeilik, M., S. Gregory, and E. Smith, *Introductory Astronomy and Astrophysics,* Second Edition, Saunders College Publishing, 1998; Acker, A. and C. Jaschek, *Astronomical Methods and Calculations,* John Wiley and Sons, 1986.)



Not to scale

(a) Approximate the tangential velocity $\mathbf{v}_t$ of Barnard's Star. (*Hint:* Use the arc length formula $s = r\theta$.)
(b) Compute the magnitude of **v**.

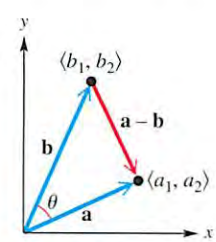
31. ***(Modeling) Measuring Rainfall*** Suppose that vector **R** models the amount of rainfall in inches and the direction it falls, and vector **A** models the area in square inches and orientation of the opening of a rain gauge, as illustrated in the figure. The total volume V of water collected in the rain gauge is given by $V = |\mathbf{R} \cdot \mathbf{A}|$. This formula calculates the volume of water collected even if the wind is blowing the rain in a slanted direction or the rain gauge is not exactly vertical. Let $\mathbf{R} = \mathbf{i} - 2\mathbf{j}$ and $\mathbf{A} = .5\mathbf{i} + \mathbf{j}$.



(a) Find $|\mathbf{R}|$ and $|\mathbf{A}|$. Interpret your results.
(b) Calculate V and interpret this result.
(c) For the rain gauge to collect the maximum amount of water, what should be true about vectors **R** and **A**?

32. ***The Dot Product*** In the figure at the right, $\mathbf{a} = \langle a_1, a_2 \rangle$, $\mathbf{b} = \langle b_1, b_2 \rangle$, and $\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$. Apply the law of cosines to the triangle and derive the equation

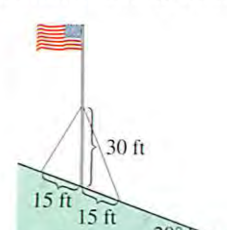
$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta.$$



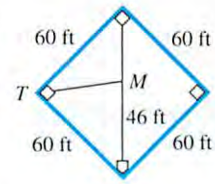
Summary Exercises on Applications of Trigonometry and Vectors

These summary exercises provide practice with applications that involve solving triangles and using vectors.

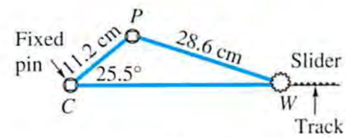
1. ***Wires Supporting a Flagpole*** A flagpole stands vertically on a hillside that makes an angle of 20° with the horizontal. Two supporting wires are attached as shown in the figure. What are the lengths of the supporting wires?



2. ***Distance between Points on a Softball Field*** The pitcher's mound on a regulation softball field is 46 ft from home plate. The distance between the bases is 60 ft, as shown in the figure. How far from third base (point T) is the pitcher's mound (point M)? Give your answer to the nearest foot.



3. ***Distance Between a Pin and a Rod*** A slider crank mechanism is shown in the figure. Find the distance between the wrist pin W and the connecting rod center C.

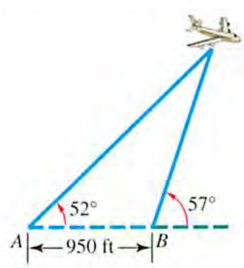


4. ***Distance Between Two Lighthouses*** Two lighthouses are located on a north-south line. From lighthouse A, the bearing of a ship 3742 m away is 129° 43′. From lighthouse B, the bearing of a ship is 39° 43′. Find the distance between the lighthouses.

5. ***Hot-Air Balloon*** A hot-air balloon is rising straight up at the speed of 15.0 ft per sec. Then a wind starts blowing horizontally at 5.00 ft per sec. What will the new speed of the balloon be and what angle with the horizontal will the balloon's path make?

6. ***Playing on a Swing*** Mary is playing with her daughter Brittany on a swing. Starting from rest, Mary pulls the swing through an angle of 40° and holds it briefly before releasing the swing. If Brittany weighs 50 lb, what horizontal force, to the nearest pound, must Mary apply while holding the swing?

7. ***Height of an Airplane*** Two observation points A and B are 950 ft apart. From these points the angles of elevation of an airplane are 52° and 57°. (See the figure.) Find the height of the airplane.

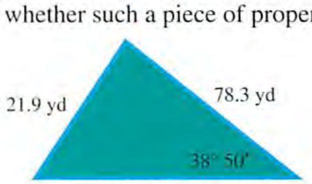


8. ***Wind and Vectors*** A wind can be described by $\mathbf{v} = 6\mathbf{i} + 8\mathbf{j}$, where vector $\mathbf{j}$ points north and represents a south wind of 1 mph.

(a) What is the speed of the wind? **(b)** Find $3\mathbf{v}$. Interpret the result.

(c) Interpret the wind if it switches to $\mathbf{u} = -8\mathbf{i} + 8\mathbf{j}$.

9. ***Property Survey*** A surveyor reported the following data about a piece of property: "The property is triangular in shape, with dimensions as shown in the figure." Use the law of sines to see whether such a piece of property could exist.



Can such a triangle exist?

10. ***Property Survey*** A second triangular piece of property has dimensions as shown. This time it turns out that the surveyor did not consider every possible case. Use the law of sines to show why.

Chapter 7 Summary

KEY TERMS

7.1 Side-Angle-Side (SAS), Angle-Side-Angle (ASA), Side-Side-Side (SSS), oblique triangle, Side-Angle-Angle (SAA)

7.2 ambiguous case

7.3 semiperimeter

7.4 scalar, vector quantity, vector, magnitude, initial point, terminal point, parallelogram rule, resultant, opposite (of a vector), zero vector, scalar product, position vector, horizontal component, vertical component, direction angle, unit vector, dot product, angle between two vectors, orthogonal vectors

7.5 equilibrant, airspeed, groundspeed

NEW SYMBOLS

OP or $\overrightarrow{OP}$	vector **OP**
$\vert$**OP**$\vert$	magnitude of vector **OP**
$\langle a, b\rangle$	position vector
i, j	unit vectors

QUICK REVIEW

CONCEPTS | EXAMPLES

7.1 Oblique Triangles and the Law of Sines

Law of Sines

In any triangle ABC, with sides a, b, and c,

$$\frac{a}{\sin A} = \frac{b}{\sin B}, \quad \frac{a}{\sin A} = \frac{c}{\sin C}, \quad \text{and} \quad \frac{b}{\sin B} = \frac{c}{\sin C}.$$

In triangle ABC, find c if $A = 44°$, $C = 62°$, and $a = 12.00$ units. Then find its area.

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{12.00}{\sin 44°} = \frac{c}{\sin 62°}$$

$$c = \frac{12.00 \sin 62°}{\sin 44°} \approx 15.25 \text{ units}$$

Area of a Triangle

In any triangle ABC, the area is half the product of the lengths of two sides and the sine of the angle between them.

$$\mathcal{A} = \frac{1}{2}bc \sin A, \quad \mathcal{A} = \frac{1}{2}ab \sin C, \quad \mathcal{A} = \frac{1}{2}ac \sin B$$

For triangle ABC above,

$$\mathcal{A} = \frac{1}{2}ac \sin B$$

$$= \frac{1}{2}(12.00)(15.25) \sin 74° \qquad B = 180° - 44° - 62°$$

$$\approx 87.96 \text{ sq units.}$$

CONCEPTS	EXAMPLES

7.2 The Ambiguous Case of the Law of Sines

Ambiguous Case
If we are given the lengths of two sides and the angle opposite one of them, for example, A, a, and b in triangle ABC, then it is possible that zero, one, or two such triangles exist. If A is acute, h is the altitude from C, and

1. $a < h < b$, then there is no triangle.
2. $a = h$ and $h < b$, then there is one triangle (a right triangle).
3. $a \geq b$, then there is one triangle.
4. $h < a < b$, then there are two triangles.

If A is obtuse and

1. $a \leq b$, then there is no triangle.
2. $a > b$, then there is one triangle.

See the table on page 313 that illustrates the possible outcomes.

Solve triangle ABC, given $A = 44.5°$, $a = 11.0$ in., and $c = 7.0$ in.

Find angle C.

$$\frac{\sin C}{7.0} = \frac{\sin 44.5°}{11.0}$$
$$\sin C \approx .4460$$
$$C \approx 26.5°$$

Another angle with this sine value is

$$180° - 26.5° = 153.5°.$$

However, $153.5° + 44.5° > 180°$, so there is only one triangle.

$$B = 180° - 44.5° - 26.5°$$
$$B = 109°$$

Using the law of sines again,

$$b \approx 14.8 \text{ in.}$$

7.3 The Law of Cosines

Law of Cosines
In any triangle ABC, with sides a, b, and c,

$$\mathbf{a^2 = b^2 + c^2 - 2bc \cos A}$$
$$\mathbf{b^2 = a^2 + c^2 - 2ac \cos B}$$
$$\mathbf{c^2 = a^2 + b^2 - 2ab \cos C.}$$

In triangle ABC, find C if $a = 11$ units, $b = 13$ units, and $c = 20$ units. Then find its area.

$$c^2 = a^2 + b^2 - 2ab \cos C$$
$$20^2 = 11^2 + 13^2 - 2(11)(13) \cos C$$
$$400 = 121 + 169 - 286 \cos C$$
$$\frac{400 - 121 - 169}{-286} = \cos C$$
$$C = \cos^{-1}\left(\frac{400 - 121 - 169}{-286}\right)$$
$$C \approx 112.6°$$

Heron's Area Formula
If a triangle has sides of lengths a, b, and c, with semiperimeter

$$\mathbf{s = \frac{1}{2}(a + b + c),}$$

then the area of the triangle is

$$\mathbf{\mathcal{A} = \sqrt{s(s - a)(s - b)(s - c)}.}$$

The semiperimeter s is

$$s = \frac{1}{2}(11 + 13 + 20) = 22,$$

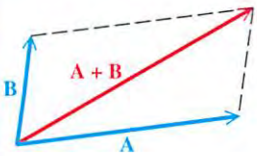
so

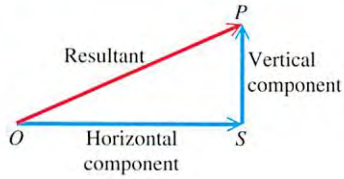
$$\mathcal{A} = \sqrt{22(22 - 11)(22 - 13)(22 - 20)} = 66 \text{ sq units.}$$

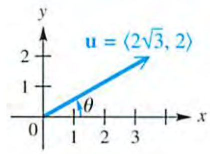
(continued)

CONCEPTS	EXAMPLES

7.4 Vectors, Operations, and the Dot Product







Magnitude and Direction Angle of a Vector

The magnitude (length) of vector $\mathbf{u} = \langle a, b\rangle$ is given by

$$|\mathbf{u}| = \sqrt{a^2 + b^2}.$$

$$|\mathbf{u}| = \sqrt{(2\sqrt{3})^2 + 2^2} = \sqrt{16} = 4$$

The direction angle θ satisfies $\tan\theta = \frac{b}{a}$, where $a \neq 0$.

Since $\tan\theta = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$, it follows that $\theta = 30°$.

Vector Operations

For any real numbers a, b, c, d, and k,

$$\langle a, b\rangle + \langle c, d\rangle = \langle a + c, b + d\rangle$$
$$k \cdot \langle a, b\rangle = \langle ka, kb\rangle.$$
$$\text{If } \mathbf{a} = \langle a_1, a_2\rangle, \text{ then } -\mathbf{a} = \langle -a_1, -a_2\rangle.$$
$$\langle a, b\rangle - \langle c, d\rangle = \langle a, b\rangle + (-\langle c, d\rangle) = \langle a - c, b - d\rangle.$$

$$\langle 4, 6\rangle + \langle -8, 3\rangle = \langle -4, 9\rangle$$
$$5\langle -2, 1\rangle = \langle -10, 5\rangle$$
$$-\langle -9, 6\rangle = \langle 9, -6\rangle$$
$$\langle 4, 6\rangle - \langle -8, 3\rangle = \langle 12, 3\rangle$$

If $\mathbf{u} = \langle x, y\rangle$ has direction angle θ, then

$$\mathbf{u} = \langle |\mathbf{u}|\cos\theta, |\mathbf{u}|\sin\theta\rangle.$$

For $\mathbf{u}$ defined above,

$$\mathbf{u} = \langle 4\cos 30°, 4\sin 30°\rangle$$
$$= \langle 2\sqrt{3}, 2\rangle$$

and $\mathbf{u} = 2\sqrt{3}\mathbf{i} + 2\mathbf{j}.$

i, j Form for Vectors

If $\mathbf{v} = \langle a, b\rangle$, then $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$, where $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$.

Dot Product

The dot product of the two vectors $\mathbf{u} = \langle a, b\rangle$ and $\mathbf{v} = \langle c, d\rangle$, denoted $\mathbf{u} \cdot \mathbf{v}$, is given by

$$\mathbf{u} \cdot \mathbf{v} = ac + bd.$$

$$\langle 2, 1\rangle \cdot \langle 5, -2\rangle = 2 \cdot 5 + 1(-2) = 8$$

If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, where $0° \leq \theta \leq 180°$, then

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta, \quad \text{or} \quad \cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}.$$

Find the angle θ between $\mathbf{u} = \langle 3, 1\rangle$ and $\mathbf{v} = \langle 2, -3\rangle$.

$$\cos\theta = \frac{\langle 3, 1\rangle \cdot \langle 2, -3\rangle}{\sqrt{3^2 + 1^2} \cdot \sqrt{2^2 + (-3)^2}}$$

$$\cos\theta = \frac{6 + (-3)}{\sqrt{10} \cdot \sqrt{13}}$$

$$\cos\theta = \frac{3}{\sqrt{130}}$$

$$\theta = \cos^{-1}\frac{3}{\sqrt{130}} \approx 74.7°$$

CHAPTER 7 ▶ Review Exercises

Use the law of sines to find the indicated part of each triangle ABC.

1. Find b if $C = 74.2°$, $c = 96.3$ m, $B = 39.5°$.

2. Find B if $A = 129.7°$, $a = 127$ ft, $b = 69.8$ ft.

3. Find B if $C = 51.3°$, $c = 68.3$ m, $b = 58.2$ m.

4. Find b if $a = 165$ m, $A = 100.2°$, $B = 25.0°$.

5. Find A if $B = 39° 50'$, $b = 268$ m, $a = 340$ m.

6. Find A if $C = 79° 20'$, $c = 97.4$ mm, $a = 75.3$ mm.

7. If we are given a, A, and C in a triangle ABC, does the possibility of the ambiguous case exist? If not, explain why.

8. Can triangle ABC exist if $a = 4.7$, $b = 2.3$, and $c = 7.0$? If not, explain why. Answer this question without using trigonometry.

9. Given $a = 10$ and $B = 30°$, determine the values of b for which A has

(a) exactly one value **(b)** two possible values **(c)** no value.

10. Explain why there can be no triangle ABC satisfying $A = 140°$, $a = 5$, and $b = 7$.

Use the law of cosines to find the indicated part of each triangle ABC.

11. Find A if $a = 86.14$ in., $b = 253.2$ in., $c = 241.9$ in.

12. Find b if $B = 120.7°$, $a = 127$ ft, $c = 69.8$ ft.

13. Find a if $A = 51° 20'$, $c = 68.3$ m, $b = 58.2$ m.

14. Find B if $a = 14.8$ m, $b = 19.7$ m, $c = 31.8$ m.

15. Find a if $A = 60°$, $b = 5.0$ cm, $c = 21$ cm.

16. Find A if $a = 13$ ft, $b = 17$ ft, $c = 8$ ft.

Solve each triangle ABC having the given information.

17. $A = 25.2°$, $a = 6.92$ yd, $b = 4.82$ yd

18. $A = 61.7°$, $a = 78.9$ m, $b = 86.4$ m

19. $a = 27.6$ cm, $b = 19.8$ cm, $C = 42° 30'$

20. $a = 94.6$ yd, $b = 123$ yd, $c = 109$ yd

Find the area of each triangle ABC with the given information.

21. $b = 840.6$ m, $c = 715.9$ m, $A = 149.3°$

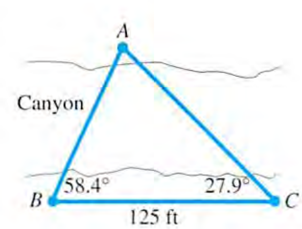
22. $a = 6.90$ ft, $b = 10.2$ ft, $C = 35° 10'$

23. $a = .913$ km, $b = .816$ km, $c = .582$ km

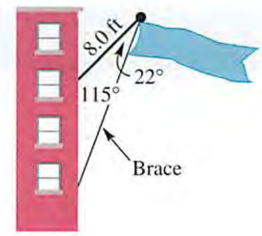
24. $a = 43$ m, $b = 32$ m, $c = 51$ m

Solve each problem.

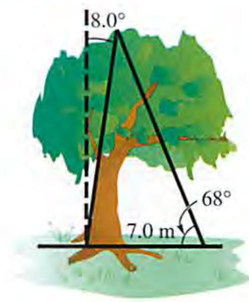
25. ***Distance Across a Canyon*** To measure the distance AB across a canyon for a power line, a surveyor measures angles B and C and the distance BC, as shown in the figure. What is the distance from A to B?



26. ***Length of a Brace*** A banner on an 8.0-ft pole is to be mounted on a building at an angle of 115°, as shown in the figure. Find the length of the brace.



27. ***Height of a Tree*** A tree leans at an angle of 8.0° from the vertical. From a point 7.0 m from the bottom of the tree, the angle of elevation to the top of the tree is 68°. How tall is the tree?



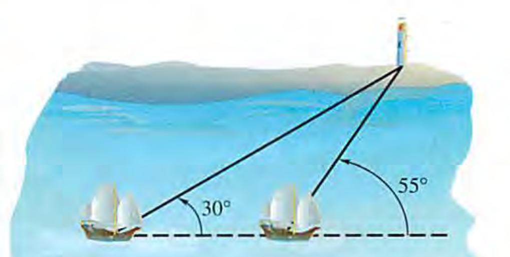
28. ***Hanging Sculpture*** A hanging sculpture in an art gallery is to be hung with two wires of lengths 15.0 ft and 12.2 ft so that the angle between them is 70.3°. How far apart should the ends of the wire be placed on the ceiling?

29. ***Height of a Tree*** A hill makes an angle of 14.3° with the horizontal. From the base of the hill, the angle of elevation to the top of a tree on top of the hill is 27.2°. The distance along the hill from the base to the tree is 212 ft. Find the height of the tree.

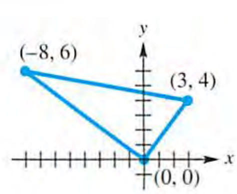
30. ***Pipeline Position*** A pipeline is to run between points A and B, which are separated by a protected wetlands area. To avoid the wetlands, the pipe will run from point A to C and then to B. The distances involved are $AB = 150$ km, $AC = 102$ km, and $BC = 135$ km. What angle should be used at point C?

31. ***Distance Between Two Boats*** Two boats leave a dock together. Each travels in a straight line. The angle between their courses measures 54° 10′. One boat travels 36.2 km per hr, and the other travels 45.6 km per hr. How far apart will they be after 3 hr?

32. ***Distance from a Ship to a Lighthouse*** A ship sailing parallel to shore sights a lighthouse at an angle of 30° from its direction of travel. After the ship travels 2.0 mi farther, the angle has increased to 55°. At that time, how far is the ship from the lighthouse?



33. ***Area of a Triangle*** Find the area of the triangle shown in the figure using Heron's area formula.

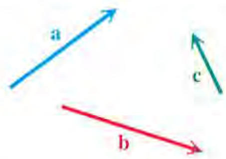


34. Show that the triangle in Exercise 33 is a right triangle. Then use the formula $\mathcal{A} = \frac{1}{2}ac \sin B$, with $B = 90°$, to find the area.

In Exercises 35 and 36, use the given vectors to sketch the following.

35. $\mathbf{a} - \mathbf{b}$

36. $\mathbf{a} + 3\mathbf{c}$



37. ***Concept Check*** Decide whether each statement is true or false.

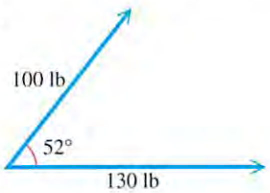
(a) Opposite angles of a parallelogram are equal.

(b) A diagonal of a parallelogram must bisect two angles of the parallelogram.

Given two forces and the angle between them, find the magnitude of the resultant force.

38. forces of 142 and 215 newtons, forming an angle of 112°

39.



Vector **v** *has the given magnitude and direction angle. Find the magnitudes of the horizontal and vertical components of* **v**.

40. $|\mathbf{v}| = 50,\ \theta = 45°$ (Give exact values.)

41. $|\mathbf{v}| = 964,\ \theta = 154°\ 20'$

Find the magnitude and direction angle for **u** *rounded to the nearest tenth.*

42. $\mathbf{u} = \langle 21, -20 \rangle$

43. $\mathbf{u} = \langle -9, 12 \rangle$

44. Let $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$ and $\mathbf{u} = -3\mathbf{i} + 2\mathbf{j}$. Express each in terms of **i** and **j**.

(a) $2\mathbf{v} + \mathbf{u}$ **(b)** $2\mathbf{v}$ **(c)** $\mathbf{v} - 3\mathbf{u}$

45. Let $\mathbf{a} = \langle 3, -2 \rangle$ and $\mathbf{b} = \langle -1, 3 \rangle$. Find $\mathbf{a} \cdot \mathbf{b}$ and the angle between **a** and **b**, rounded to the nearest tenth of a degree.

Find the vector of magnitude 1 *having the same direction angle as the given vector.*

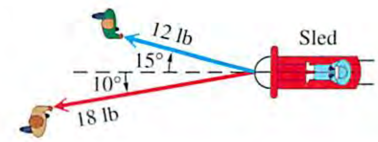
46. $\mathbf{u} = \langle -4, 3 \rangle$

47. $\mathbf{u} = \langle 5, 12 \rangle$

Solve each problem.

48. ***Force Placed on a Barge*** One rope pulls a barge directly east with a force of 100 newtons. Another rope pulls the barge to the northeast with a force of 200 newtons. Find the resultant force acting on the barge, to the nearest unit, and the angle between the resultant and the first rope, to the nearest tenth.

49. ***Weight of a Sled and Passenger*** Paula and Steve are pulling their daughter Jessie on a sled. Steve pulls with a force of 18 lb at an angle of 10°. Paula pulls with a force of 12 lb at an angle of 15°. Find the magnitude of the resultant force on Jessie and the sled.

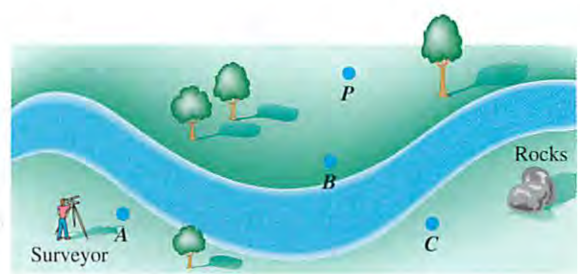


50. ***Angle of a Hill*** A 186-lb force just keeps a 2800-lb car from rolling down a hill. What angle does the hill make with the horizontal?

51. ***Direction and Speed of a Plane*** A plane has an airspeed of 520 mph. The pilot wishes to fly on a bearing of 310°. A wind of 37 mph is blowing from a bearing of 212°. What direction should the pilot fly, and what will be her actual speed?

52. ***Speed and Direction of a Boat*** A boat travels 15 km per hr in still water. The boat is traveling across a large river, on a bearing of 130°. The current in the river, coming from the west, has a speed of 7 km per hr. Find the resulting speed of the boat and its resulting direction of travel.

53. ***Control Points*** To obtain accurate aerial photographs, ground control must determine the coordinates of **control points** located on the ground that can be identified in the photographs. Using these known control points, the orientation and scale of each photograph can be found. Then, unknown positions and distances can easily be determined. The figure shows three consecutive control points A, B, and C.

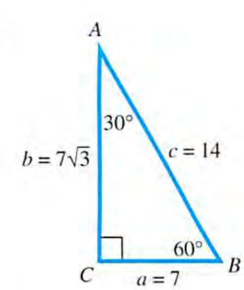


A surveyor measures a baseline distance of 92.1300 ft from B to an arbitrary point P. Angles BAP and BCP are found to be 2° 22′ 47″ and 5° 13′ 11″, respectively. Then, angles APB and CPB are determined to be 63° 4′ 25″ and 74° 19′ 49″, respectively. Determine the distance between control points A and B and between B and C. (*Source:* Moffitt, F. and E. Mikhail, *Photogrammetry,* Third Edition, Harper & Row, 1980.)

The following identities involve all six parts of a triangle ABC and are thus useful for checking answers.

$$\frac{a+b}{c} = \frac{\cos \frac{1}{2}(A-B)}{\sin \frac{1}{2}C} \quad \text{Newton's formula}$$

$$\frac{a-b}{c} = \frac{\sin \frac{1}{2}(A-B)}{\cos \frac{1}{2}C} \quad \text{Mollweide's formula}$$



54. Apply Newton's formula to the given triangle to verify the accuracy of the information.

55. Apply Mollweide's formula to the given triangle to verify the accuracy of the information.

CHAPTER 7 ▶ Test

Find the indicated part of each triangle ABC.

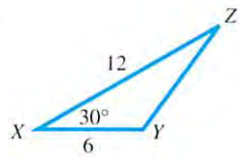
1. Find C if $A = 25.2°$, $a = 6.92$ yd, and $b = 4.82$ yd.

2. Find c if $C = 118°$, $a = 75.0$ km, and $b = 131$ km.

3. Find B if $a = 17.3$ ft, $b = 22.6$ ft, $c = 29.8$ ft.

4. Find the area of triangle ABC if $a = 14$, $b = 30$, and $c = 40$.

5. Find the area of triangle XYZ shown here.



6. Given $a = 10$ and $B = 150°$ in triangle ABC, determine the values of b for which A has

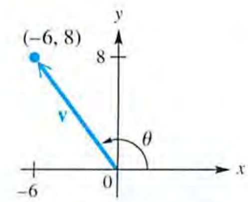
(a) exactly one value **(b)** two possible values **(c)** no value.

Solve each triangle ABC.

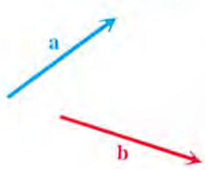
7. $A = 60°$, $b = 30$ m, $c = 45$ m

8. $b = 1075$ in., $c = 785$ in., $C = 38° \, 30'$

9. Find the magnitude and the direction angle for the vector shown in the figure.



10. Use the given vectors to sketch $\mathbf{a} + \mathbf{b}$.

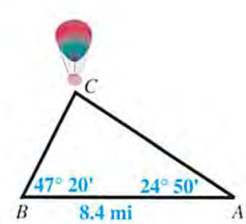


11. For the vectors $\mathbf{u} = \langle -1, 3 \rangle$ and $\mathbf{v} = \langle 2, -6 \rangle$, find each of the following.

(a) $\mathbf{u} + \mathbf{v}$ **(b)** $-3\mathbf{v}$ **(c)** $\mathbf{u} \cdot \mathbf{v}$ **(d)** $|\mathbf{u}|$

Solve each problem.

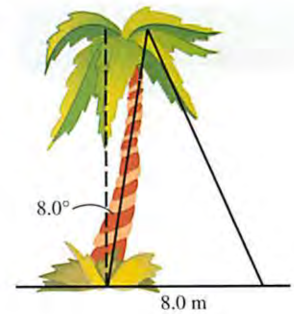
12. ***Height of a Balloon*** The angles of elevation of a balloon from two points A and B on level ground are $24° \, 50'$ and $47° \, 20'$, respectively. As shown in the figure, points A, B, and C are in the same vertical plane and points A and B are 8.4 mi apart. Approximate the height of the balloon above the ground to the nearest tenth of a mile.



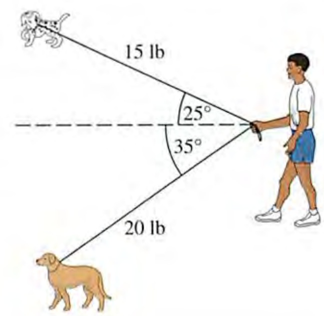
13. ***Horizontal and Vertical Components*** Find the horizontal and vertical components of the vector with magnitude 569 that is inclined 127.5° from the horizontal. Give your answer in the form $\langle a, b \rangle$.

14. ***Radio Direction Finders*** Radio direction finders are placed at points A and B, which are 3.46 mi apart on an east-west line, with A west of B. From A, the bearing of a certain illegal pirate radio transmitter is 48°, and from B the bearing is 302°. Find the distance between the transmitter and A to the nearest hundredth of a mile.

15. ***Height of a Tree*** A tree leans at an angle of 8.0° from the vertical, as shown in the figure. From a point 8.0 m from the bottom of the tree, the angle of elevation to the top of the tree is 66°. How tall is the tree?



16. ***Walking Dogs on Leashes*** While Michael is walking his two dogs, Duke and Prince, they reach a corner and must wait for a WALK sign. Michael is holding the two leashes in the same hand, and the dogs are pulling on their leashes at the angles and forces shown in the figure. Find the magnitude of the force (to the nearest tenth of a pound) Michael must apply to restrain the dogs.



CHAPTER 7 ▶ Quantitative Reasoning

Just how much does the U.S. flag "show its colors"?

The flag of the United States includes the colors red, white, and blue. Which color is predominant? Clearly the answer is either red or white. (It can be shown that only 18.73% of the total area is blue.) (*Source:* Banks, R., *Slicing Pizzas, Racing Turtles, and Further Adventures in Applied Mathematics,* Princeton University Press, 1999.)

1. Let R denote the radius of the circumscribing circle of a five-pointed star appearing on the American flag. The star can be decomposed into ten congruent triangles. In the figure, r is the radius of the circumscribing circle of the pentagon in the interior of the star. Show that the area of a star is

$$A = \left[5\frac{\sin A \sin B}{\sin(A + B)}\right]R^2.$$

(*Hint:* $\sin C = \sin[180° - (A + B)] = \sin(A + B)$.)

2. Angles A and B have values 18° and 36°, respectively. Express the area of a star in terms of its radius, R.

3. To determine whether red or white is predominant, we must know the measurements of the flag. Consider a flag of width 10 in., length 19 in., length of each upper stripe 11.4 in., and radius R of the circumscribing circle of each star .308 in. The thirteen stripes consist of six matching pairs of red and white stripes and one additional red, upper stripe. Therefore, we must compare the area of a red, upper stripe with the total area of the 50 white stars.

(a) Compute the area of the red, upper stripe.

(b) Compute the total area of the 50 white stars.

(c) Which color occupies the greatest area on the flag?