8 Complex Numbers, Polar Equations, and Parametric Equations

High-resolution computer graphics and *complex numbers* make it possible to produce beautiful shapes called *fractals*. Benoit B. Mandelbrot first used the term *fractal* in 1975. At its basic level, a fractal is a unique, enchanting geometric figure with an endless self-similarity property. A fractal image repeats itself infinitely with ever-decreasing dimensions. If you look at smaller and smaller portions of a fractal image, you will continue to see the whole—much like looking into two parallel mirrors that are facing each other.

The fractal called *Newton's basins of attraction for the cube roots of unity* is discussed in Exercise 50, Section 8.4. (*Source:* Crownover, R., *Introduction to Fractals and Chaos,* Jones and Bartlett, 1995; Lauwerier, H., *Fractals,* Princeton University Press, 1991.)

8.1 Complex Numbers

Basic Concepts of Complex Numbers ▪ Complex Solutions of Equations ▪ Operations on Complex Numbers

Basic Concepts of Complex Numbers

The set of real numbers does not include all numbers needed in mathematics. For example, there is no real number solution of the equation

$$x^2 = -1,$$

since -1 has no real square root. Square roots of negative numbers were not incorporated into an integrated number system until the 16th century. They were then used as solutions of equations and, later in the 18th century, in surveying. Today, such numbers are used extensively in science and engineering.

▼ LOOKING AHEAD TO CALCULUS
The letters j and k are also used to represent $\sqrt{-1}$ in calculus and some applications (electronics, for example).

To extend the real number system to include numbers such as $\sqrt{-1}$, the number i is defined to have the following property.

$$i^2 = -1$$

Thus, $\boldsymbol{i = \sqrt{-1}}$. The number i is called the **imaginary unit.** Numbers of the form $a + bi$, where a and b are real numbers, are called **complex numbers.** In the complex number $a + bi$, a is the **real part** and b is the **imaginary part.***

The relationships among the various sets of numbers are shown in Figure 1.

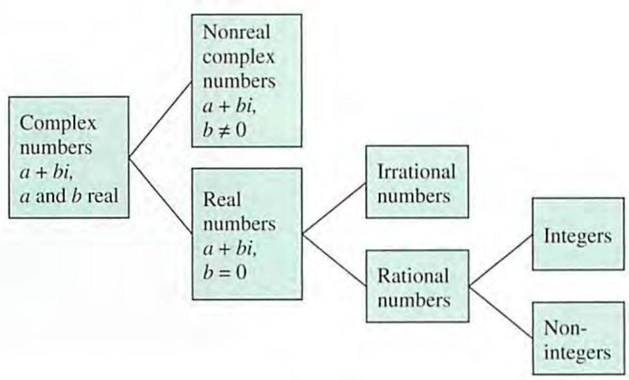


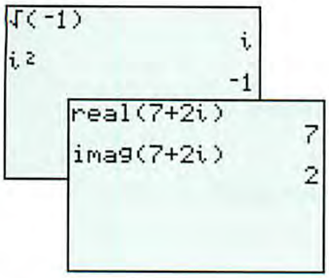
Figure 1

Two complex numbers $a + bi$ and $c + di$ are equal provided that their real parts are equal and their imaginary parts are equal; that is,

$$a + bi = c + di \quad \text{if and only if} \quad a = c \quad \text{and} \quad b = d.$$

For a complex number $a + bi$, if $b = 0$, then $a + bi = a$, which is a real number. Thus, the set of real numbers is a subset of the set of complex numbers. If $a = 0$ and $b \neq 0$, the complex number is said to be a **pure imaginary number.** For example, $3i$ is a pure imaginary number. A pure imaginary number, or a number such as $7 + 2i$ with $a \neq 0$ and $b \neq 0$, is a **nonreal complex number.** A complex number written in the form $a + bi$ (or $a + ib$) is in

*In some texts, the term bi is defined to be the imaginary part.



The calculator is in complex number mode.

Figure 2

standard form. (The form $a + ib$ is used to write expressions such as $i\sqrt{5}$, since $\sqrt{5}i$ could be mistaken for $\sqrt{5i}$.)

NOW TRY EXERCISES 9, 11, AND 13. ◀

Some graphing calculators, such as the TI-83/84 Plus, are capable of working with complex numbers, as seen in Figure 2. The top screen supports the definition of i. The bottom screen shows how the calculator returns the real and imaginary parts of $7 + 2i$. ■

For a positive real number a, $\sqrt{-a}$ is defined as follows.

THE EXPRESSION $\sqrt{-a}$

If $a > 0$, then $$\sqrt{-a} = i\sqrt{a}.$$

▶ EXAMPLE 1 WRITING $\sqrt{-a}$ AS $i\sqrt{a}$

Write as the product of a real number and i, using the definition of $\sqrt{-a}$.

(a) $\sqrt{-16}$ **(b)** $\sqrt{-70}$ **(c)** $\sqrt{-48}$

Solution

(a) $\sqrt{-16} = i\sqrt{16} = 4i$ **(b)** $\sqrt{-70} = i\sqrt{70}$

(c) $\sqrt{-48} = i\sqrt{48} = i\sqrt{16 \cdot 3} = 4i\sqrt{3}$ Product rule for radicals: $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$

NOW TRY EXERCISES 17, 19, AND 21. ◀

Complex Solutions of Equations

▶ EXAMPLE 2 SOLVING QUADRATIC EQUATIONS FOR COMPLEX SOLUTIONS

Solve each equation.

(a) $x^2 = -9$ **(b)** $x^2 + 24 = 0$

Solution

(a) Take the square root on both sides, remembering that we must find both roots, indicated by the $\pm$ sign.

$$x^2 = -9$$

$$x = \pm\sqrt{-9} \quad \text{Square root property (Appendix A)}$$

Take *both* square roots.

$$x = \pm i\sqrt{9} \quad \sqrt{-a} = i\sqrt{a}$$

$$x = \pm 3i \quad \sqrt{9} = 3$$

The solution set is $\{\pm 3i\}$.

(b) $x^2 + 24 = 0$

$x^2 = -24$ Subtract 24.

$x = \pm\sqrt{-24}$ Square root property

$x = \pm i\sqrt{24}$ $\sqrt{-a} = i\sqrt{a}$

$x = \pm i\sqrt{4 \cdot 6} = \pm 2i\sqrt{6}$ Product rule for radicals

The solution set is $\{\pm 2i\sqrt{6}\}$.

NOW TRY EXERCISES 25 AND 27.

▶ EXAMPLE 3 SOLVING A QUADRATIC EQUATION FOR COMPLEX SOLUTIONS

Solve $9x^2 + 5 = 6x$.

Solution Write the equation in standard form, $9x^2 - 6x + 5 = 0$. Then use the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{Quadratic formula (Appendix A)}$$

The fraction bar extends under $-b$.

$$= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(9)(5)}}{2(9)} \quad a = 9,\ b = -6,\ c = 5$$

$$= \frac{6 \pm \sqrt{-144}}{18}$$

$$= \frac{6 \pm 12i}{18} \quad \sqrt{-144} = 12i$$

Factor first, then divide out the common factor.

$$x = \frac{6(1 \pm 2i)}{6 \cdot 3} = \frac{1 \pm 2i}{3} \quad \text{Factor; write in lowest terms.}$$

The solution set is $\left\{\frac{1}{3} \pm \frac{2}{3}i\right\}$.

NOW TRY EXERCISE 29.

Operations on Complex Numbers

Products or quotients with negative radicands are simplified by first rewriting $\sqrt{-a}$ as $i\sqrt{a}$ for a positive number a. Then the properties of real numbers are applied, together with the fact that $i^2 = -1$.

▶ **Caution** ***When working with negative radicands, use the definition $\sqrt{-a} = i\sqrt{a}$ before using any of the other rules for radicals.*** In particular, the rule $\sqrt{c} \cdot \sqrt{d} = \sqrt{cd}$ is valid only when c and d are *not* both negative. For example,

$$\sqrt{(-4)(-9)} = \sqrt{36} = 6,$$

while $\sqrt{-4} \cdot \sqrt{-9} = 2i(3i) = 6i^2 = -6,$

so $\sqrt{-4} \cdot \sqrt{-9} \neq \sqrt{(-4)(-9)}.$

▶ EXAMPLE 4 **FINDING PRODUCTS AND QUOTIENTS INVOLVING NEGATIVE RADICANDS**

Multiply or divide, as indicated. Simplify each answer.

(a) $\sqrt{-7} \cdot \sqrt{-7}$ **(b)** $\sqrt{-6} \cdot \sqrt{-10}$ **(c)** $\dfrac{\sqrt{-20}}{\sqrt{-2}}$ **(d)** $\dfrac{\sqrt{-48}}{\sqrt{24}}$

Solution

(a)
$$\begin{aligned} \sqrt{-7} \cdot \sqrt{-7} &= i\sqrt{7} \cdot i\sqrt{7} \\ &= i^2 \cdot (\sqrt{7})^2 \\ &= -1 \cdot 7 \qquad i^2 = -1 \\ &= -7 \end{aligned}$$

First write all square roots in terms of i.

(b)
$$\begin{aligned} \sqrt{-6} \cdot \sqrt{-10} &= i\sqrt{6} \cdot i\sqrt{10} \\ &= i^2 \cdot \sqrt{60} \\ &= -1\sqrt{4 \cdot 15} \\ &= -1 \cdot 2\sqrt{15} \\ &= -2\sqrt{15} \end{aligned}$$

(c) $$\frac{\sqrt{-20}}{\sqrt{-2}} = \frac{i\sqrt{20}}{i\sqrt{2}} = \sqrt{\frac{20}{2}} = \sqrt{10} \qquad \text{Quotient rule for radicals; } \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

(d) $$\frac{\sqrt{-48}}{\sqrt{24}} = \frac{i\sqrt{48}}{\sqrt{24}} = i\sqrt{\frac{48}{24}} = i\sqrt{2}$$

NOW TRY EXERCISES 37, 39, 41, AND 43. ◀

▶ EXAMPLE 5 **SIMPLIFYING A QUOTIENT INVOLVING A NEGATIVE RADICAND**

Write $\dfrac{-8 + \sqrt{-128}}{4}$ in standard form $a + bi$.

Solution

$$\begin{aligned} \frac{-8 + \sqrt{-128}}{4} &= \frac{-8 + \sqrt{-64 \cdot 2}}{4} \\ &= \frac{-8 + 8i\sqrt{2}}{4} \qquad \sqrt{-64} = 8i \\ &= \frac{4(-2 + 2i\sqrt{2})}{4} \qquad \text{Factor.} \\ &= -2 + 2i\sqrt{2} \qquad \text{Lowest terms} \end{aligned}$$

Be sure to factor before simplifying.

NOW TRY EXERCISE 49. ◀

With the definitions $i^2 = -1$ and $\sqrt{-a} = i\sqrt{a}$ for $a > 0$, all properties of real numbers are extended to complex numbers. As a result, complex numbers are added, subtracted, multiplied, and divided using the definitions on the following pages and real number properties.

ADDITION AND SUBTRACTION OF COMPLEX NUMBERS

For complex numbers $a + bi$ and $c + di$,

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

and
$$(a + bi) - (c + di) = (a - c) + (b - d)i.$$

That is, to add or subtract complex numbers, add or subtract the real parts and add or subtract the imaginary parts.

▶ EXAMPLE 6 ADDING AND SUBTRACTING COMPLEX NUMBERS

Find each sum or difference.

(a) $(3 - 4i) + (-2 + 6i)$ **(b)** $(-9 + 7i) + (3 - 15i)$

(c) $(-4 + 3i) - (6 - 7i)$ **(d)** $(12 - 5i) - (8 - 3i) + (-4 + 2i)$

Solution

(a) $(3 - 4i) + (-2 + 6i) = \underbrace{[3 + (-2)]}_{\text{Add real parts.}} + \underbrace{[-4 + 6]}_{\text{Add imaginary parts.}}i$ Commutative, associative, distributive properties

$$= 1 + 2i$$

(b) $(-9 + 7i) + (3 - 15i) = -6 - 8i$

(c) $(-4 + 3i) - (6 - 7i) = (-4 - 6) + [3 - (-7)]i$
$$= -10 + 10i$$

(d) $(12 - 5i) - (8 - 3i) + (-4 + 2i) = (12 - 8 - 4) + (-5 + 3 + 2)i$
$$= 0, \quad \text{or} \quad 0 + 0i$$

NOW TRY EXERCISES 55 AND 57. ◀

The product of two complex numbers is found by multiplying as if the numbers were binomials and using the fact that $i^2 = -1$, as follows.

$(a + bi)(c + di) = ac + adi + bic + bidi$ FOIL (multiply, First, Outer, Inner, Last terms)

$= ac + adi + bci + bdi^2$

$= ac + (ad + bc)i + bd(-1)$ Distributive property; $i^2 = -1$

$= (ac - bd) + (ad + bc)i$

MULTIPLICATION OF COMPLEX NUMBERS

For complex numbers $a + bi$ and $c + di$,

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

This definition is not practical in routine calculations. To find a given product, it is easier just to multiply as with binomials.

▶ EXAMPLE 7 MULTIPLYING COMPLEX NUMBERS

Find each product.

(a) $(2 - 3i)(3 + 4i)$ **(b)** $(4 + 3i)^2$

(c) $(2 + i)(-2 - i)$ **(d)** $(6 + 5i)(6 - 5i)$

Solution

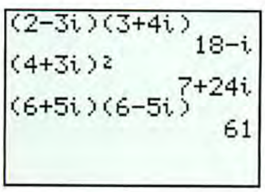
(a)
$$\begin{aligned}(2 - 3i)(3 + 4i) &= 2(3) + 2(4i) - 3i(3) - 3i(4i) && \text{FOIL} \\ &= 6 + 8i - 9i - 12i^2 \\ &= 6 - i - 12(-1) && i^2 = -1 \\ &= 18 - i\end{aligned}$$

(b)
$$\begin{aligned}(4 + 3i)^2 &= 4^2 + 2(4)(3i) + (3i)^2 && \text{Square of a binomial: } (x+y)^2 = x^2 + 2xy + y^2 \\ &= 16 + 24i + 9i^2 \\ &= 16 + 24i + 9(-1) && i^2 = -1 \\ &= 7 + 24i\end{aligned}$$

Remember to add twice the product of the two terms.

(c)
$$\begin{aligned}(2 + i)(-2 - i) &= -4 - 2i - 2i - i^2 && \text{FOIL} \\ &= -4 - 4i - (-1) && \text{Combine terms; } i^2 = -1 \\ &= -4 - 4i + 1 \\ &= -3 - 4i && \text{Standard form}\end{aligned}$$

(d)
$$\begin{aligned}(6 + 5i)(6 - 5i) &= 6^2 - (5i)^2 && \text{Product of the sum and difference of two terms: } (x+y)(x-y) = x^2 - y^2 \\ &= 36 - 25(-1) && i^2 = -1 \\ &= 36 + 25 \\ &= 61, \text{ or } 61 + 0i && \text{Standard form}\end{aligned}$$



This screen shows how the TI–83/84 Plus displays the results found in Example 7(a), (b), and (d).

NOW TRY EXERCISES 63, 67, AND 71. ◀

Powers of i can be simplified using the facts

$$i^2 = -1 \quad \text{and} \quad i^4 = (i^2)^2 = (-1)^2 = 1.$$

▶ EXAMPLE 8 SIMPLIFYING POWERS OF *i*

Simplify each power of i.

(a) i^{15} **(b)** i^{-3} **(c)** $\dfrac{1}{i^{-13}}$

Solution

(a) Since $i^2 = -1$ and $i^4 = 1$, write the given power as a product involving i^2 or i^4. For example,

$$i^3 = i^2 \cdot i = (-1) \cdot i = -i.$$

Alternatively, using i^4 and i^3 to rewrite i^{15} gives

$$i^{15} = i^{12} \cdot i^3 = (i^4)^3 \cdot i^3 = 1^3(-i) = -i.$$

(b) $i^{-3} = i^{-4} \cdot i = (i^4)^{-1} \cdot i = (1)^{-1} \cdot i = i$

(c) $\dfrac{1}{i^{-13}} = i^{13} = i^{12} \cdot i = (i^4)^3 \cdot i = 1^3 \cdot i = i$

NOW TRY EXERCISES 81, 89, AND 91.

We can use the method of Example 8 to construct the following list of powers of i.

Powers of i can be found on the TI–83/84 Plus calculator.

POWERS OF i

$i^1 = i$	$i^5 = i$	$i^9 = i$	
$i^2 = -1$	$i^6 = -1$	$i^{10} = -1$	
$i^3 = -i$	$i^7 = -i$	$i^{11} = -i$	
$i^4 = 1$	$i^8 = 1$	$i^{12} = 1,$	and so on.

Example 7(d) showed that $(6 + 5i)(6 - 5i) = 61$. The numbers $6 + 5i$ and $6 - 5i$ differ only in the sign of their imaginary parts, and are called **complex conjugates.** ***The product of a complex number $a + bi$ and its conjugate $a - bi$ is always a real number, $a^2 + b^2$.*** This product is the sum of the squares of the real and imaginary parts.

PROPERTY OF COMPLEX CONJUGATES

For real numbers a and b,

$$(a + bi)(a - bi) = a^2 + b^2.$$

To find the quotient of two complex numbers in standard form, we multiply both the numerator and the denominator by the complex conjugate of the denominator.

▶ EXAMPLE 9 DIVIDING COMPLEX NUMBERS

Write each quotient in standard form $a + bi$.

(a) $\dfrac{3 + 2i}{5 - i}$ **(b)** $\dfrac{3}{i}$

Solution

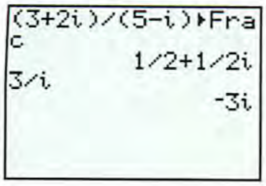
(a) $\dfrac{3 + 2i}{5 - i} = \dfrac{(3 + 2i)(5 + i)}{(5 - i)(5 + i)}$ Multiply by the complex conjugate of the denominator in both the numerator and the denominator.

$= \dfrac{15 + 3i + 10i + 2i^2}{25 - i^2}$ Multiply.

$= \dfrac{13 + 13i}{26}$ $i^2 = -1$

$= \dfrac{13}{26} + \dfrac{13i}{26}$ $\dfrac{a + bi}{c} = \dfrac{a}{c} + \dfrac{bi}{c}$

$= \dfrac{1}{2} + \dfrac{1}{2}i$ Lowest terms; standard form



This screen supports the results in Example 9. Notice that the answer to part (a) must be interpreted $\frac{1}{2}+\frac{1}{2}i$, *not* $\frac{1}{2}+\frac{1}{2i}$.

To *check* this answer, show that

$$\left(\frac{1}{2}+\frac{1}{2}i\right)(5-i)=3+2i. \quad \text{Quotient} \times \text{Divisor} = \text{Dividend}$$

(b) $\frac{3}{i}=\frac{3(-i)}{i(-i)}$ $\quad$ $-i$ is the conjugate of i.

$=\frac{-3i}{-i^2}$

$=\frac{-3i}{1}$ $\quad$ $-i^2=-(-1)=1$

$=-3i$, or $0-3i$ $\quad$ Standard form

NOW TRY EXERCISES 95 AND 101.

8.1 Exercises

Concept Check *Determine whether each statement is* true *or* false. *If it is false, tell why.*

1. Every real number is a complex number.
2. No real number is a pure imaginary number.
3. Every pure imaginary number is a complex number.
4. A number can be both real and complex.
5. There is no real number that is a complex number.
6. A complex number might not be a pure imaginary number.

Identify each number as real, complex, pure imaginary, *or* nonreal complex. *(More than one of these descriptions may apply.)*

7. -4 **8.** 0 **9.** $13i$ **10.** $-7i$ **11.** $5+i$

12. $-6-2i$ **13.** π **14.** $\sqrt{24}$ **15.** $\sqrt{-25}$ **16.** $\sqrt{-36}$

Write each number as the product of a real number and i. See Example 1.

17. $\sqrt{-25}$ **18.** $\sqrt{-36}$ **19.** $\sqrt{-10}$ **20.** $\sqrt{-15}$

21. $\sqrt{-288}$ **22.** $\sqrt{-500}$ **23.** $-\sqrt{-18}$ **24.** $-\sqrt{-80}$

Solve each quadratic equation and express all nonreal complex solutions in terms of i. See Examples 2 and 3.

25. $x^2=-16$ **26.** $x^2=-36$ **27.** $x^2+12=0$

28. $x^2+48=0$ **29.** $3x^2+2=-4x$ **30.** $2x^2+3x=-2$

31. $x^2-6x+14=0$ **32.** $x^2+4x+11=0$ **33.** $4(x^2-x)=-7$

34. $3(3x^2-2x)=-7$ **35.** $x^2+1=-x$ **36.** $x^2+2=2x$

Multiply or divide, as indicated. Simplify each answer. See Example 4.

37. $\sqrt{-13}\cdot\sqrt{-13}$ **38.** $\sqrt{-17}\cdot\sqrt{-17}$ **39.** $\sqrt{-3}\cdot\sqrt{-8}$

40. $\sqrt{-5}\cdot\sqrt{-15}$ **41.** $\frac{\sqrt{-30}}{\sqrt{-10}}$ **42.** $\frac{\sqrt{-70}}{\sqrt{-7}}$

43. $\dfrac{\sqrt{-24}}{\sqrt{8}}$ 44. $\dfrac{\sqrt{-54}}{\sqrt{27}}$ 45. $\dfrac{\sqrt{-10}}{\sqrt{-40}}$

46. $\dfrac{\sqrt{-40}}{\sqrt{20}}$ 47. $\dfrac{\sqrt{-6}\cdot\sqrt{-2}}{\sqrt{3}}$ 48. $\dfrac{\sqrt{-12}\cdot\sqrt{-6}}{\sqrt{8}}$

Write each number in standard form a + bi. See Example 5.

49. $\dfrac{-6-\sqrt{-24}}{2}$ 50. $\dfrac{-9-\sqrt{-18}}{3}$ 51. $\dfrac{10+\sqrt{-200}}{5}$

52. $\dfrac{20+\sqrt{-8}}{2}$ 53. $\dfrac{-3+\sqrt{-18}}{24}$ 54. $\dfrac{-5+\sqrt{-50}}{10}$

Find each sum or difference. Write the answer in standard form. See Example 6.

55. $(3+2i)+(9-3i)$ 56. $(4-i)+(8+5i)$

57. $(-2+4i)-(-4+4i)$ 58. $(-3+2i)-(-4+2i)$

59. $(2-5i)-(3+4i)-(-1-9i)$ 60. $(-4-i)-(2+3i)+(6+4i)$

61. $-i-2-(6-4i)-(5-2i)$ 62. $3-(4-i)-4i+(-2+5i)$

Find each product. Write the answer in standard form. See Example 7.

63. $(2+i)(3-2i)$ 64. $(-2+3i)(4-2i)$ 65. $(2+4i)(-1+3i)$

66. $(1+3i)(2-5i)$ 67. $(3-2i)^2$ 68. $(2+i)^2$

69. $(3+i)(3-i)$ 70. $(5+i)(5-i)$ 71. $(-2-3i)(-2+3i)$

72. $(6-4i)(6+4i)$ 73. $(\sqrt{6}+i)(\sqrt{6}-i)$ 74. $(\sqrt{2}-4i)(\sqrt{2}+4i)$

75. $i(3-4i)(3+4i)$ 76. $i(2+7i)(2-7i)$ 77. $3i(2-i)^2$

78. $-5i(4-3i)^2$ 79. $(2+i)(2-i)(4+3i)$ 80. $(3-i)(3+i)(2-6i)$

Simplify each power of i. See Example 8.

81. i^{25} 82. i^{29} 83. i^{22} 84. i^{26}

85. i^{23} 86. i^{27} 87. i^{32} 88. i^{40}

89. i^{-13} 90. i^{-14} 91. $\dfrac{1}{i^{-11}}$ 92. $\dfrac{1}{i^{-12}}$

93. Suppose that your friend, Leah Goldberg, tells you that she has discovered a method of simplifying a positive power of i. "Just divide the exponent by 4," she says, "and then look at the remainder. Then refer to the table of powers of i in this section. The large power of i is equal to the power indicated by the remainder. And if the remainder is 0, the result is $i^0 = 1$." Explain why her method works.

94. Explain why the following method of simplifying i^{-42} works.

$$i^{-42} = i^{-42}\cdot i^{44} = i^{-42+44} = i^2 = -1$$

Find each quotient. Write the answer in standard form a + bi. See Example 9.

95. $\dfrac{6+2i}{1+2i}$ 96. $\dfrac{14+5i}{3+2i}$ 97. $\dfrac{2-i}{2+i}$

98. $\frac{4-3i}{4+3i}$ **99.** $\frac{1-3i}{1+i}$ **100.** $\frac{-3+4i}{2-i}$

101. $\frac{-5}{i}$ **102.** $\frac{-6}{i}$ **103.** $\frac{8}{-i}$

104. $\frac{12}{-i}$ **105.** $\frac{2}{3i}$ **106.** $\frac{5}{9i}$

107. Show that $\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$ is a square root of i.

108. Show that $\frac{\sqrt{3}}{2}+\frac{1}{2}i$ is a cube root of i.

109. Evaluate $3z - z^2$ if $z = 3 - 2i$.

110. Evaluate $-2z + z^3$ if $z = -6i$.

(Modeling) Alternating Current *Complex numbers are used to describe current, I, voltage, E, and impedance, Z (the opposition to current). These three quantities are related by the equation $E = IZ$. Thus, if any two of these quantities are known, the third can be found. In each problem, solve the equation $E = IZ$ for the missing variable.*

111. $I = 8 + 6i,\quad Z = 6 + 3i$

112. $I = 10 + 6i,\quad Z = 8 + 5i$

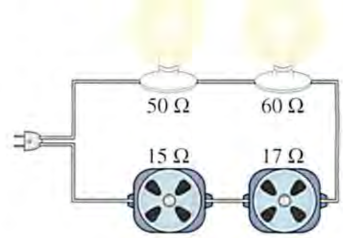
113. $I = 7 + 5i,\quad E = 28 + 54i$

114. $E = 35 + 55i,\quad Z = 6 + 4i$

(Modeling) Impedance ***Impedance*** *is a measure of the opposition to the flow of alternating electrical current found in common electrical outlets. It consists of two parts called* **resistance** *and* **reactance**. *Resistance occurs when a lightbulb is turned on, while reactance is produced when electricity passes through a coil of wire like that found in electric motors. Impedance Z in* ohms (Ω) *can be expressed as a complex number, where the real part represents resistance and the imaginary part represents reactance.*

For example, if the resistive part is 3 ohms *and the reactive part is* 4 ohms, *then the impedance could be described by the complex number $Z = 3 + 4i$. In the series circuit shown in the figure, the total impedance will be the sum of the individual impedances.* (*Source:* Wilcox, G. and C. Hesselberth, *Electricity for Engineering Technology*, Allyn & Bacon, 1970.)

115. The circuit contains two light bulbs and two electric motors. Assuming that the light bulbs are pure resistive and the motors are pure reactive, find the total impedance in this circuit and express it in the form $Z = a + bi$.



116. The phase angle θ measures the phase difference between the voltage and the current in an electrical circuit. θ (in degrees) can be determined by the equation $\tan\theta = \frac{b}{a}$. Find θ for this circuit.

8.2 Trigonometric (Polar) Form of Complex Numbers

The Complex Plane and Vector Representation ▪ Trigonometric (Polar) Form ▪ Converting Between Rectangular and Trigonometric (Polar) Forms ▪ An Application of Complex Numbers to Fractals

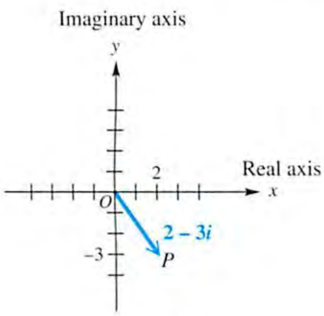


Figure 3

The Complex Plane and Vector Representation Unlike real numbers, complex numbers cannot be ordered. One way to organize and illustrate them is by using a graph. To graph a complex number such as $2 - 3i$, we modify the familiar coordinate system by calling the horizontal axis the **real axis** and the vertical axis the **imaginary axis.** Then complex numbers can be graphed in this **complex plane,** as shown in Figure 3. Each complex number $a + bi$ determines a unique position vector with initial point $(0, 0)$ and terminal point (a, b). This relationship shows the close connection between vectors and complex numbers.

> ▶ **Note** This geometric representation is the reason that $a + bi$ is called the **rectangular form** of a complex number. (*Rectangular form* is also called *standard form.*)

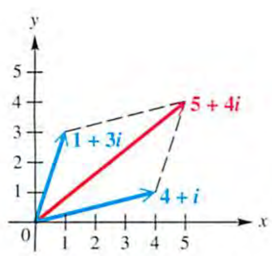


Figure 4

Recall that the sum of the two complex numbers $4 + i$ and $1 + 3i$ is

$$(4 + i) + (1 + 3i) = 5 + 4i. \quad \text{(Section 8.1)}$$

Graphically, the sum of two complex numbers is represented by the vector that is the resultant of the vectors corresponding to the two numbers, as shown in Figure 4.

▶ EXAMPLE 1 EXPRESSING THE SUM OF COMPLEX NUMBERS GRAPHICALLY

Find the sum of $6 - 2i$ and $-4 - 3i$. Graph both complex numbers and their resultant.

Solution The sum is found by adding the two numbers.

$$(6 - 2i) + (-4 - 3i) = 2 - 5i$$

The graphs are shown in Figure 5.

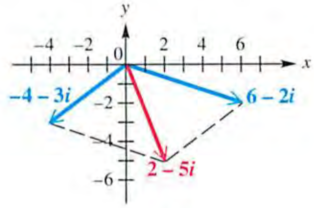


Figure 5

NOW TRY EXERCISE 13. ◀

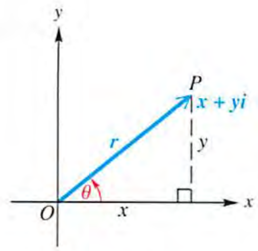


Figure 6

Trigonometric (Polar) Form Figure 6 shows the complex number $x + yi$ that corresponds to a vector **OP** with direction angle θ and magnitude r. The following relationships among x, y, r, and θ can be verified from Figure 6.

RELATIONSHIPS AMONG $x, y, r,$ AND θ

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2} \qquad \tan\theta = \frac{y}{x}, \quad \text{if } x \neq 0$$

Substituting $x = r\cos\theta$ and $y = r\sin\theta$ into $x + yi$ gives

$$\begin{aligned} x + yi &= r\cos\theta + (r\sin\theta)i \\ &= r(\cos\theta + i\sin\theta). \quad \text{Factor out } r. \end{aligned}$$

TRIGONOMETRIC (POLAR) FORM OF A COMPLEX NUMBER

The expression

$$r(\cos\theta + i\sin\theta)$$

is called the **trigonometric form** (or **polar form**) of the complex number $x + yi$. The expression $\cos\theta + i\sin\theta$ is sometimes abbreviated **cis θ.** Using this notation, $r(\cos\theta + i\sin\theta)$ is written **r cis θ.**

The number r is the **absolute value** (or **modulus**) of $x + yi$, and θ is the **argument** of $x + yi$. In this section, we choose the value of θ in the interval $[0°, 360°)$. However, any angle coterminal with θ also could serve as the argument.

▶ EXAMPLE 2 CONVERTING FROM TRIGONOMETRIC FORM TO RECTANGULAR FORM

Express $2(\cos 300° + i\sin 300°)$ in rectangular form.

Algebraic Solution

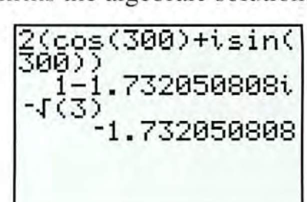
From **Chapter 2,** $\cos 300° = \frac{1}{2}$ and $\sin 300° = -\frac{\sqrt{3}}{2}$, so

$$\begin{aligned} 2(\cos 300° + i\sin 300°) &= 2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) \\ &= 1 - i\sqrt{3}. \end{aligned}$$

Notice that the real part is positive and the imaginary part is negative; this is consistent with 300° being a quadrant IV angle.

Graphing Calculator Solution

Figure 7 confirms the algebraic solution.



The imaginary part is an approximation for $-\sqrt{3}$.

Figure 7

NOW TRY EXERCISE 29. ◀

Converting Between Rectangular and Trigonometric (Polar) Forms To convert from rectangular form to trigonometric form, we use the following procedure.

CONVERTING FROM RECTANGULAR FORM TO TRIGONOMETRIC FORM

Step 1 Sketch a graph of the number $x + yi$ in the complex plane.

Step 2 Find r by using the equation $r = \sqrt{x^2 + y^2}$.

Step 3 Find θ by using the equation $\tan\theta = \frac{y}{x}$, $x \neq 0$, choosing the quadrant indicated in Step 1.

▶ **Caution** Errors often occur in Step 3. ***Be sure to choose the correct quadrant for θ by referring to the graph sketched in Step 1.***

▶ EXAMPLE 3 CONVERTING FROM RECTANGULAR FORM TO TRIGONOMETRIC FORM

Write each complex number in trigonometric form.

(a) $-\sqrt{3} + i$ (Use radian measure.) **(b)** $-3i$ (Use degree measure.)

Solution

(a) We start by sketching the graph of $-\sqrt{3} + i$ in the complex plane, as shown in Figure 8. Since $x = -\sqrt{3}$ and $y = 1$,

$$r = \sqrt{x^2 + y^2} = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{3 + 1} = 2,$$

and $$\tan\theta = \frac{y}{x} = \frac{1}{-\sqrt{3}} = -\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = -\frac{\sqrt{3}}{3}.$$

Rationalize the denominator.

Since $\tan\theta = -\frac{\sqrt{3}}{3}$, the reference angle for θ in radians is $\frac{\pi}{6}$. From the graph, we see that θ is in quadrant II, so $\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$. Therefore,

$$-\sqrt{3} + i = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = 2 \text{ cis } \frac{5\pi}{6}.$$

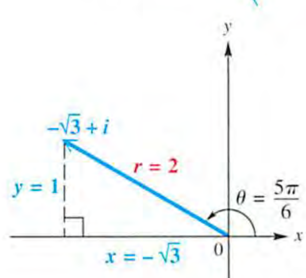


Figure 8

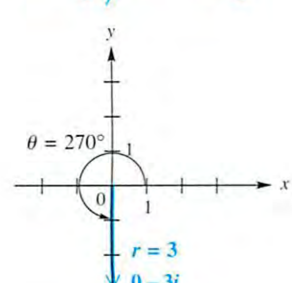
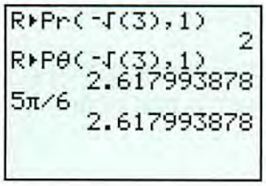


Figure 9



Choices 5 and 6 in the top screen show how to convert from rectangular (x, y) form to trigonometric form. The calculator is in radian mode. The results agree with our algebraic results in Example 3(a).

(b) The sketch of $-3i$ is shown in Figure 9 on the preceding page. Since $-3i = 0 - 3i$, we have $x = 0$ and $y = -3$ and

$$r = \sqrt{0^2 + (-3)^2} = \sqrt{0 + 9} = \sqrt{9} = 3.$$

We cannot find θ by using $\tan \theta = \frac{y}{x}$, because $x = 0$. From the graph, a value for θ is 270°. In trigonometric form,

$$-3i = 3(\cos 270° + i \sin 270°) = 3 \text{ cis } 270°.$$

NOW TRY EXERCISES 41 AND 47.

▶ **Note** In Example 3, we gave answers in both forms: $r(\cos \theta + i \sin \theta)$ and r cis θ. These forms will be used interchangeably from now on.

▶ EXAMPLE 4 CONVERTING BETWEEN TRIGONOMETRIC AND RECTANGULAR FORMS USING CALCULATOR APPROXIMATIONS

Write each complex number in its alternative form, using calculator approximations as necessary.

(a) $6(\cos 115° + i \sin 115°)$ **(b)** $5 - 4i$

Solution

(a) Since 115° does not have a special angle as a reference angle, we cannot find exact values for cos 115° and sin 115°. Use a calculator set in degree mode to find $\cos 115° \approx -.4226182617$ and $\sin 115° \approx .906307787$. Therefore, in rectangular form,

$$6(\cos 115° + i \sin 115°) \approx 6(-.4226182617 + .906307787i)$$
$$\approx -2.5357 + 5.4378i.$$

(b) A sketch of $5 - 4i$ shows that θ must be in quadrant IV. See Figure 10. Here

$$r = \sqrt{5^2 + (-4)^2} = \sqrt{41} \quad \text{and} \quad \tan \theta = -\frac{4}{5}.$$

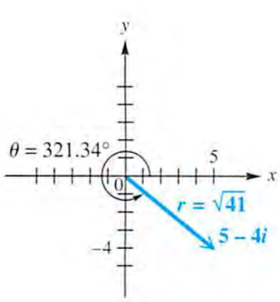


Figure 10

Use a calculator to find that one measure of θ is $-38.66°$. In order to express θ in the interval $[0, 360°)$, we find $\theta = 360° - 38.66° = 321.34°$. Use these results to get

$$5 - 4i = \sqrt{41} \text{ cis } 321.34°.$$

NOW TRY EXERCISES 53 AND 57.

An Application of Complex Numbers to Fractals

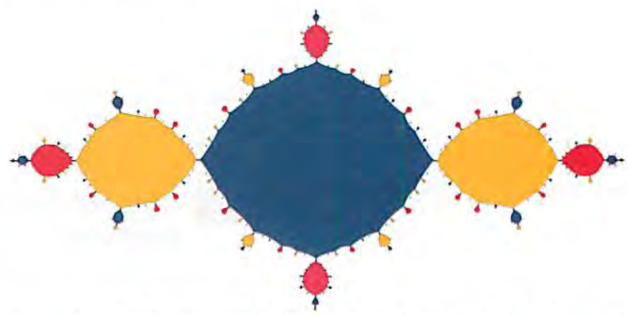
As mentioned in the chapter opener, a **fractal** is a unique, enchanting geometric figure with an endless self-similarity property.

▶ EXAMPLE 5 **DECIDING WHETHER A COMPLEX NUMBER IS IN THE JULIA SET**

The fractal called the **Julia set** is shown in Figure 11. To determine if a complex number $z = a + bi$ is in this Julia set, perform the following sequence of calculations. Repeatedly compute the values of

$$z^2 - 1, \quad (z^2 - 1)^2 - 1, \quad [(z^2 - 1)^2 - 1]^2 - 1, \quad \ldots .$$

If the absolute values of any of the resulting complex numbers exceed 2, then the complex number z is not in the Julia set. Otherwise z is part of this set and the point (a, b) should be shaded in the graph.



Source: Figure from Crownover, R., *Introduction to Fractals and Chaos.* Copyright © 1995. Boston: Jones and Bartlett Publishers. Reprinted with permission.

Figure 11

Determine whether each number belongs to the Julia set.

(a) $z = 0 + 0i$ **(b)** $z = 1 + 1i$

Solution

(a) Here

$$\begin{aligned} z &= 0 + 0i = 0, \\ z^2 - 1 &= 0^2 - 1 = -1, \\ (z^2 - 1)^2 - 1 &= (-1)^2 - 1 = 0, \\ [(z^2 - 1)^2 - 1]^2 - 1 &= 0^2 - 1 = -1, \end{aligned}$$

and so on. We see that the calculations repeat as $0, -1, 0, -1$, and so on. The absolute values are either 0 or 1, which do not exceed 2, so $0 + 0i$ is in the Julia set and the point $(0, 0)$ is part of the graph.

(b) We have $z^2 - 1 = (1 + i)^2 - 1 = (1 + 2i + i^2) - 1 = -1 + 2i$. The absolute value is $\sqrt{(-1)^2 + 2^2} = \sqrt{5}$. Since $\sqrt{5}$ is greater than 2, $1 + 1i$ is not in the Julia set and $(1, 1)$ is not part of the graph.

NOW TRY EXERCISE 63. ◀

8.2 Exercises

1. *Concept Check* The absolute value of a complex number represents the __________ of the vector representing it in the complex plane.

2. *Concept Check* What is the geometric interpretation of the argument of a complex number?

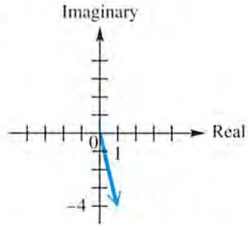
Graph each complex number. See Example 1.

3. $-3 + 2i$ **4.** $6 - 5i$ **5.** $\sqrt{2} + \sqrt{2}i$ **6.** $2 - 2i\sqrt{3}$

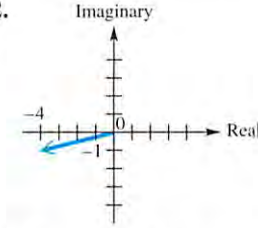
7. $-4i$ **8.** $3i$ **9.** -8 **10.** 2

Concept Check *Give the rectangular form of the complex number represented in each graph.*

11.



12.



Find the sum of each pair of complex numbers. In Exercises 13–16, graph both complex numbers and their resultant. See Example 1.

13. $4 - 3i, -1 + 2i$ **14.** $2 + 3i, -4 - i$ **15.** $5 - 6i, -5 + 3i$

16. $7 - 3i, -4 + 3i$ **17.** $-3, 3i$ **18.** $6, -2i$

19. $2 + 6i, -2i$ **20.** $4 - 2i, 5$ **21.** $7 + 6i, 3i$

22. $-5 - 8i, -1$ **23.** $\frac{1}{2} + \frac{2}{3}i, \frac{2}{3} + \frac{1}{2}i$ **24.** $-\frac{1}{5} + \frac{2}{7}i, \frac{3}{7} - \frac{3}{4}i$

Write each complex number in rectangular form. See Example 2.

25. $2(\cos 45° + i \sin 45°)$ **26.** $4(\cos 60° + i \sin 60°)$

27. $10(\cos 90° + i \sin 90°)$ **28.** $8(\cos 270° + i \sin 270°)$

29. $4(\cos 240° + i \sin 240°)$ **30.** $2(\cos 330° + i \sin 330°)$

31. $3 \text{ cis } 150°$ **32.** $2 \text{ cis } 30°$

33. $5 \text{ cis } 300°$ **34.** $6 \text{ cis } 135°$

35. $\sqrt{2} \text{ cis } 225°$ **36.** $\sqrt{3} \text{ cis } 315°$

37. $4(\cos(-30°) + i \sin(-30°))$ **38.** $\sqrt{2}(\cos(-60°) + i \sin(-60°))$

Write each complex number in trigonometric form $r(\cos \theta + i \sin \theta)$, with θ in the interval $[0°, 360°)$. See Example 3.

39. $-3 - 3i\sqrt{3}$ **40.** $1 + i\sqrt{3}$ **41.** $\sqrt{3} - i$

42. $4\sqrt{3} + 4i$ **43.** $-5 - 5i$ **44.** $-2 + 2i$

45. $2 + 2i$ **46.** $4 + 4i$ **47.** $5i$

48. $-2i$ **49.** -4 **50.** 7

Perform each conversion, using a calculator to approximate answers as necessary. See Example 4.

	Rectangular Form	Trigonometric Form
51.	$2 + 3i$	________
52.	________	$\cos 35° + i \sin 35°$
53.	________	$3(\cos 250° + i \sin 250°)$
54.	$-4 + i$	________
55.	$12i$	________
56.	________	$3 \text{ cis } 180°$
57.	$3 + 5i$	________
58.	________	$\text{cis } 110.5°$

Concept Check *The complex number z, where $z = x + yi$, can be graphed in the plane as (x, y). Describe the graphs of all complex numbers z satisfying the conditions in Exercises 59–62.*

59. The absolute value of z is 1.

60. The real and imaginary parts of z are equal.

61. The real part of z is 1.

62. The imaginary part of z is 1.

Julia Set *Refer to Example 5 to solve Exercises 63 and 64.*

63. Is $z = -.2i$ in the Julia set?

64. The graph of the Julia set in Figure 11 appears to be symmetric with respect to both the x-axis and y-axis. Complete the following to show that this is true.

(a) Show that complex conjugates have the same absolute value.
(b) Compute $z_1^2 - 1$ and $z_2^2 - 1$, where $z_1 = a + bi$ and $z_2 = a - bi$.
(c) Discuss why if (a, b) is in the Julia set then so is $(a, -b)$.
(d) Conclude that the graph of the Julia set must be symmetric with respect to the x-axis.
(e) Using a similar argument, show that the Julia set must also be symmetric with respect to the y-axis.

In Exercises 65 and 66, suppose $z = r(\cos\theta + i\sin\theta)$.

65. Use vectors to show that the conjugate of z is

$$r[\cos(360° - \theta) + i\sin(360° - \theta)], \quad \text{or} \quad r(\cos\theta - i\sin\theta).$$

66. Use vectors to show that $-z = r[\cos(\theta + \pi) + i\sin(\theta + \pi)]$.

Concept Check *In Exercises 67–69, identify the geometric condition (A, B, or C) that implies the situation.*

A. *The corresponding vectors have opposite directions.*
B. *The terminal points of the vectors corresponding to $a + bi$ and $c + di$ lie on a horizontal line.*
C. *The corresponding vectors have the same direction.*

67. The difference between two nonreal complex numbers $a + bi$ and $c + di$ is a real number.

68. The absolute value of the sum of two complex numbers $a + bi$ and $c + di$ is equal to the sum of their absolute values.

69. The absolute value of the difference of two complex numbers $a + bi$ and $c + di$ is equal to the sum of their absolute values.

70. Show that z and iz have the same absolute value. How are the graphs of these two numbers related?

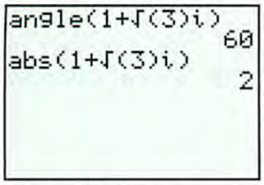
8.3 The Product and Quotient Theorems

Products of Complex Numbers in Trigonometric Form ▪ Quotients of Complex Numbers in Trigonometric Form

Products of Complex Numbers in Trigonometric Form

Using the FOIL method to multiply complex numbers in rectangular form, we find the product of $1 + i\sqrt{3}$ and $-2\sqrt{3} + 2i$ as follows.

$$\begin{aligned}(1 + i\sqrt{3})(-2\sqrt{3} + 2i) &= -2\sqrt{3} + 2i - 2i(3) + 2i^2\sqrt{3} && \text{(Section 8.1)}\\ &= -2\sqrt{3} + 2i - 6i - 2\sqrt{3} && i^2 = -1\\ &= -4\sqrt{3} - 4i && \text{Combine terms.}\end{aligned}$$



With the TI–83/84 Plus calculator in complex and degree modes, the MATH menu can be used to find the angle and the magnitude (absolute value) of the vector that corresponds to a given complex number.

We can also find this same product by first converting the complex numbers $1 + i\sqrt{3}$ and $-2\sqrt{3} + 2i$ to trigonometric form. Using the method explained in the preceding section,

$$1 + i\sqrt{3} = 2(\cos 60° + i \sin 60°) \qquad \text{(Section 8.2)}$$

and
$$-2\sqrt{3} + 2i = 4(\cos 150° + i \sin 150°).$$

If we multiply the trigonometric forms, and if we use identities for the cosine and the sine of the sum of two angles, then the result is

$$\begin{aligned}
&[2(\cos 60° + i \sin 60°)][4(\cos 150° + i \sin 150°)] \\
&= 2 \cdot 4(\cos 60° \cdot \cos 150° + i \sin 60° \cdot \cos 150° \\
&\quad + i \cos 60° \cdot \sin 150° + i^2 \sin 60° \cdot \sin 150°) \qquad \text{(Section 5.3)} \\
&= 8[(\cos 60° \cdot \cos 150° - \sin 60° \cdot \sin 150°) \\
&\quad + i(\sin 60° \cdot \cos 150° + \cos 60° \cdot \sin 150°)] \\
&= 8[\cos(60° + 150°) + i \sin(60° + 150°)] \\
&= 8(\cos 210° + i \sin 210°).
\end{aligned}$$

The absolute value of the product, 8, is equal to the product of the absolute values of the factors, $2 \cdot 4$, and the argument of the product, 210°, is equal to the sum of the arguments of the factors, $60° + 150°$.

As we would expect, the product obtained when multiplying by the first method is the rectangular form of the product obtained when multiplying by the second method.

$$\begin{aligned}
8(\cos 210° + i \sin 210°) &= 8\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) \\
&= -4\sqrt{3} - 4i
\end{aligned}$$

We can generalize this work in the following *product theorem*.

PRODUCT THEOREM

If $r_1(\cos \theta_1 + i \sin \theta_1)$ and $r_2(\cos \theta_2 + i \sin \theta_2)$ are any two complex numbers, then

$$\begin{aligned}
&[r_1(\cos \theta_1 + i \sin \theta_1)] \cdot [r_2(\cos \theta_2 + i \sin \theta_2)] \\
&= r_1 r_2[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)].
\end{aligned}$$

In compact form, this is written

$$(r_1 \operatorname{cis} \theta_1)(r_2 \operatorname{cis} \theta_2) = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2).$$

That is, to multiply complex numbers in trigonometric form, multiply their absolute values and add their arguments.

▶ EXAMPLE 1 USING THE PRODUCT THEOREM

Find the product of $3(\cos 45° + i \sin 45°)$ and $2(\cos 135° + i \sin 135°)$. Write the result in rectangular form.

Solution

$$
\begin{aligned}
&[3(\cos 45° + i \sin 45°)][2(\cos 135° + i \sin 135°)] \\
&= 3 \cdot 2[\cos(45° + 135°) + i \sin(45° + 135°)] && \text{Product theorem} \\
&= 6(\cos 180° + i \sin 180°) && \text{Multiply and add.} \\
&= 6(-1 + i \cdot 0) \\
&= 6(-1) = -6 && \text{Rectangular form}
\end{aligned}
$$

NOW TRY EXERCISE 7. ◀

Quotients of Complex Numbers in Trigonometric Form The rectangular form of the quotient of the complex numbers $1 + i\sqrt{3}$ and $-2\sqrt{3} + 2i$ is

$$
\begin{aligned}
\frac{1 + i\sqrt{3}}{-2\sqrt{3} + 2i} &= \frac{(1 + i\sqrt{3})(-2\sqrt{3} - 2i)}{(-2\sqrt{3} + 2i)(-2\sqrt{3} - 2i)} && \text{Multiply by the conjugate of the denominator. (Section 8.1)} \\
&= \frac{-2\sqrt{3} - 2i - 6i - 2i^2\sqrt{3}}{12 - 4i^2} && \text{FOIL; } (x + y)(x - y) = x^2 - y^2 \\
&= \frac{-8i}{16} = -\frac{1}{2}i. && \text{Simplify.}
\end{aligned}
$$

Writing $1 + i\sqrt{3}$, $-2\sqrt{3} + 2i$, and $-\frac{1}{2}i$ in trigonometric form gives

$$1 + i\sqrt{3} = 2(\cos 60° + i \sin 60°),$$

$$-2\sqrt{3} + 2i = 4(\cos 150° + i \sin 150°),$$

and

$$-\frac{1}{2}i = \frac{1}{2}[\cos(-90°) + i \sin(-90°)].$$

The absolute value of the quotient, $\frac{1}{2}$, is the quotient of the two absolute values, $\frac{2}{4} = \frac{1}{2}$. The argument of the quotient, $-90°$, is the difference of the two arguments, $60° - 150° = -90°$. Generalizing leads to the *quotient theorem*.

QUOTIENT THEOREM

If $r_1(\cos \theta_1 + i \sin \theta_1)$ and $r_2(\cos \theta_2 + i \sin \theta_2)$ are any two complex numbers, where $r_2(\cos \theta_2 + i \sin \theta_2) \neq 0$, then

$$\frac{r_1(\cos \theta_1 + i \sin \theta_1)}{r_2(\cos \theta_2 + i \sin \theta_2)} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)].$$

In compact form, this is written

$$\frac{r_1 \operatorname{cis} \theta_1}{r_2 \operatorname{cis} \theta_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2).$$

That is, to divide complex numbers in trigonometric form, divide their absolute values and subtract their arguments.

▶ EXAMPLE 2 USING THE QUOTIENT THEOREM

Find the quotient $\frac{10 \text{ cis}(-60°)}{5 \text{ cis } 150°}$. Write the result in rectangular form.

Solution

$$\frac{10 \text{ cis}(-60°)}{5 \text{ cis } 150°} = \frac{10}{5} \text{cis}(-60° - 150°) \quad \text{Quotient theorem}$$

$$= 2 \text{ cis}(-210°) \quad \text{Divide and subtract.}$$

$$= 2[\cos(-210°) + i \sin(-210°)] \quad \text{Rewrite.}$$

$$= 2\left[-\frac{\sqrt{3}}{2} + i\left(\frac{1}{2}\right)\right] \quad \cos(-210°) = -\tfrac{\sqrt{3}}{2};\ \sin(-210°) = \tfrac{1}{2} \text{ (Section 2.2)}$$

$$= -\sqrt{3} + i \quad \text{Rectangular form}$$

NOW TRY EXERCISE 17. ◀

▶ EXAMPLE 3 USING THE PRODUCT AND QUOTIENT THEOREMS WITH A CALCULATOR

Use a calculator to find the following. Write the results in rectangular form.

(a) $(9.3 \text{ cis } 125.2°)(2.7 \text{ cis } 49.8°)$ **(b)** $\frac{10.42\left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right)}{5.21\left(\cos \frac{\pi}{5} + i \sin \frac{\pi}{5}\right)}$

Solution

(a) $(9.3 \text{ cis } 125.2°)(2.7 \text{ cis } 49.8°) = 9.3(2.7) \text{ cis}(125.2° + 49.8°)$ Product theorem

$$= 25.11 \text{ cis } 175°$$

$$= 25.11(\cos 175° + i \sin 175°)$$

$$\approx 25.11[-.99619470 + i(.08715574)]$$

$$\approx -25.0144 + 2.1885i$$

(b) $\frac{10.42\left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right)}{5.21\left(\cos \frac{\pi}{5} + i \sin \frac{\pi}{5}\right)} = \frac{10.42}{5.21}\left[\cos\left(\frac{3\pi}{4} - \frac{\pi}{5}\right) + i \sin\left(\frac{3\pi}{4} - \frac{\pi}{5}\right)\right]$ Quotient theorem

$$= 2\left(\cos \frac{11\pi}{20} + i \sin \frac{11\pi}{20}\right)$$

$$\approx -.3129 + 1.9754i$$

NOW TRY EXERCISES 27 AND 29. ◀

8.3 Exercises

Concept Check *Fill in the blanks with the correct responses.*

1. When multiplying two complex numbers in trigonometric form, we ______ their absolute values and ______ their arguments.

2. When dividing two complex numbers in trigonometric form, we ______ their absolute values and ______ their arguments.

Find each product and write it in rectangular form. See Example 1.

3. $[3(\cos 60^\circ + i \sin 60^\circ)][2(\cos 90^\circ + i \sin 90^\circ)]$

4. $[4(\cos 30^\circ + i \sin 30^\circ)][5(\cos 120^\circ + i \sin 120^\circ)]$

5. $[4(\cos 60^\circ + i \sin 60^\circ)][6(\cos 330^\circ + i \sin 330^\circ)]$

6. $[8(\cos 300^\circ + i \sin 300^\circ)][5(\cos 120^\circ + i \sin 120^\circ)]$

7. $[2(\cos 45^\circ + i \sin 45^\circ)][2(\cos 225^\circ + i \sin 225^\circ)]$

8. $[8(\cos 210^\circ + i \sin 210^\circ)][2(\cos 330^\circ + i \sin 330^\circ)]$

9. $(\sqrt{3} \text{ cis } 45^\circ)(\sqrt{3} \text{ cis } 225^\circ)$

10. $(6 \text{ cis } 120^\circ)[5 \text{ cis}(-30^\circ)]$

11. $(5 \text{ cis } 90^\circ)(3 \text{ cis } 45^\circ)$

12. $(\sqrt{2} \text{ cis } 300^\circ)(\sqrt{2} \text{ cis } 270^\circ)$

Find each quotient and write it in rectangular form. In Exercises 19–24, first convert the numerator and the denominator to trigonometric form. See Example 2.

13. $\dfrac{4(\cos 120^\circ + i \sin 120^\circ)}{2(\cos 150^\circ + i \sin 150^\circ)}$

14. $\dfrac{24(\cos 150^\circ + i \sin 150^\circ)}{2(\cos 30^\circ + i \sin 30^\circ)}$

15. $\dfrac{10(\cos 230^\circ + i \sin 230^\circ)}{5(\cos 50^\circ + i \sin 50^\circ)}$

16. $\dfrac{12(\cos 293^\circ + i \sin 293^\circ)}{6(\cos 23^\circ + i \sin 23^\circ)}$

17. $\dfrac{3 \text{ cis } 305^\circ}{9 \text{ cis } 65^\circ}$

18. $\dfrac{16 \text{ cis } 310^\circ}{8 \text{ cis } 70^\circ}$

19. $\dfrac{8}{\sqrt{3} + i}$

20. $\dfrac{2i}{-1 - i\sqrt{3}}$

21. $\dfrac{-i}{1 + i}$

22. $\dfrac{1}{2 - 2i}$

23. $\dfrac{2\sqrt{6} - 2i\sqrt{2}}{\sqrt{2} - i\sqrt{6}}$

24. $\dfrac{4 + 4i}{2 - 2i}$

Use a calculator to perform the indicated operations, expressing real and imaginary parts to four decimal places. Give answers in rectangular form. See Example 3.

25. $[2.5(\cos 35^\circ + i \sin 35^\circ)][3.0(\cos 50^\circ + i \sin 50^\circ)]$

26. $[4.6(\cos 12^\circ + i \sin 12^\circ)][2.0(\cos 13^\circ + i \sin 13^\circ)]$

27. $(12 \text{ cis } 18.5^\circ)(3 \text{ cis } 12.5^\circ)$

28. $(4 \text{ cis } 19.25^\circ)(7 \text{ cis } 41.75^\circ)$

29. $\dfrac{45(\cos 127^\circ + i \sin 127^\circ)}{22.5(\cos 43^\circ + i \sin 43^\circ)}$

30. $\dfrac{30(\cos 130^\circ + i \sin 130^\circ)}{10(\cos 21^\circ + i \sin 21^\circ)}$

31. $\left[2 \text{ cis } \dfrac{5\pi}{9}\right]^2$

32. $\left[24.3 \text{ cis } \dfrac{7\pi}{12}\right]^2$

RELATING CONCEPTS

***For individual or collaborative investigation* (Exercises 33–39)**

Consider the complex numbers

$$w = -1 + i \quad \text{and} \quad z = -1 - i.$$

Work Exercises 33–39 in order.

33. Multiply w and z using their rectangular forms and the FOIL method from **Section 8.1.** Leave the product in rectangular form.

34. Find the trigonometric forms of w and z.

35. Multiply w and z using their trigonometric forms and the method described in this section.

36. Use the result of Exercise 35 to find the rectangular form of wz. How does this compare to your result in Exercise 33?

37. Find the quotient $\frac{w}{z}$ using their rectangular forms and multiplying both the numerator and the denominator by the conjugate of the denominator. Leave the quotient in rectangular form.

38. Use the trigonometric forms of w and z, found in Exercise 34, to divide w by z using the method described in this section.

39. Use the result of Exercise 38 to find the rectangular form of $\frac{w}{z}$. How does this compare to your result in Exercise 37?

40. Notice that $(r \text{ cis } \theta)^2 = (r \text{ cis } \theta)(r \text{ cis } \theta) = r^2 \text{ cis}(\theta + \theta) = r^2 \text{ cis } 2\theta$. State in your own words how we can square a complex number in trigonometric form. (In the next section, we will develop this idea more fully.)

41. Without actually performing the operations, state why the following products are the same.

$$[2(\cos 45^\circ + i \sin 45^\circ)] \cdot [5(\cos 90^\circ + i \sin 90^\circ)]$$

and $$\{2[\cos(-315^\circ) + i \sin(-315^\circ)]\} \cdot \{5[\cos(-270^\circ) + i \sin(-270^\circ)]\}$$

42. Show that $\frac{1}{z} = \frac{1}{r}(\cos \theta - i \sin \theta)$, where $z = r(\cos \theta + i \sin \theta)$.

(Modeling) Solve each problem.

43. *Electrical Current* The alternating current in an electric inductor is $I = \frac{E}{Z}$ amperes, where E is voltage and $Z = R + X_L i$ is impedance. If $E = 8(\cos 20^\circ + i \sin 20^\circ)$, $R = 6$, and $X_L = 3$, find the current. Give the answer in rectangular form, with real and imaginary parts to the nearest hundredth.

44. *Electrical Current* The current I in a circuit with voltage E, resistance R, capacitive reactance X_c, and inductive reactance X_L is

$$I = \frac{E}{R + (X_L - X_c)i}.$$

Find I if $E = 12(\cos 25^\circ + i \sin 25^\circ)$, $R = 3$, $X_L = 4$, and $X_c = 6$. Give the answer in rectangular form, with real and imaginary parts to the nearest tenth.

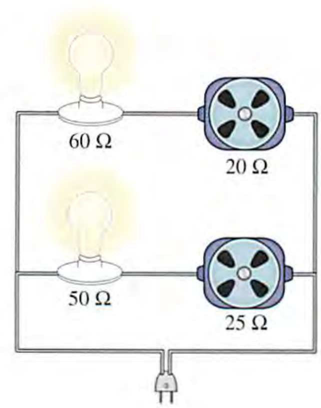
(Modeling) Impedance *In the parallel electrical circuit shown in the figure, the impedance Z can be calculated using the equation*

$$Z = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2}},$$

where Z_1 and Z_2 are the impedances for the branches of the circuit.

45. If $Z_1 = 50 + 25i$ and $Z_2 = 60 + 20i$, calculate Z.

46. Determine the angle θ for the value of Z found in Exercise 45.



8.4 De Moivre's Theorem; Powers and Roots of Complex Numbers

Powers of Complex Numbers (De Moivre's Theorem) ▪ Roots of Complex Numbers

Powers of Complex Numbers (De Moivre's Theorem)

In the previous section, we studied the product theorem for complex numbers in trigonometric form. Because raising a number to a positive integer power is a repeated application of the product rule, it would seem likely that a theorem for finding powers of complex numbers exists. For example,

$$\begin{aligned}[r(\cos\theta + i\sin\theta)]^2 &= [r(\cos\theta + i\sin\theta)][r(\cos\theta + i\sin\theta)] \\ &= r \cdot r[\cos(\theta+\theta) + i\sin(\theta+\theta)] \\ &= r^2(\cos 2\theta + i\sin 2\theta).\end{aligned}$$

In the same way,

$$[r(\cos\theta + i\sin\theta)]^3 = r^3(\cos 3\theta + i\sin 3\theta).$$

These results suggest the following theorem for positive integer values of n. Although the theorem is stated and can be proved for all n, we use it only for positive integer values of n and their reciprocals.


Abraham De Moivre (1667–1754)

DE MOIVRE'S THEOREM

If $r(\cos\theta + i\sin\theta)$ is a complex number, and if n is any real number, then

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta).$$

In compact form, this is written

$$[r \text{ cis } \theta]^n = r^n(\text{cis } n\theta).$$

This theorem is named after the French expatriate friend of Isaac Newton, Abraham De Moivre, although he never explicitly stated it.

▶ EXAMPLE 1 FINDING A POWER OF A COMPLEX NUMBER

Find $\left(1 + i\sqrt{3}\right)^8$ and express the result in rectangular form.

Solution First write $1 + i\sqrt{3}$ in trigonometric form.

$$1 + i\sqrt{3} = 2(\cos 60° + i \sin 60°) \quad \text{(Section 8.2)}$$

Now, apply De Moivre's theorem.

$$\begin{aligned}
\left(1 + i\sqrt{3}\right)^8 &= [2(\cos 60° + i \sin 60°)]^8 \\
&= 2^8[\cos(8 \cdot 60°) + i \sin(8 \cdot 60°)] && \text{De Moivre's theorem} \\
&= 256(\cos 480° + i \sin 480°) \\
&= 256(\cos 120° + i \sin 120°) && \text{480° and 120° are coterminal. (Section 1.1)} \\
&= 256\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) && \cos 120° = -\tfrac{1}{2};\ \sin 120° = \tfrac{\sqrt{3}}{2} \text{ (Section 2.2)} \\
&= -128 + 128i\sqrt{3} && \text{Rectangular form}
\end{aligned}$$

NOW TRY EXERCISE 7. ◀

Roots of Complex Numbers Every nonzero complex number has exactly n distinct complex nth roots. De Moivre's theorem can be extended to find all nth roots of a complex number.

nth ROOT

For a positive integer n, the complex number $a + bi$ is an ***n*th root** of the complex number $x + yi$ if

$$(a + bi)^n = x + yi.$$

To find the three complex cube roots of $8(\cos 135° + i \sin 135°)$, for example, look for a complex number, say $r(\cos \alpha + i \sin \alpha)$, that will satisfy

$$[r(\cos \alpha + i \sin \alpha)]^3 = 8(\cos 135° + i \sin 135°).$$

By De Moivre's theorem, this equation becomes

$$r^3(\cos 3\alpha + i \sin 3\alpha) = 8(\cos 135° + i \sin 135°).$$

Set $r^3 = 8$ and $\cos 3\alpha + i \sin 3\alpha = \cos 135° + i \sin 135°$, to satisfy this equation. The first of these conditions implies that $r = 2$, and the second implies that

$$\cos 3\alpha = \cos 135° \qquad \text{and} \qquad \sin 3\alpha = \sin 135°.$$

For these equations to be satisfied, 3α must represent an angle that is coterminal with $135°$. Therefore, we must have

$$3\alpha = 135° + 360° \cdot k, \quad k \text{ any integer}$$

or

$$\alpha = \frac{135° + 360° \cdot k}{3}, \quad k \text{ any integer.}$$

Now, let k take on the integer values 0, 1, and 2.

$$\text{If } k = 0, \text{ then} \qquad \alpha = \frac{135° + 0°}{3} = 45°.$$

$$\text{If } k = 1, \text{ then} \qquad \alpha = \frac{135° + 360°}{3} = \frac{495°}{3} = 165°.$$

$$\text{If } k = 2, \text{ then} \qquad \alpha = \frac{135° + 720°}{3} = \frac{855°}{3} = 285°.$$

In the same way, $\alpha = 405°$ when $k = 3$. But note that $405° = 45° + 360°$ so $\sin 405° = \sin 45°$ and $\cos 405° = \cos 45°$. Similarly, if $k = 4$, $\alpha = 525°$, which has the same sine and cosine values as $165°$. To continue with larger values of k would just be repeating solutions already found. Therefore, all of the cube roots (three of them) can be found by letting $k = 0$, 1, or 2.

When $k = 0$, the root is $\quad 2(\cos 45° + i \sin 45°)$.
When $k = 1$, the root is $\quad 2(\cos 165° + i \sin 165°)$.
When $k = 2$, the root is $\quad 2(\cos 285° + i \sin 285°)$.

In summary, we see that $2(\cos 45° + i \sin 45°)$, $2(\cos 165° + i \sin 165°)$, and $2(\cos 285° + i \sin 285°)$ are the three cube roots of $8(\cos 135° + i \sin 135°)$.

Generalizing our results, we state the following theorem.

*n*th ROOT THEOREM

If n is any positive integer, r is a positive real number, and θ is in degrees, then the nonzero complex number $r(\cos \theta + i \sin \theta)$ has exactly n distinct nth roots, given by

$$\sqrt[n]{r}(\cos \alpha + i \sin \alpha) \quad \text{or} \quad \sqrt[n]{r} \text{ cis } \alpha,$$

where

$$\alpha = \frac{\theta + 360° \cdot k}{n}, \quad \text{or} \quad \alpha = \frac{\theta}{n} + \frac{360° \cdot k}{n}, \qquad k = 0, 1, 2, \ldots, n - 1.$$

▶ **Note** In the statement of the nth root theorem, if θ is in radians, then

$$\alpha = \frac{\theta + 2\pi k}{n}, \quad \text{or} \quad \alpha = \frac{\theta}{n} + \frac{2\pi k}{n}.$$

▶ EXAMPLE 2 FINDING COMPLEX ROOTS

Find the two square roots of $4i$. Write the roots in rectangular form.

Solution First write $4i$ in trigonometric form as

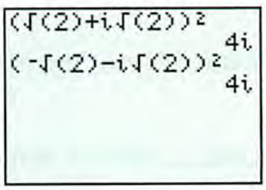
$$4i = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right).$$

Here $r = 4$ and $\theta = \frac{\pi}{2}$. The square roots have absolute value $\sqrt{4} = 2$ and arguments as follows.

$$\alpha = \frac{\frac{\pi}{2}}{2} + \frac{2\pi k}{2} = \frac{\pi}{4} + \pi k$$

Be careful simplifying here.

Since there are two square roots, let $k = 0$ and 1.



This screen confirms the result of Example 2.

If $k = 0$, then $\quad \alpha = \frac{\pi}{4} + \pi \cdot 0 = \frac{\pi}{4}.$

If $k = 1$, then $\quad \alpha = \frac{\pi}{4} + \pi \cdot 1 = \frac{5\pi}{4}.$

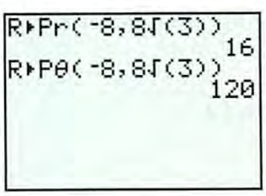
Using these values for α, the square roots are $2 \text{ cis } \frac{\pi}{4}$ and $2 \text{ cis } \frac{5\pi}{4}$, which can be written in rectangular form as $\sqrt{2} + i\sqrt{2}$ and $-\sqrt{2} - i\sqrt{2}$.

NOW TRY EXERCISE 17(a). ◀

▶ EXAMPLE 3 FINDING COMPLEX ROOTS

Find all fourth roots of $-8 + 8i\sqrt{3}$. Write the roots in rectangular form.

Solution $\quad -8 + 8i\sqrt{3} = 16 \text{ cis } 120°$ Write in trigonometric form.



Degree mode

This screen shows how a calculator finds r and θ for the number in Example 3.

Here $r = 16$ and $\theta = 120°$. The fourth roots of this number have absolute value $\sqrt[4]{16} = 2$ and arguments as follows.

$$\alpha = \frac{120°}{4} + \frac{360° \cdot k}{4} = 30° + 90° \cdot k$$

Since there are four fourth roots, let $k = 0, 1, 2,$ and 3.

If $k = 0$, then $\quad \alpha = 30° + 90° \cdot 0 = 30°.$

If $k = 1$, then $\quad \alpha = 30° + 90° \cdot 1 = 120°.$

If $k = 2$, then $\quad \alpha = 30° + 90° \cdot 2 = 210°.$

If $k = 3$, then $\quad \alpha = 30° + 90° \cdot 3 = 300°.$

Using these angles, the fourth roots are

$$2 \text{ cis } 30°, \quad 2 \text{ cis } 120°, \quad 2 \text{ cis } 210°, \quad \text{and} \quad 2 \text{ cis } 300°.$$

These four roots can be written in rectangular form as

$$\sqrt{3} + i, \quad -1 + i\sqrt{3}, \quad -\sqrt{3} - i, \quad \text{and} \quad 1 - i\sqrt{3}.$$

The graphs of these roots lie on a circle with center at the origin and radius 2. See Figure 12. The roots are equally spaced about the circle, 90° apart.

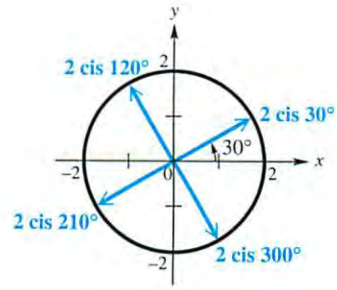


Figure 12

NOW TRY EXERCISES 23(a) AND (b).

▶ EXAMPLE 4 SOLVING AN EQUATION BY FINDING COMPLEX ROOTS

Find all complex number solutions of $x^5 - 1 = 0$. Graph them as vectors in the complex plane.

Solution Write the equation as

$$x^5 - 1 = 0, \qquad \text{or} \qquad x^5 = 1.$$

While there is only one real number solution, 1, there are five complex number solutions. To find these solutions, first write 1 in trigonometric form as

$$1 = 1 + 0i = 1(\cos 0° + i \sin 0°).$$

The absolute value of the fifth roots is $\sqrt[5]{1} = 1$, and the arguments are given by

$$0° + 72° \cdot k, \qquad k = 0, 1, 2, 3, \text{ and } 4.$$

By using these arguments, the fifth roots are

$$1(\cos 0° + i \sin 0°), \qquad k = 0$$
$$1(\cos 72° + i \sin 72°), \qquad k = 1$$
$$1(\cos 144° + i \sin 144°), \qquad k = 2$$
$$1(\cos 216° + i \sin 216°), \qquad k = 3$$

and

$$1(\cos 288° + i \sin 288°). \qquad k = 4$$

The solution set of the equation can be written as

$$\{\text{cis } 0°, \text{cis } 72°, \text{cis } 144°, \text{cis } 216°, \text{cis } 288°\}.$$

The first of these roots equals 1; the others cannot easily be expressed in rectangular form but can be approximated with a calculator.

The tips of the arrows representing the five fifth roots all lie on a unit circle and are equally spaced around it every 72°, as shown in Figure 13.

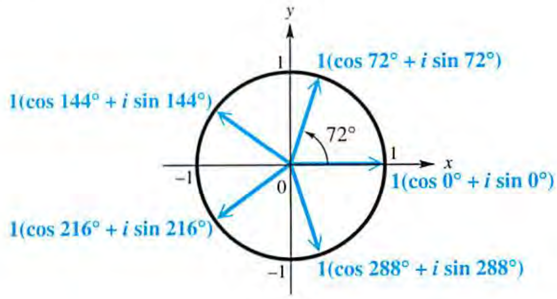


Figure 13

NOW TRY EXERCISE 35.

8.4 Exercises

Find each power. Write each answer in rectangular form. See Example 1.

1. $[3(\cos 30° + i \sin 30°)]^3$
2. $[2(\cos 135° + i \sin 135°)]^4$
3. $(\cos 45° + i \sin 45°)^8$
4. $[2(\cos 120° + i \sin 120°)]^3$
5. $[3 \text{ cis } 100°]^3$
6. $[3 \text{ cis } 40°]^3$
7. $(\sqrt{3} + i)^5$
8. $(2 - 2i\sqrt{3})^4$
9. $(2\sqrt{2} - 2i\sqrt{2})^6$
10. $\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)^8$
11. $(-2 - 2i)^5$
12. $(-1 + i)^7$

*In Exercises 13–24, **(a)** find all cube roots of each complex number. Leave answers in trigonometric form. **(b)** Graph each cube root as a vector in the complex plane. See Examples 2 and 3.*

13. $\cos 0° + i \sin 0°$
14. $\cos 90° + i \sin 90°$
15. $8 \text{ cis } 60°$
16. $27 \text{ cis } 300°$
17. $-8i$
18. $27i$
19. -64
20. 27
21. $1 + i\sqrt{3}$
22. $2 - 2i\sqrt{3}$
23. $-2\sqrt{3} + 2i$
24. $\sqrt{3} - i$

Find and graph all specified roots of 1.

25. second (square)
26. fourth
27. sixth

Find and graph all specified roots of i.

28. second (square)
29. third (cube)
30. fourth

Find all complex number solutions of each equation. Leave answers in trigonometric form. See Example 4.

31. $x^3 - 1 = 0$
32. $x^3 + 1 = 0$
33. $x^3 + i = 0$
34. $x^4 + i = 0$
35. $x^3 - 8 = 0$
36. $x^3 + 27 = 0$
37. $x^4 + 1 = 0$
38. $x^4 + 16 = 0$
39. $x^4 - i = 0$
40. $x^5 - i = 0$
41. $x^3 - (4 + 4i\sqrt{3}) = 0$
42. $x^4 - (8 + 8i\sqrt{3}) = 0$

43. Solve the equation $x^3 - 1 = 0$ by factoring the left side as the difference of two cubes and setting each factor equal to 0. Apply the quadratic formula as needed. Then compare your solutions to those of Exercise 31.

44. Solve the equation $x^3 + 27 = 0$ by factoring the left side as the sum of two cubes and setting each factor equal to 0. Apply the quadratic formula as needed. Then compare your solutions to those of Exercise 36.

RELATING CONCEPTS

For individual or collaborative investigation
(Exercises 45–48)

In ***Chapter 5*** *we derived identities, or formulas, for* $\cos 2\theta$ *and* $\sin 2\theta$*. These identities can also be derived using De Moivre's theorem.* ***Work Exercises 45–48 in order,*** *to see how this is done.*

45. De Moivre's theorem states that $(\cos \theta + i \sin \theta)^2 =$ ____________.

46. Expand the left side of the equation in Exercise 45 as a binomial and collect terms to write the left side in the form $a + bi$.

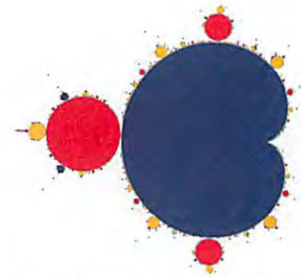
47. Use the result of Exercise 46 to obtain the double-angle formula for cosine.

48. Repeat Exercise 47, but find the double-angle formula for sine.

Solve each problem.

49. *Mandelbrot Set* The fractal called the **Mandelbrot set** is shown in the figure. To determine if a complex number $z = a + bi$ is in this set, perform the following sequence of calculations. Repeatedly compute

$$z, \quad z^2 + z, \quad (z^2 + z)^2 + z,$$
$$[(z^2 + z)^2 + z]^2 + z, \ldots .$$



Source: Figure from Crownover, R., *Introduction to Fractals and Chaos.* Copyright © 1995. Boston: Jones and Bartlett Publishers. Reprinted with permission.

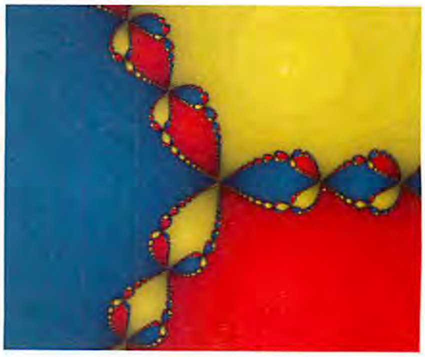
In a manner analogous to the Julia set, the complex number z does not belong to the Mandelbrot set if any of the resulting absolute values exceed 2. Otherwise z is in the set and the point (a, b) should be shaded in the graph. Determine whether or not the following numbers belong to the Mandelbrot set. (*Source:* Lauwerier, H., *Fractals,* Princeton University Press, 1991.)

(a) $z = 0 + 0i$ **(b)** $z = 1 - 1i$ **(c)** $z = -.5i$

50. *Basins of Attraction* The fractal shown in the figure is the solution to Cayley's problem of determining the basins of attraction for the cube roots of unity. The three cube roots of unity are

$$w_1 = 1, \quad w_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i,$$

and $$w_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$



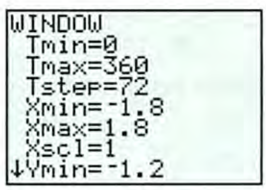
This fractal can be generated by repeatedly evaluating the function defined by

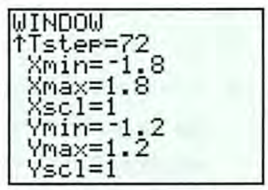
$$f(z) = \frac{2z^3 + 1}{3z^2},$$

where z is a complex number. One begins by picking $z_1 = a + bi$ and then successively computing $z_2 = f(z_1)$, $z_3 = f(z_2)$, $z_4 = f(z_3)$, ... If the resulting values of $f(z)$ approach w_1, color the pixel at (a, b) red. If it approaches w_2, color it blue, and if it approaches w_3, color it yellow. If this process continues for a large number of different z_1, the fractal in the figure will appear. Determine the appropriate color of the pixel for each value of z_1. (*Source:* Crownover, R., *Introduction to Fractals and Chaos,* Jones and Bartlett Publishers, 1995.)

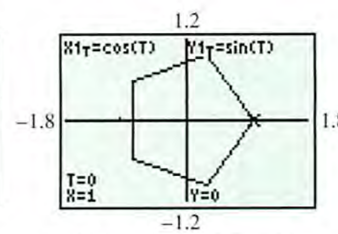
(a) $z_1 = i$ **(b)** $z_1 = 2 + i$ **(c)** $z_1 = -1 - i$

51. The screens here illustrate how a pentagon can be graphed using a graphing calculator. Note that a pentagon has five sides, and the T-step is $\frac{360}{5} = 72$. The display at the bottom of the graph screen indicates that one fifth root of 1 is $1 + 0i = 1$. Use this technique to find all fifth roots of 1, and express the real and imaginary parts in decimal form.





This is a continuation of the previous screen.



The calculator is in parametric, degree, and connected graph modes.

52. Use the method of Exercise 51 to find the first three of the ten 10th roots of 1.

53. One of the three cube roots of a complex number is $2 + 2i\sqrt{3}$. Determine the rectangular form of its other two cube roots.

Use a calculator to find all solutions of each equation in rectangular form.

54. $x^2 + 2 - i = 0$

55. $x^2 - 3 + 2i = 0$

56. $x^3 + 4 - 5i = 0$

57. $x^5 + 2 + 3i = 0$

58. ***Concept Check*** How many complex 64th roots does 1 have? How many are real? How many are not?

59. ***Concept Check*** *True* or *false*: Every real number must have two distinct real square roots.

60. ***Concept Check*** *True* or *false*: Some real numbers have three real cube roots.

61. Show that if z is an nth root of 1, then so is $\frac{1}{z}$.

62. Explain why a real number can have only one real cube root.

63. Explain why the n nth roots of 1 are equally spaced around the unit circle.

64. Refer to Figure 13. A regular pentagon can be created by joining the tips of the arrows. Explain how you can use this principle to create a regular octagon.

CHAPTER 8 ▶ Quiz (Sections 8.1–8.4)

1. Multiply or divide as indicated. Simplify each answer.

(a) $\sqrt{-24} \cdot \sqrt{-3}$ **(b)** $\dfrac{\sqrt{-8}}{\sqrt{72}}$

2. For the complex numbers

$$w = 3 + 5i \quad \text{and} \quad z = -4 + i,$$

find each of the following in rectangular form.

(a) $w + z$ (and give a geometric representation)

(b) $w - z$ **(c)** wz **(d)** $\dfrac{w}{z}$

3. Express each of the following in rectangular form.

(a) $(1 - i)^3$ **(b)** i^{33}

4. Solve $3x^2 - x + 4 = 0$ over the complex number system.

5. Write each complex number in trigonometric (polar) form, where $0° \le \theta < 360°$.

(a) $-4i$ **(b)** $1 - i\sqrt{3}$ **(c)** $-3 - i$

6. Write each complex number in rectangular form.

(a) $4(\cos 60° + i \sin 60°)$ **(b)** $5 \text{ cis } 130°$

(c) $7(\cos 270° + i \sin 270°)$

7. For the complex numbers

$$w = 12(\cos 80° + i \sin 80°) \quad \text{and} \quad z = 3(\cos 50° + i \sin 50°),$$

find each of the following in the form specified.

(a) wz (trigonometric form) **(b)** $\dfrac{w}{z}$ (rectangular form)

(c) z^3 (rectangular form)

8. Find the four complex fourth roots of -16. Express them in both trigonometric and rectangular forms.

8.5 Polar Equations and Graphs

Polar Coordinate System ▪ Graphs of Polar Equations ▪ Converting from Polar to Rectangular Equations ▪ Classifying Polar Equations

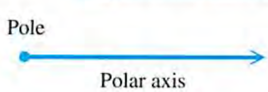


Figure 14

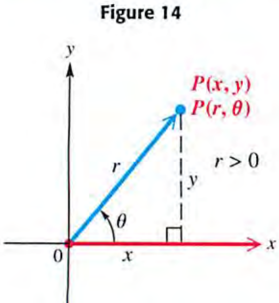


Figure 15

Polar Coordinate System

We have been using the rectangular coordinate system to graph equations. The **polar coordinate system** is based on a point, called the **pole,** and a ray, called the **polar axis.** The polar axis is usually drawn in the direction of the positive x-axis, as shown in Figure 14.

In Figure 15, the pole has been placed at the origin of a rectangular coordinate system so that the polar axis coincides with the positive x-axis. Point P has rectangular coordinates (x, y). Point P can also be located by giving the directed angle θ from the positive x-axis to ray OP and the *directed* distance r from the pole to point P. The ordered pair (r, θ) gives the **polar coordinates** of point P. If $r > 0$ then point P lies on the terminal side of θ, and if $r < 0$ then point P lies on the ray pointing in the opposite direction of the terminal side of θ, a distance $|r|$ from the pole. See Figure 16 on the next page.

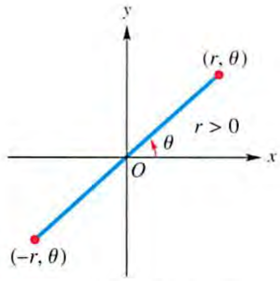


Figure 16

Using Figure 15 and trigonometry, we can establish the following relationships between rectangular and polar coordinates.

RECTANGULAR AND POLAR COORDINATES

If a point has rectangular coordinates (x, y) and polar coordinates (r, θ), then these coordinates are related as follows.

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}, \quad \text{if } x \neq 0$$

▶ EXAMPLE 1 PLOTTING POINTS WITH POLAR COORDINATES

Plot each point by hand in the polar coordinate system. Then, determine the rectangular coordinates of each point.

(a) $P(2, 30°)$ **(b)** $Q\left(-4, \frac{2\pi}{3}\right)$ **(c)** $R\left(5, -\frac{\pi}{4}\right)$

Solution

(a) In this case, $r = 2$ and $\theta = 30°$, so the point P is located 2 units from the origin in the positive direction on a ray making a 30° angle with the polar axis, as shown in Figure 17 below.

Using the conversion equations, we find the rectangular coordinates as follows.

$x = r\cos\theta$	$y = r\sin\theta$
$x = 2\cos 30°$	$y = 2\sin 30°$
$x = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$	$y = 2\left(\frac{1}{2}\right) = 1$ (Section 2.1)

The rectangular coordinates are $(\sqrt{3}, 1)$.

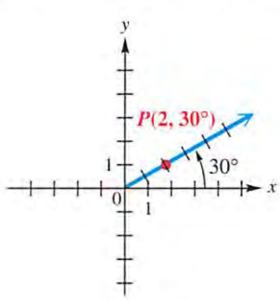


Figure 17

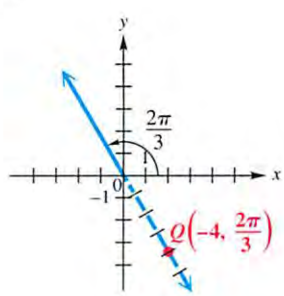


Figure 18

(b) Since r is *negative*, Q is 4 units in the *opposite* direction from the pole on an extension of the $\frac{2\pi}{3}$ ray. See Figure 18. The rectangular coordinates are

$$x = -4\cos\frac{2\pi}{3} = -4\left(-\frac{1}{2}\right) = 2$$

and

$$y = -4\sin\frac{2\pi}{3} = -4\left(\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}.$$

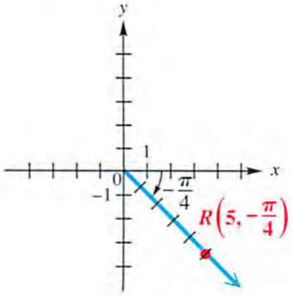


Figure 19

(c) Point R is shown in Figure 19. Since θ is negative, the angle is measured in the clockwise direction. Furthermore, we have

$$x = 5\cos\left(-\frac{\pi}{4}\right) = \frac{5\sqrt{2}}{2} \quad \text{and} \quad y = 5\sin\left(-\frac{\pi}{4}\right) = -\frac{5\sqrt{2}}{2}.$$

NOW TRY EXERCISES 3(a), (c), 5(a), (c), AND 11(a), (c).

While a given point in the plane can have only one pair of rectangular coordinates, this same point can have an infinite number of pairs of polar coordinates. For example, $(2, 30°)$ locates the same point as

$$(2, 390°), \quad (2, -330°), \quad \text{and} \quad (-2, 210°).$$

LOOKING AHEAD TO CALCULUS

Techniques studied in calculus associated with derivatives and integrals provide methods of finding slopes of tangent lines to polar curves, areas bounded by such curves, and lengths of their arcs.

EXAMPLE 2 GIVING ALTERNATIVE FORMS FOR COORDINATES OF A POINT

(a) Give three other pairs of polar coordinates for the point $P(3, 140°)$.

(b) Determine two pairs of polar coordinates for the point with rectangular coordinates $(-1, 1)$.

Solution

(a) Three pairs that could be used for the point are $(3, -220°)$, $(-3, 320°)$, and $(-3, -40°)$. See Figure 20.

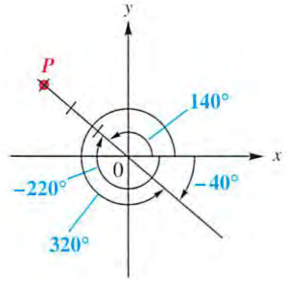


Figure 20

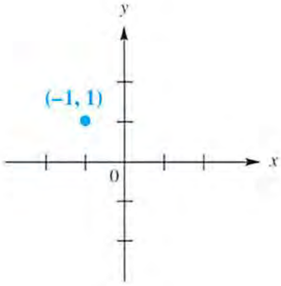


Figure 21

(b) As shown in Figure 21, the point $(-1, 1)$ lies in the second quadrant. Since $\tan\theta = \frac{1}{-1} = -1$, one possible value for θ is $135°$. Also,

$$r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + 1^2} = \sqrt{2}.$$

Two pairs of polar coordinates are $\left(\sqrt{2}, 135°\right)$ and $\left(-\sqrt{2}, 315°\right)$.

NOW TRY EXERCISES 3(b), 5(b), 11(b), AND 13.

Graphs of Polar Equations Equations in x and y are called **rectangular** (or **Cartesian**) **equations.** An equation in which r and θ are the variables is a **polar equation.**

$$r = 3 \sin \theta, \qquad r = 2 + \cos \theta, \qquad r = \theta \quad \text{Polar equations}$$

While the rectangular forms of lines and circles are the ones most often encountered, they can also be defined in terms of polar coordinates. The polar equation of the line $ax + by = c$ can be derived as follows.

$$ax + by = c \quad \text{Rectangular equation}$$

$$a(r \cos \theta) + b(r \sin \theta) = c \quad \text{Convert from rectangular to polar coordinates.}$$

$$r(a \cos \theta + b \sin \theta) = c \quad \text{Factor out } r.$$

This is the polar equation of $ax + by = c$.

$$\boldsymbol{r = \frac{c}{a \cos \theta + b \sin \theta}}$$

For the circle $x^2 + y^2 = a^2$, we have

$$x^2 + y^2 = a^2$$

$$\left(\sqrt{x^2 + y^2}\right)^2 = a^2 \qquad \left(\sqrt{k}\right)^2 = k$$

$$r^2 = a^2 \qquad \sqrt{x^2 + y^2} = r$$

These are polar equations of $x^2 + y^2 = a^2$.

$$\boldsymbol{r = \pm a.} \qquad r \text{ can be negative in polar coordinates.}$$

▶ EXAMPLE 3 **EXAMINING POLAR AND RECTANGULAR EQUATIONS OF LINES AND CIRCLES**

For each rectangular equation, give the equivalent polar equation and sketch its graph.

(a) $y = x - 3$ **(b)** $x^2 + y^2 = 4$

Solution

(a) This is the equation of a line. Rewrite $y = x - 3$ as

$$x - y = 3. \quad \text{Standard form } ax + by = c$$

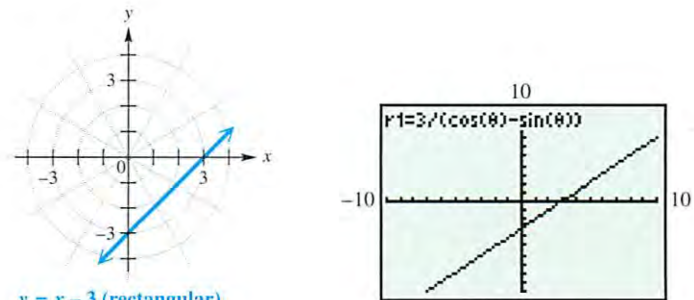
Using the general form for the polar equation of a line,

$$r = \frac{c}{a \cos \theta + b \sin \theta}$$

$$r = \frac{3}{1 \cdot \cos \theta + (-1) \cdot \sin \theta} \qquad a = 1, b = -1, c = 3$$

$$r = \frac{3}{\cos \theta - \sin \theta}.$$

A traditional graph is shown in Figure 22(a), and a calculator graph is shown in Figure 22(b).



$y = x - 3$ (rectangular)
$r = \dfrac{3}{\cos\theta - \sin\theta}$ (polar)
(a)

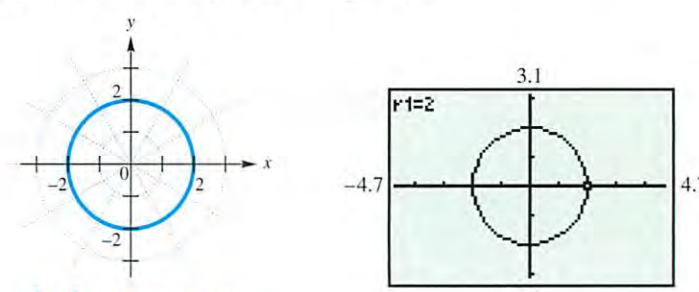
Polar graphing mode
(b)

Figure 22

(b) The graph of $x^2 + y^2 = 4$ is a circle with center at the origin and radius 2.

$$x^2 + y^2 = 4 \qquad \text{(Appendix B)}$$
$$\left(\sqrt{x^2 + y^2}\right)^2 = (\pm 2)^2$$
$$\sqrt{x^2 + y^2} = 2 \quad \text{or} \quad \sqrt{x^2 + y^2} = -2$$
$$r = 2 \quad \text{or} \quad r = -2$$

(In polar coordinates, we may have $r < 0$.) The graphs of $r = 2$ and $r = -2$ coincide. See the graphs in Figure 23.



$x^2 + y^2 = 4$ (rectangular)
$r = 2$ (polar)
(a)

Polar graphing mode
(b)

Figure 23

NOW TRY EXERCISES 23 AND 25.

To graph polar equations, evaluate r for various values of θ until a pattern appears, and then join the points with a smooth curve.

A graphing calculator can be used to graph an equation in the form $r = f(\theta)$. (See the screens in Example 3.) Refer to your owner's manual to see how your model handles polar graphs. As always, you must set an appropriate window and choose the correct angle mode (radians or degrees). You will need to decide on maximum and minimum values of θ. Keep in mind the periods of the functions, so a complete set of ordered pairs is generated. ■

▶ EXAMPLE 4 GRAPHING A POLAR EQUATION (CARDIOID)

Graph $r = 1 + \cos \theta$.

Algebraic Solution

To graph this equation, find some ordered pairs (as in the table). Once the pattern of values of r becomes clear, it is not necessary to find more ordered pairs. That is why we stopped with the ordered pair $(1.9, 330°)$ in the table.

θ	$\cos \theta$	$r = 1 + \cos \theta$	θ	$\cos \theta$	$r = 1 + \cos \theta$
0°	1	2	135°	−.7	.3
30°	.9	1.9	150°	−.9	.1
45°	.7	1.7	180°	−1	0
60°	.5	1.5	270°	0	1
90°	0	1	315°	.7	1.7
120°	−.5	.5	330°	.9	1.9

Connect the points in order—from $(2, 0°)$ to $(1.9, 30°)$ to $(1.7, 45°)$ and so on. See Figure 24. This curve is called a **cardioid** because of its heart shape. The curve has been graphed on a **polar grid.**

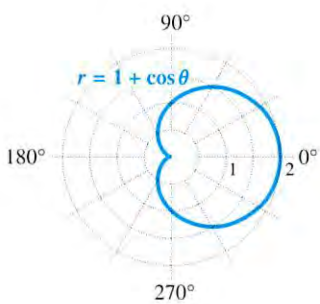


Figure 24

Graphing Calculator Solution

We choose degree mode and graph values of θ in the interval $[0°, 360°]$. The screens in Figure 25(a) show the choices needed to generate the graph in Figure 25(b).

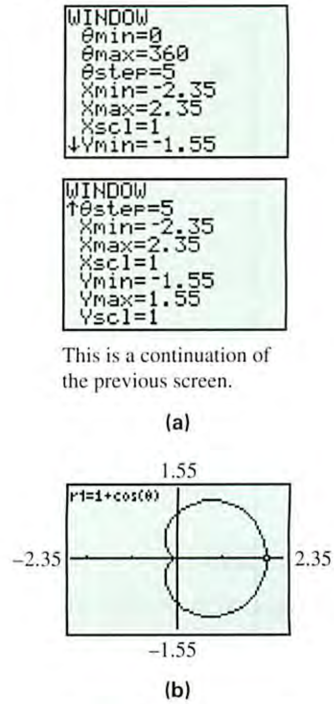


Figure 25

NOW TRY EXERCISE 41. ◀

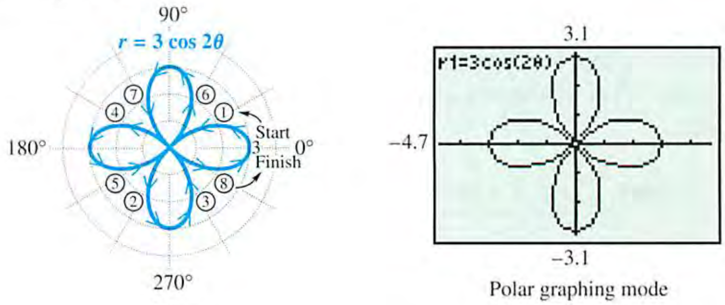
▶ EXAMPLE 5 GRAPHING A POLAR EQUATION (ROSE)

Graph $r = 3 \cos 2\theta$.

Solution Because of the argument 2θ, the graph requires a larger number of points than when the argument is just θ. A few ordered pairs are given in the table. You should complete the table similarly through the first 180° so that 2θ has values up to 360°.

θ	0°	15°	30°	45°	60°	75°	90°
2θ	0°	30°	60°	90°	120°	150°	180°
$\cos 2\theta$	1	.9	.5	0	−.5	−.9	−1
$r = 3 \cos 2\theta$	3	2.6	1.5	0	−1.5	−2.6	−3

Plotting the points from the table in order gives the graph, called a **four-leaved rose.** Notice in Figure 26(a) on the next page how the graph is developed with a continuous curve, beginning with the upper half of the right horizontal leaf and ending with the lower half of that leaf. As the graph is traced, the curve goes through the pole four times. This can actually be seen as a calculator graphs the curve. See Figure 26(b).



(a) (b)

Figure 26

NOW TRY EXERCISE 45.

The equation $r = 3 \cos 2\theta$ in Example 5 has a graph that belongs to a family of curves called **roses.** The graphs of $\boldsymbol{r = a \sin n\theta}$ and $\boldsymbol{r = a \cos n\theta}$ are roses, with n leaves if n is odd, and $2n$ leaves if n is even. The value of a determines the length of the leaves.

▶ EXAMPLE 6 GRAPHING A POLAR EQUATION (LEMNISCATE)

Graph $r^2 = \cos 2\theta$.

Algebraic Solution

Complete a table of ordered pairs, and sketch the graph, as in Figure 27. The point $(-1, 0°)$, with r negative, may be plotted as $(1, 180°)$. Also, $(-.7, 30°)$ may be plotted as $(.7, 210°)$, and so on.

Values of θ for $45° < \theta < 135°$ are not included in the table because the corresponding values of $\cos 2\theta$ are negative (quadrants II and III) and so do not have real square roots. Values of θ larger than 180° give 2θ larger than 360° and would repeat the points already found. This curve is called a **lemniscate.**

$\boldsymbol{\theta}$	0°	30°	45°	135°	150°	180°
$\boldsymbol{2\theta}$	0°	60°	90°	270°	300°	360°
$\boldsymbol{\cos 2\theta}$	1	.5	0	0	.5	1
$\boldsymbol{r = \pm\sqrt{\cos 2\theta}}$	± 1	$\pm .7$	0	0	$\pm .7$	± 1

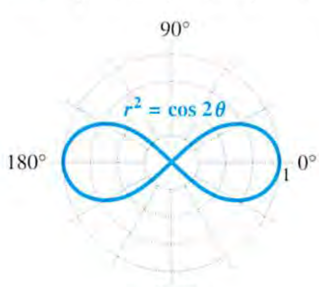


Figure 27

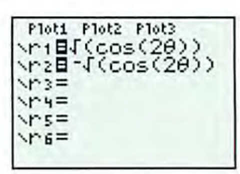
Graphing Calculator Solution

To graph $r^2 = \cos 2\theta$ with a graphing calculator, let

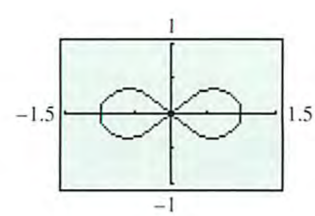
$$r_1 = \sqrt{\cos 2\theta}$$

and $$r_2 = -\sqrt{\cos 2\theta}.$$

See Figures 28(a) and (b).



(a)



(b)

Figure 28

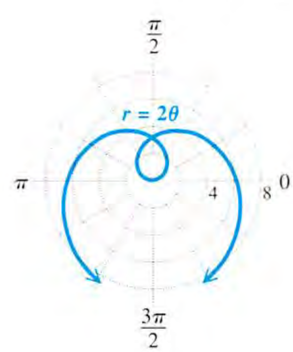
NOW TRY EXERCISE 47.

▶ EXAMPLE 7 **GRAPHING A POLAR EQUATION (SPIRAL OF ARCHIMEDES)**

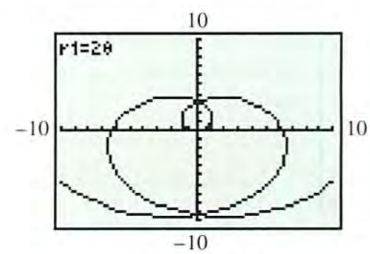
Graph $r = 2\theta$ (θ measured in radians).

Solution Some ordered pairs are shown in the table. Since $r = 2\theta$, rather than a trigonometric function of θ, we must also consider negative values of θ. Radian measures have been rounded. The graph in Figure 29(a), is called a **spiral of Archimedes.** Figure 29(b) shows a calculator graph of this spiral.

θ (radians)	$r = 2\theta$	θ (radians)	$r = 2\theta$
$-\pi$	-6.3	$\frac{\pi}{3}$	2.1
$-\frac{\pi}{2}$	-3.1	$\frac{\pi}{2}$	3.1
$-\frac{\pi}{4}$	-1.6	π	6.3
0	0	$\frac{3\pi}{2}$	9.4
$\frac{\pi}{6}$	1	2π	12.6



(a)



$-2\pi \le \theta \le 2\pi$

More of the spiral can be seen in this calculator graph.

(b)

Figure 29

NOW TRY EXERCISE 63. ◀

Converting from Polar to Rectangular Equations

▶ EXAMPLE 8 **CONVERTING A POLAR EQUATION TO A RECTANGULAR EQUATION**

Convert the equation $r = \frac{4}{1 + \sin\theta}$ to rectangular coordinates, and graph.

Solution

$$r = \frac{4}{1 + \sin\theta} \quad \text{Polar equation}$$

$$r + r\sin\theta = 4 \quad \text{Multiply by } 1 + \sin\theta.$$

$$\sqrt{x^2 + y^2} + y = 4 \quad \text{Let } r = \sqrt{x^2 + y^2} \text{ and } r\sin\theta = y.$$

$$\sqrt{x^2 + y^2} = 4 - y \quad \text{Subtract } y.$$

$$x^2 + y^2 = (4 - y)^2 \quad \text{Square each side.}$$

$$x^2 + y^2 = 16 - 8y + y^2 \quad \text{Expand the right side.}$$

$$x^2 = -8y + 16 \quad \text{Subtract } y^2.$$

$$x^2 = -8(y - 2) \quad \text{Rectangular equation}$$

The final equation represents a parabola and is graphed in Figure 30.

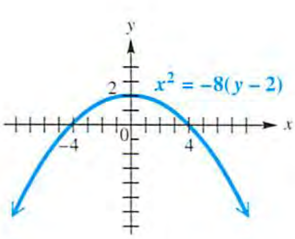
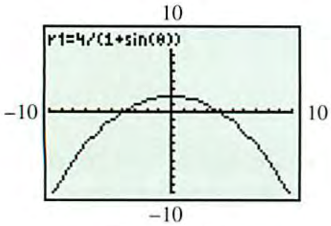


Figure 30

NOW TRY EXERCISE 55. ◀



$0° \le \theta \le 360°$

Figure 31

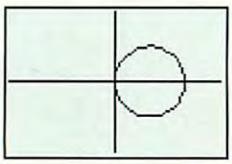
The conversion in Example 8 is not necessary when using a graphing calculator. Figure 31 shows the graph of $r = \frac{4}{1 + \sin \theta}$, graphed directly with the calculator in polar mode. ■

Classifying Polar Equations

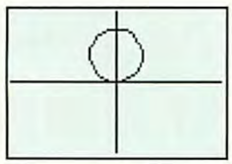
The following table summarizes some common polar graphs and forms of their equations. (In addition to circles, lemniscates, and roses, we include **limaçons.** Cardioids are a special case of limaçons, where $\left|\frac{a}{b}\right| = 1$.)

Circles and Lemniscates

Circles

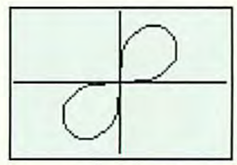


$r = a \cos \theta$

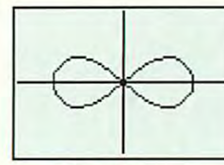


$r = a \sin \theta$

Lemniscates



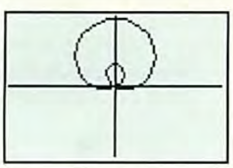
$r^2 = a^2 \sin 2\theta$



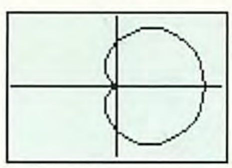
$r^2 = a^2 \cos 2\theta$

Limaçons

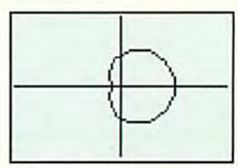
$r = a \pm b \sin \theta$ or $r = a \pm b \cos \theta$



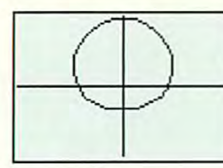
$\frac{a}{b} < 1$



$\frac{a}{b} = 1$



$1 < \frac{a}{b} < 2$

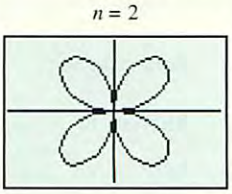


$\frac{a}{b} \ge 2$

Rose Curves

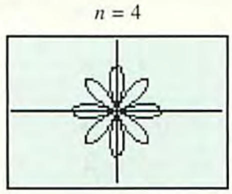
$2n$ leaves if n is even, $n \ge 2$

$n = 2$



$r = a \sin n\theta$

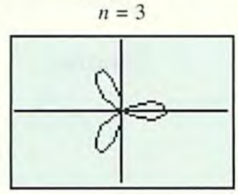
$n = 4$



$r = a \cos n\theta$

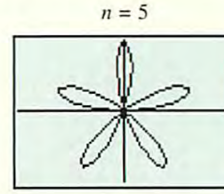
n leaves if n is odd

$n = 3$



$r = a \cos n\theta$

$n = 5$



$r = a \sin n\theta$

▶ **Note** Some other polar curves are the **cissoid, kappa curve, conchoid, trisectrix, cruciform, strophoid,** and **lituus.** Refer to older textbooks on analytic geometry to investigate them.

8.5 Exercises

1. *Concept Check* For each point given in polar coordinates, state the quadrant in which the point lies if it is graphed in a rectangular coordinate system.
   (a) $(5, 135^\circ)$ **(b)** $(2, 60^\circ)$ **(c)** $(6, -30^\circ)$ **(d)** $(4.6, 213^\circ)$

2. *Concept Check* For each point given in polar coordinates, state the axis on which the point lies if it is graphed in a rectangular coordinate system. Also state whether it is on the positive portion or the negative portion of the axis. (For example, $(5, 0^\circ)$ lies on the positive x-axis.)
   (a) $(7, 360^\circ)$ **(b)** $(4, 180^\circ)$ **(c)** $(2, -90^\circ)$ **(d)** $(8, 450^\circ)$

For each pair of polar coordinates, **(a)** *plot the point,* **(b)** *give two other pairs of polar coordinates for the point, and* **(c)** *give the rectangular coordinates for the point. See Examples 1 and 2.*

3. $(1, 45^\circ)$ **4.** $(3, 120^\circ)$ **5.** $(-2, 135^\circ)$ **6.** $(-4, 30^\circ)$
7. $(5, -60^\circ)$ **8.** $(2, -45^\circ)$ **9.** $(-3, -210^\circ)$ **10.** $(-1, -120^\circ)$
11. $\left(3, \frac{5\pi}{3}\right)$ **12.** $\left(4, \frac{3\pi}{2}\right)$

For each pair of rectangular coordinates, **(a)** *plot the point and* **(b)** *give two pairs of polar coordinates for the point, where* $0^\circ \le \theta < 360^\circ$. *See Example 2(b).*

13. $(1, -1)$ **14.** $(1, 1)$ **15.** $(0, 3)$ **16.** $(0, -3)$
17. $(\sqrt{2}, \sqrt{2})$ **18.** $(-\sqrt{2}, \sqrt{2})$ **19.** $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$ **20.** $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
21. $(3, 0)$ **22.** $(-2, 0)$

For each rectangular equation, give its equivalent polar equation and sketch its graph. See Example 3.

23. $x - y = 4$ **24.** $x + y = -7$ **25.** $x^2 + y^2 = 16$
26. $x^2 + y^2 = 9$ **27.** $2x + y = 5$ **28.** $3x - 2y = 6$

RELATING CONCEPTS

For individual or collaborative investigation
(Exercises 29–36)

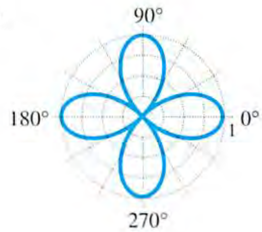
In rectangular coordinates, the graph of $ax + by = c$ *is a horizontal line if* $a = 0$ *or a vertical line if* $b = 0$. ***Work Exercises 29–36 in order,*** *to determine the general forms of polar equations for horizontal and vertical lines.*

29. Begin with the equation $y = k$, whose graph is a horizontal line. Make a trigonometric substitution for y using r and θ.
30. Solve the equation in Exercise 29 for r.
31. Rewrite the equation in Exercise 30 using the appropriate reciprocal function.
32. Sketch the graph of $r = 3 \csc \theta$. What is the corresponding rectangular equation?
33. Begin with the equation $x = k$, whose graph is a vertical line. Make a trigonometric substitution for x using r and θ.
34. Solve the equation in Exercise 33 for r.
35. Rewrite the equation in Exercise 34 using the appropriate reciprocal function.
36. Sketch the graph of $r = 3 \sec \theta$. What is the corresponding rectangular equation?

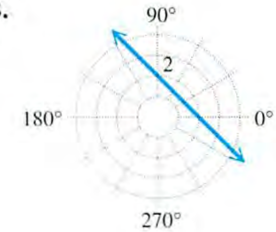
Concept Check *In Exercises 37–40, match each equation with its polar graph from choices A–D.*

37. $r = 3$ **38.** $r = \cos 3\theta$ **39.** $r = \cos 2\theta$ **40.** $r = \dfrac{2}{\cos\theta + \sin\theta}$

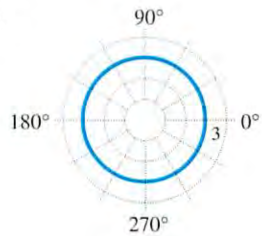
A.



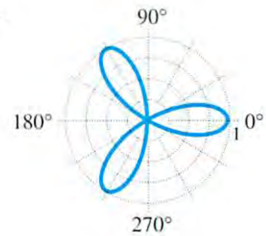
B.



C.



D.



Give a complete graph of each polar equation. In Exercises 41–50, also identify the type of polar graph. See Examples 4–6.

41. $r = 2 + 2\cos\theta$

42. $r = 8 + 6\cos\theta$

43. $r = 3 + \cos\theta$

44. $r = 2 - \cos\theta$

45. $r = 4\cos 2\theta$

46. $r = 3\cos 5\theta$

47. $r^2 = 4\cos 2\theta$

48. $r^2 = 4\sin 2\theta$

49. $r = 4 - 4\cos\theta$

50. $r = 6 - 3\cos\theta$

51. $r = 2\sin\theta\tan\theta$
(This is a **cissoid.**)

52. $r = \dfrac{\cos 2\theta}{\cos\theta}$
(This is a **cissoid with a loop.**)

For each equation, find an equivalent equation in rectangular coordinates and graph. See Example 8.

53. $r = 2\sin\theta$

54. $r = 2\cos\theta$

55. $r = \dfrac{2}{1 - \cos\theta}$

56. $r = \dfrac{3}{1 - \sin\theta}$

57. $r = -2\cos\theta - 2\sin\theta$

58. $r = \dfrac{3}{4\cos\theta - \sin\theta}$

59. $r = 2\sec\theta$

60. $r = -5\csc\theta$

61. $r = \dfrac{2}{\cos\theta + \sin\theta}$

62. $r = \dfrac{2}{2\cos\theta + \sin\theta}$

63. Graph $r = \theta$, a spiral of Archimedes. (See Example 7.) Use both positive and nonpositive values for θ.

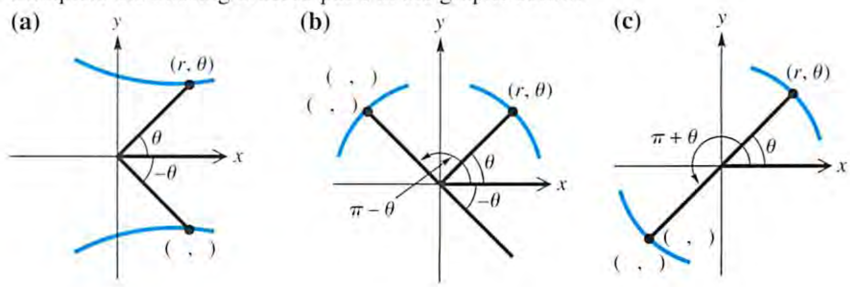
64. Use a graphing calculator window of $[-1250, 1250]$ by $[-1250, 1250]$, in degree mode, to graph more of $r = 2\theta$ (a spiral of Archimedes) than what is shown in Figure 29. Use $-1250° \le \theta \le 1250°$.

65. Find the polar equation of the line that passes through the points $(1, 0°)$ and $(2, 90°)$.

66. Explain how to plot a point (r, θ) in polar coordinates, if $r < 0$.

Concept Check *The polar graphs in this section exhibit symmetry. (See* **Appendix D.**) *Visualize an xy-plane superimposed on the polar coordinate system, with the pole at the origin and the polar axis on the positive x-axis. Then a polar graph may be symmetric with respect to the x-axis (the polar axis), the y-axis (the line $\theta = \frac{\pi}{2}$), or the origin (the pole). Use this information to work Exercises 67 and 68.*

67. Complete the missing ordered pairs in the graphs below.



68. Based on your results in Exercise 67, fill in the blanks with the correct responses.

(a) The graph of $r = f(\theta)$ is symmetric with respect to the polar axis if substitution of __________ for θ leads to an equivalent equation.

(b) The graph of $r = f(\theta)$ is symmetric with respect to the vertical line $\theta = \frac{\pi}{2}$ if substitution of __________ for θ leads to an equivalent equation.

(c) Alternatively, the graph of $r = f(\theta)$ is symmetric with respect to the vertical line $\theta = \frac{\pi}{2}$ if substitution of __________ for r and __________ for θ leads to an equivalent equation.

(d) The graph of $r = f(\theta)$ is symmetric with respect to the pole if substitution of __________ for r leads to an equivalent equation.

(e) Alternatively, the graph of $r = f(\theta)$ is symmetric with respect to the pole if substitution of __________ for θ leads to an equivalent equation.

(f) In general, the completed statements in parts (a)–(e) mean that the graphs of polar equations of the form $r = a \pm b \cos \theta$ (where a may be 0) are symmetric with respect to __________.

(g) In general, the completed statements in parts (a)–(e) mean that the graphs of polar equations of the form $r = a \pm b \sin \theta$ (where a may be 0) are symmetric with respect to __________.

The graph of $r = a\theta$ in polar coordinates is an example of the spiral of Archimedes. With your calculator set to radian mode, use the given value of a and interval of θ to graph the spiral in the window specified.

69. $a = 1, 0 \le \theta \le 4\pi, [-15, 15]$ by $[-15, 15]$

70. $a = 2, -4\pi \le \theta \le 4\pi, [-30, 30]$ by $[-30, 30]$

71. $a = 1.5, -4\pi \le \theta \le 4\pi, [-20, 20]$ by $[-20, 20]$

72. $a = -1, 0 \le \theta \le 12\pi, [-40, 40]$ by $[-40, 40]$

Find the polar coordinates of the points of intersection of the given curves for the specified interval of θ.

73. $r = 4 \sin \theta, r = 1 + 2 \sin \theta; 0 \le \theta < 2\pi$

74. $r = 3, r = 2 + 2 \cos \theta; 0° \le \theta < 360°$

75. $r = 2 + \sin \theta, r = 2 + \cos \theta; 0 \le \theta < 2\pi$

76. $r = \sin 2\theta, r = \sqrt{2} \cos \theta; 0 \le \theta < \pi$

(Modeling) *Solve each problem.*

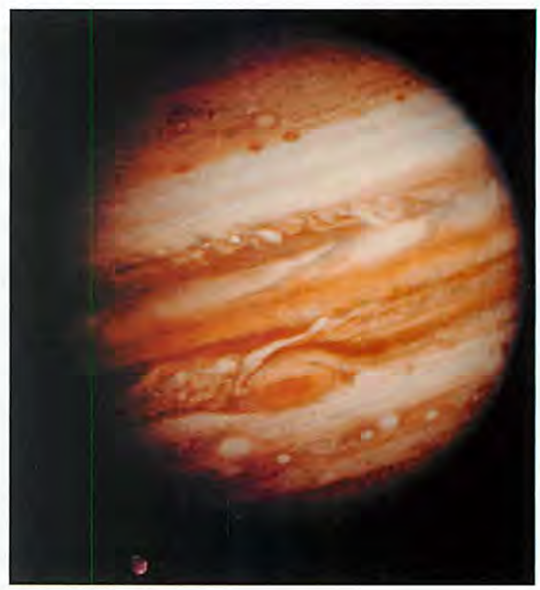
77. *Orbits of Satellites* The polar equation

$$r = \frac{a(1 - e^2)}{1 + e \cos \theta}$$

can be used to graph the orbits of the satellites of our sun, where a is the average distance in astronomical units from the sun and e is a constant called the **eccentricity**. The sun will be located at the pole. The table lists the values of a and e.

Satellite	a	e
Mercury	.39	.206
Venus	.78	.007
Earth	1.00	.017
Mars	1.52	.093
Jupiter	5.20	.048
Saturn	9.54	.056
Uranus	19.20	.047
Neptune	30.10	.009
Pluto	39.40	.249

Source: Karttunen, H., P. Kröger, H. Oja, M. Putannen, and K. Donners (Editors), *Fundamental Astronomy, 4th edition,* Springer-Verlag, 2003; Zeilik, M., S. Gregory, and E. Smith, *Introductory Astronomy and Astrophysics,* Saunders College Publishers, 1992.



(a) Graph the orbits of the four closest satellites on the same polar grid. Choose a viewing window that results in a graph with nearly circular orbits.

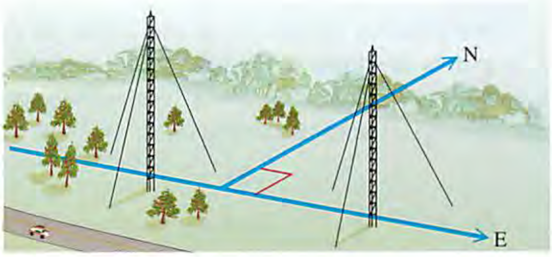
(b) Plot the orbits of Earth, Jupiter, Uranus, and Pluto on the same polar grid. How does Earth's distance from the sun compare to the others' distances from the sun?

(c) Use graphing to determine whether or not Pluto is always farthest from the sun.

78. *Radio Towers and Broadcasting Patterns* Many times radio stations do not broadcast in all directions with the same intensity. To avoid interference with an existing station to the north, a new station may be licensed to broadcast only east and west. To create an east-west signal, two radio towers are sometimes used, as illustrated in the figure. Locations where the radio signal is received correspond to the interior of the curve defined by

$$r^2 = 40{,}000 \cos 2\theta,$$

where the polar axis (or positive x-axis) points east.



(a) Graph $r^2 = 40{,}000 \cos 2\theta$ for $0° \leq \theta \leq 360°$, where units are in miles. Assuming the radio towers are located near the pole, use the graph to describe the regions where the signal can be received and where the signal cannot be received.

(b) Suppose a radio signal pattern is given by $r^2 = 22{,}500 \sin 2\theta$. Graph this pattern and interpret the results.

8.6 Parametric Equations, Graphs, and Applications

Basic Concepts ▪ Parametric Graphs and Their Rectangular Equivalents ▪ The Cycloid ▪ Applications of Parametric Equations

Basic Concepts Throughout this text, we have graphed sets of ordered pairs of real numbers that correspond to a function of the form $y = f(x)$ or $r = g(\theta)$. Another way to determine a set of ordered pairs involves two functions f and g defined by $x = f(t)$ and $y = g(t)$, where t is a real number in some interval I. Each value of t leads to a corresponding x-value and a corresponding y-value, and thus to an ordered pair (x, y).

PARAMETRIC EQUATIONS OF A PLANE CURVE

A **plane curve** is a set of points (x, y) such that $x = f(t)$, $y = g(t)$, and f and g are both defined on an interval I. The equations $x = f(t)$ and $y = g(t)$ are **parametric equations** with **parameter t.**

Graphing calculators are capable of graphing plane curves defined by parametric equations. The calculator must be set in parametric mode, and the window requires intervals for the parameter t, as well as for x and y. ■

Parametric Graphs and Their Rectangular Equivalents

▶ EXAMPLE 1 GRAPHING A PLANE CURVE DEFINED PARAMETRICALLY

Let $x = t^2$ and $y = 2t + 3$, for t in $[-3, 3]$. Graph the set of ordered pairs (x, y).

Algebraic Solution

Make a table of corresponding values of t, x, and y over the domain of t. Then plot the points as shown in Figure 32. The graph is a portion of a parabola with horizontal axis $y = 3$. The arrowheads indicate the direction the curve traces as t increases.

t	x	y
−3	9	−3
−2	4	−1
−1	1	1
0	0	3
1	1	5
2	4	7
3	9	9

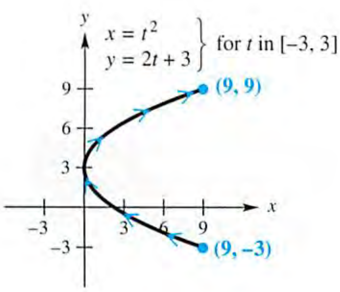
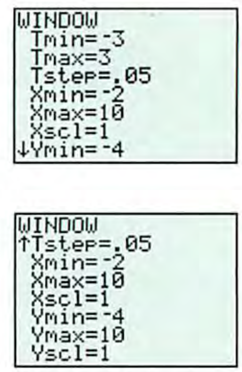


Figure 32

Graphing Calculator Solution

We set the parameters of the TI-83/84 Plus as shown in the top two screens to obtain the bottom screen in Figure 33.



This is a continuation of the previous screen.

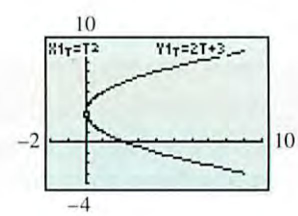


Figure 33

NOW TRY EXERCISE 5(a). ◀

▶ EXAMPLE 2 FINDING AN EQUIVALENT RECTANGULAR EQUATION

Find a rectangular equation for the plane curve of Example 1 defined as follows:

$$x = t^2, \quad y = 2t + 3, \qquad \text{for } t \text{ in } [-3, 3].$$

Solution To eliminate the parameter t, solve either equation for t. Here, only the second equation, $y = 2t + 3$, leads to a unique solution for t, so we choose it.

$$y = 2t + 3 \qquad \text{Solve for } t. \text{ (Appendix A)}$$
$$2t = y - 3$$
$$t = \frac{y - 3}{2}$$

Now substitute this result into the first equation to get

$$x = t^2 = \left(\frac{y - 3}{2}\right)^2 = \frac{(y - 3)^2}{4}, \qquad \text{or} \qquad 4x = (y - 3)^2.$$

This is the equation of a horizontal parabola opening to the right, which agrees with the graph given in Figure 32. Because t is in $[-3, 3]$, x is in $[0, 9]$ and y is in $[-3, 9]$. The rectangular equation must be given with its restricted domain as

$$4x = (y - 3)^2, \qquad \text{for } x \text{ in } [0, 9].$$

NOW TRY EXERCISE 5(b).

▶ EXAMPLE 3 GRAPHING A PLANE CURVE DEFINED PARAMETRICALLY

Graph the plane curve defined by $x = 2 \sin t$, $y = 3 \cos t$, for t in $[0, 2\pi]$.

Solution To convert to a rectangular equation, it is not productive here to solve either equation for t. Instead, we use the fact that $\sin^2 t + \cos^2 t = 1$ to apply another approach. Square both sides of each equation; solve one for $\sin^2 t$, the other for $\cos^2 t$.

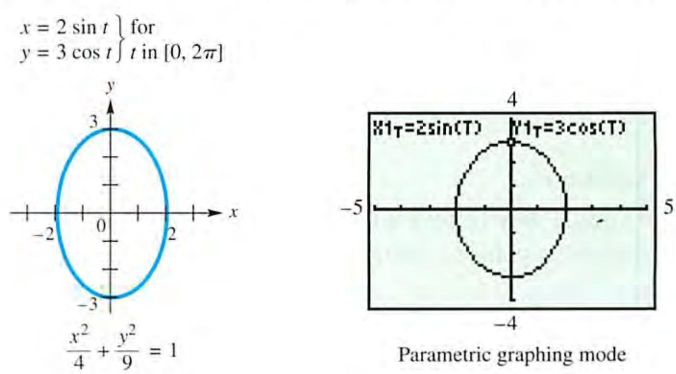
$x = 2 \sin t$	$y = 3 \cos t$	Given equations
$x^2 = 4 \sin^2 t$	$y^2 = 9 \cos^2 t$	Square both sides.
$\frac{x^2}{4} = \sin^2 t$	$\frac{y^2}{9} = \cos^2 t$	Divide.

Now add corresponding sides of the two equations.

$$\frac{x^2}{4} + \frac{y^2}{9} = \sin^2 t + \cos^2 t$$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1 \qquad \sin^2 t + \cos^2 t = 1 \text{ (Section 5.1)}$$

This is the equation of an **ellipse.** See Figure 34 for traditional and calculator graphs. (Ellipses are covered in more detail in college algebra courses.)



(a) (b)

Figure 34

NOW TRY EXERCISE 27.

Parametric representations of a curve are not unique. In fact, there are infinitely many parametric representations of a given curve. If the curve can be described by a rectangular equation $y = f(x)$, with domain X, then one simple parametric representation is

$$x = t, \quad y = f(t), \qquad \text{for } t \text{ in } X.$$

▶ EXAMPLE 4 **FINDING ALTERNATIVE PARAMETRIC EQUATION FORMS**

Give two parametric representations for the equation of the parabola

$$y = (x - 2)^2 + 1.$$

Solution The simplest choice is to let

$$x = t, \quad y = (t - 2)^2 + 1, \qquad \text{for } t \text{ in } (-\infty, \infty).$$

Another choice, which leads to a simpler equation for y, is

$$x = t + 2, \quad y = t^2 + 1, \qquad \text{for } t \text{ in } (-\infty, \infty).$$

NOW TRY EXERCISE 29. ◀

▶ **Note** Sometimes trigonometric functions are desirable. One choice in Example 4 might be

$$x = 2 + \tan t, \quad y = \sec^2 t, \qquad \text{for } t \text{ in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

The Cycloid

The path traced by a fixed point on the circumference of a circle rolling along a line is called a *cycloid*. A **cycloid** is defined by

$$\boldsymbol{x = at - a\sin t, \quad y = a - a\cos t, \qquad \text{for } t \text{ in } (-\infty, \infty).}$$

The cycloid is a special case of a curve traced out by a point at a given distance from the center of a circle as the circle rolls along a straight line. Such a curve is called a **trochoid. Bezier curves** are used in manufacturing, and **Conchoids of Nicodemes** are so named because the shape of their outer branches resembles a conch shell. Other examples are **hypocycloids, epicycloids, the witch of Agnesi, swallowtail catastrophe curves,** and **Lissajou figures.** (*Source:* Stewart, J., *Calculus*, Fifth Edition, Brooks/Cole Publishing Co., 2003.)

▶ EXAMPLE 5 **GRAPHING A CYCLOID**

Graph the cycloid $x = t - \sin t$, $y = 1 - \cos t$, for t in $[0, 2\pi]$.

Algebraic Solution

There is no simple way to find a rectangular equation for the cycloid from its parametric equations. Instead, begin with a table of values.

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
x	0	.08	.6	π	5.7	2π
y	0	.3	1	2	1	0

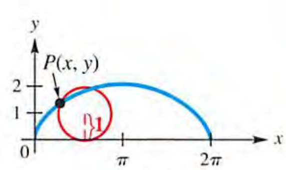


Figure 35

Plotting the ordered pairs (x, y) from the table of values leads to the portion of the graph in Figure 35 from 0 to 2π.

Graphing Calculator Solution

It is easier to graph a cycloid with a graphing calculator in parametric mode than with traditional methods. See Figure 36.

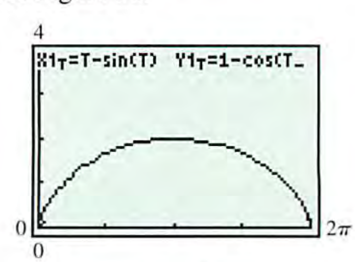


Figure 36

NOW TRY EXERCISE 33. ◀

Figure 37

The cycloid has an interesting physical property. If a flexible cord or wire goes through points P and Q as in Figure 37, and a bead is allowed to slide due to the force of gravity without friction along this path from P to Q, the path that requires the shortest time takes the shape of the graph of an inverted cycloid.

LOOKING AHEAD TO CALCULUS

At any time t, the velocity of an object is given by the vector $\mathbf{v} = \langle f'(t), g'(t)\rangle$. The object's speed at time t is

$$|\mathbf{v}| = \sqrt{(f'(t))^2 + (g'(t))^2}.$$

Applications of Parametric Equations Parametric equations are used to simulate motion. If a ball is thrown with a velocity of v feet per second at an angle θ with the horizontal, its flight can be modeled by the parametric equations

$$x = (v \cos \theta)t \quad \text{and} \quad y = (v \sin \theta)t - 16t^2 + h,$$

where t is in seconds and h is the ball's initial height in feet above the ground. The term $-16t^2$ occurs because gravity is pulling downward. See Figure 38. These equations ignore air resistance.

Figure 38

EXAMPLE 6 SIMULATING MOTION WITH PARAMETRIC EQUATIONS

Three golf balls are hit simultaneously into the air at 132 ft per sec (90 mph) at angles of 30°, 50°, and 70° with the horizontal.

(a) Assuming the ground is level, determine graphically which ball travels the farthest. Estimate this distance.

(b) Which ball reaches the greatest height? Estimate this height.

Solution

(a) The three sets of parametric equations determined by the three golf balls are as follows since $h = 0$.

$$x_1 = (132 \cos 30°)t, \quad y_1 = (132 \sin 30°)t - 16t^2$$
$$x_2 = (132 \cos 50°)t, \quad y_2 = (132 \sin 50°)t - 16t^2$$
$$x_3 = (132 \cos 70°)t, \quad y_3 = (132 \sin 70°)t - 16t^2$$

The graphs of the three sets of parametric equations are shown on the next page in Figure 39(a), where $0 \leq t \leq 9$. From the graph in Figure 39(b), we can see that the ball hit at 50° travels the farthest distance. Using the TRACE feature of the TI-83/84 Plus, we estimate this distance to be about 540 ft.

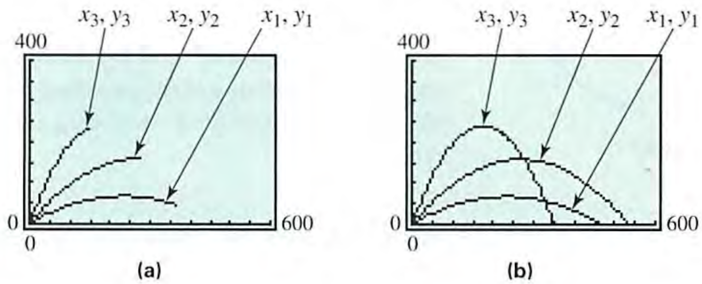


Figure 39

(b) Again, use the TRACE feature to find that the ball hit at 70° reaches the greatest height, about 240 ft.

NOW TRY EXERCISE 39.

▶ **Note** The TI-83/84 Plus graphing calculator allows the user to view the graphing of more than one equation either *sequentially* or *simultaneously*. By choosing the latter, the three balls in Figure 39 can be viewed in flight at the same time.

▶ EXAMPLE 7 EXAMINING PARAMETRIC EQUATIONS OF FLIGHT

Jack Lukas launches a small rocket from a table that is 3.36 ft above the ground. Its initial velocity is 64 ft per sec, and it is launched at an angle of 30° with respect to the ground. Find the rectangular equation that models its path. What type of path does the rocket follow?

Solution The path of the rocket is defined by the parametric equations

$$x = (64 \cos 30°)t \quad \text{and} \quad y = (64 \sin 30°)t - 16t^2 + 3.36$$

or, equivalently,

$$x = 32\sqrt{3}t \quad \text{and} \quad y = -16t^2 + 32t + 3.36.$$

From $x = 32\sqrt{3}t$, we obtain

$$t = \frac{x}{32\sqrt{3}}.$$

Substituting into the other parametric equation for t yields

$$y = -16\left(\frac{x}{32\sqrt{3}}\right)^2 + 32\left(\frac{x}{32\sqrt{3}}\right) + 3.36$$

$$y = -\frac{1}{192}x^2 + \frac{\sqrt{3}}{3}x + 3.36. \quad \text{Simplify.}$$

Because this equation defines a parabola, the rocket follows a parabolic path.

NOW TRY EXERCISE 43(a).

▶ EXAMPLE 8 **ANALYZING THE PATH OF A PROJECTILE**

Determine the total flight time and the horizontal distance traveled by the rocket in Example 7.

Algebraic Solution

The equation $y = -16t^2 + 32t + 3.36$ tells the vertical position of the rocket at time t. We need to determine those values of t for which $y = 0$ since these values correspond to the rocket at ground level. This yields

$$0 = -16t^2 + 32t + 3.36.$$

Using the quadratic formula, the solutions are $t = -.1$ or $t = 2.1$. Since t represents time, $t = -.1$ is an unacceptable answer. Therefore, the flight time is 2.1 sec.

The rocket was in the air for 2.1 sec, so we can use $t = 2.1$ and the parametric equation that models the horizontal position, $x = 32\sqrt{3}t$, to obtain

$$x = 32\sqrt{3}(2.1) \approx 116.4 \text{ ft}.$$

Graphing Calculator Solution

Figure 40 shows that when T = 2.1, the horizontal distance X covered is approximately 116.4 ft, which agrees with the algebraic solution.

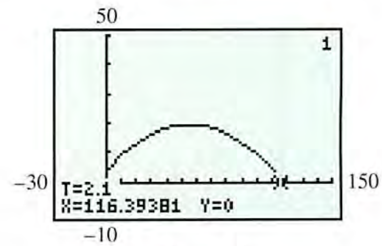


Figure 40

NOW TRY EXERCISE 43(b). ◀

8.6 Exercises

Concept Check *Match the ordered pair from Column II with the pair of parametric equations in Column I on whose graph the point lies. In each case, consider the given value of t.*

I	II
1. $x = 3t + 6,\ y = -2t + 4;\quad t = 2$	**A.** $(5, 25)$
2. $x = \cos t,\ y = \sin t;\quad t = \frac{\pi}{4}$	**B.** $(7, 2)$
3. $x = t,\ y = t^2;\quad t = 5$	**C.** $(12, 0)$
4. $x = t^2 + 3,\ y = t^2 - 2;\quad t = 2$	**D.** $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

For each plane curve **(a)** *graph the curve, and* **(b)** *find a rectangular equation for the curve. See Examples 1 and 2.*

5. $x = t + 2,\ y = t^2,\quad$ for t in $[-1, 1]$

6. $x = 2t,\ y = t + 1,\quad$ for t in $[-2, 3]$

7. $x = \sqrt{t},\ y = 3t - 4,\quad$ for t in $[0, 4]$

8. $x = t^2,\ y = \sqrt{t},\quad$ for t in $[0, 4]$

9. $x = t^3 + 1,\ y = t^3 - 1,\quad$ for t in $(-\infty, \infty)$

10. $x = 2t - 1,\ y = t^2 + 2,\quad$ for t in $(-\infty, \infty)$

11. $x = 2 \sin t,\ y = 2 \cos t,\quad$ for t in $[0, 2\pi]$

12. $x = \sqrt{5} \sin t, y = \sqrt{3} \cos t, \quad \text{for } t \text{ in } [0, 2\pi]$

13. $x = 3 \tan t, y = 2 \sec t, \quad \text{for } t \text{ in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

14. $x = \cot t, y = \csc t, \quad \text{for } t \text{ in } (0, \pi)$

15. $x = \sin t, y = \csc t, \quad \text{for } t \text{ in } (0, \pi)$

16. $x = \tan t, y = \cot t, \quad \text{for } t \text{ in } \left(0, \frac{\pi}{2}\right)$

17. $x = t, y = \sqrt{t^2 + 2}, \quad \text{for } t \text{ in } (-\infty, \infty)$

18. $x = \sqrt{t}, y = t^2 - 1, \quad \text{for } t \text{ in } [0, \infty)$

19. $x = 2 + \sin t, y = 1 + \cos t, \quad \text{for } t \text{ in } [0, 2\pi]$

20. $x = 1 + 2 \sin t, y = 2 + 3 \cos t, \quad \text{for } t \text{ in } [0, 2\pi]$

21. $x = t + 2, y = \frac{1}{t + 2}, \quad \text{for } t \neq -2$

22. $x = t - 3, y = \frac{2}{t - 3}, \quad \text{for } t \neq 3$

23. $x = t + 2, y = t - 4, \quad \text{for } t \text{ in } (-\infty, \infty)$

24. $x = t^2 + 2, y = t^2 - 4, \quad \text{for } t \text{ in } (-\infty, \infty)$

Graph each plane curve defined by the parametric equations for t in $[0, 2\pi]$. *Then find a rectangular equation for the plane curve. See Example 3.*

25. $x = 3 \cos t, y = 3 \sin t$

26. $x = 2 \cos t, y = 2 \sin t$

27. $x = 3 \sin t, y = 2 \cos t$

28. $x = 4 \sin t, y = 3 \cos t$

Give two parametric representations for the equation of each parabola. See Example 4.

29. $y = (x + 3)^2 - 1$

30. $y = (x + 4)^2 + 2$

31. $y = x^2 - 2x + 3$

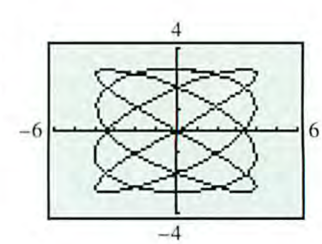
32. $y = x^2 - 4x + 6$

Graph each cycloid defined by the given equations for t in the specified interval. See Example 5.

33. $x = 2t - 2 \sin t, y = 2 - 2 \cos t, \quad \text{for } t \text{ in } [0, 4\pi]$

34. $x = t - \sin t, y = 1 - \cos t, \quad \text{for } t \text{ in } [0, 4\pi]$

Lissajous Figures *The screen shown here is an example of a* ***Lissajous figure.*** *Lissajous figures occur in electronics and may be used to find the frequency of an unknown voltage. Graph each Lissajous figure for t in* $[0, 6.5]$ *in the window* $[-6, 6]$ *by* $[-4, 4]$.



35. $x = 2 \cos t, y = 3 \sin 2t$

36. $x = 3 \cos 2t, y = 3 \sin 3t$

37. $x = 3 \sin 4t, y = 3 \cos 3t$

38. $x = 4 \sin 4t, y = 3 \sin 5t$

(Modeling) *In Exercises 39–42, do the following.*

(a) *Determine the parametric equations that model the path of the projectile.*

(b) *Determine the rectangular equation that models the path of the projectile.*

(c) *Determine approximately how long the projectile is in flight and the horizontal distance covered.*

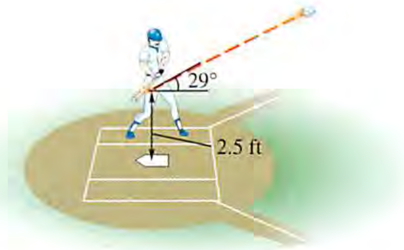
See Examples 6–8.

39. ***Flight of a Model Rocket*** A model rocket is launched from the ground with velocity 48 ft per sec at an angle of 60° with respect to the ground.

40. ***Flight of a Golf Ball*** Tiger is playing golf. He hits a golf ball from the ground at an angle of 60° with respect to the ground at velocity 150 ft per sec.

41. ***Flight of a Softball*** Sally hits a softball when it is 2 ft above the ground. The ball leaves her bat at an angle of 20° with respect to the ground at velocity 88 ft per sec.

42. ***Flight of a Baseball*** Pronk hits a baseball when it is 2.5 ft above the ground. The ball leaves his bat at an angle of 29° from the horizontal with velocity 136 ft per sec.



(Modeling) *Solve each problem. See Examples 7 and 8.*

43. ***Path of a Rocket*** A rocket is launched from the top of an 8-ft ladder. Its initial velocity is 128 ft per sec, and it is launched at an angle of 60° with respect to the ground.

(a) Find the rectangular equation that models its path. What type of path does the rocket follow?

(b) Determine the total flight time and the horizontal distance the rocket travels.

44. ***Simulating Gravity on the Moon*** If an object is thrown on the moon, then the parametric equations of flight are

$$x = (v \cos \theta)t \qquad \text{and} \qquad y = (v \sin \theta)t - 2.66t^2 + h.$$

Estimate the distance that a golf ball hit at 88 ft per sec (60 mph) at an angle of 45° with the horizontal travels on the moon if the moon's surface is level.

45. ***Flight of a Baseball*** A baseball is hit from a height of 3 ft at a 60° angle above the horizontal. Its initial velocity is 64 ft per sec.

(a) Write parametric equations that model the flight of the baseball.

(b) Determine the horizontal distance traveled by the ball in the air. Assume that the ground is level.

(c) What is the maximum height of the baseball? At that time, how far has the ball traveled horizontally?

(d) Would the ball clear a 5-ft-high fence that is 100 ft from the batter?

(Modeling) Path of a Projectile *In Exercises 46 and 47, a projectile has been launched from the ground with initial velocity 88 ft per sec. You are supplied with the parametric equations modeling the path of the projectile.*

(a) *Graph the parametric equations.*

(b) *Approximate θ, the angle the projectile makes with the horizontal at launch, to the nearest tenth of a degree.*

(c) *Based on your answer to part (b), write parametric equations for the projectile using the cosine and sine functions.*

46. $x = 82.69295063t,\ y = -16t^2 + 30.09777261t$

47. $x = 56.56530965t,\ y = -16t^2 + 67.41191099t$

48. Give two parametric representations of the line through the point (x_1, y_1) with slope m.

49. Give two parametric representations of the parabola $y = a(x - h)^2 + k$.

50. Give a parametric representation of the rectangular equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

51. Give a parametric representation of the rectangular equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

52. The spiral of Archimedes has polar equation $r = a\theta$, where $r^2 = x^2 + y^2$. Show that a parametric representation of the spiral of Archimedes is

$$x = a\theta \cos \theta, \quad y = a\theta \sin \theta, \qquad \text{for } \theta \text{ in } (-\infty, \infty).$$

53. Show that the **hyperbolic spiral** $r\theta = a$, where $r^2 = x^2 + y^2$, is given parametrically by

$$x = \frac{a \cos \theta}{\theta}, \quad y = \frac{a \sin \theta}{\theta}, \qquad \text{for } \theta \text{ in } (-\infty, 0) \cup (0, \infty).$$

54. The parametric equations $x = \cos t$, $y = \sin t$, for t in $[0, 2\pi]$ and the parametric equations $x = \cos t$, $y = -\sin t$, for t in $[0, 2\pi]$ both have the unit circle as their graph. However, in one case the circle is traced out clockwise (as t moves from 0 to 2π) and in the other case the circle is traced out counterclockwise. For which pair of equations is the circle traced out in the clockwise direction?

Concept Check *Consider the parametric equations $x = f(t)$, $y = g(t)$, for t in $[a, b]$, with $c > 0$, $d > 0$.*

55. How is the graph affected if the equation $x = f(t)$ is replaced by $x = c + f(t)$?

56. How is the graph affected if the equation $y = g(t)$ is replaced by $y = d + g(t)$?

Chapter 8 Summary

KEY TERMS

8.1 imaginary unit, complex number, real part, imaginary part, pure imaginary number, nonreal complex number, standard form, complex conjugates

8.2 real axis, imaginary axis, complex plane, rectangular form of a complex number, trigonometric (polar) form of a complex number, absolute value (modulus), argument

8.4 *n*th root of a complex number

8.5 polar coordinate system, pole, polar axis, polar coordinates, rectangular (Cartesian) equation, polar equation, cardioid, polar grid, rose curve, lemniscate, spiral of Archimedes, limaçon

8.6 plane curve, parametric equations of a plane curve, parameter, cycloid

NEW SYMBOLS

i imaginary unit	$a + bi$ complex number

QUICK REVIEW

CONCEPTS	EXAMPLES

8.1 Complex Numbers

Definition of i

$$i^2 = -1 \quad \text{or} \quad i = \sqrt{-1}$$

Definition of $\sqrt{-a}$

For $a > 0$,

$$\sqrt{-a} = i\sqrt{a}.$$

Examples:

$$\sqrt{-4} = 2i$$

$$\sqrt{-12} = i\sqrt{12} = 2i\sqrt{3}$$

Adding and Subtracting Complex Numbers

Add or subtract the real parts and add or subtract the imaginary parts.

Example:

$$(2 + 3i) + (3 + i) - (2 - i)$$
$$= (2 + 3 - 2) + (3 + 1 + 1)i$$
$$= 3 + 5i$$

Multiplying and Dividing Complex Numbers

Multiply complex numbers as with binomials, and use the fact that $i^2 = -1$.

Example:

$$(6 + i)(3 - 2i) = 18 - 12i + 3i - 2i^2 \quad \text{FOIL}$$
$$= (18 + 2) + (-12 + 3)i \quad i^2 = -1$$
$$= 20 - 9i$$

Divide complex numbers by multiplying the numerator and denominator by the complex conjugate of the denominator.

Example:

$$\frac{3 + i}{1 + i} = \frac{(3 + i)(1 - i)}{(1 + i)(1 - i)} = \frac{3 - 3i + i - i^2}{1 - i^2}$$
$$= \frac{4 - 2i}{2} = \frac{2(2 - i)}{2} = 2 - i$$

(continued)

CONCEPTS | EXAMPLES

8.2 Trigonometric (Polar) Form of Complex Numbers

Trigonometric (Polar) Form of Complex Numbers

If the complex number $x + yi$ corresponds to the vector with direction angle θ and magnitude r, then

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2} \qquad \tan\theta = \frac{y}{x}, \quad \text{if } x \neq 0.$$

The expression

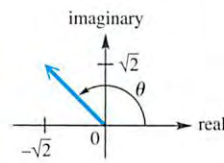
$$r(\cos\theta + i\sin\theta) \quad \text{or} \quad r\operatorname{cis}\theta$$

is the trigonometric form (or polar form) of $x + yi$.

Write $2(\cos 60° + i\sin 60°)$ in rectangular form.

$$2(\cos 60° + i\sin 60°) = 2\left(\frac{1}{2} + i\cdot\frac{\sqrt{3}}{2}\right) = 1 + i\sqrt{3}$$

Write $-\sqrt{2} + i\sqrt{2}$ in trigonometric form.



$$r = \sqrt{(-\sqrt{2})^2 + (\sqrt{2})^2} = 2$$

$\tan\theta = -1$ and θ is in quadrant II, so $\theta = 180° - 45° = 135°$. Therefore,

$$-\sqrt{2} + i\sqrt{2} = 2\operatorname{cis}135°.$$

8.3 The Product and Quotient Theorems

Product and Quotient Theorems

For any two complex numbers $r_1(\cos\theta_1 + i\sin\theta_1)$ and $r_2(\cos\theta_2 + i\sin\theta_2)$,

$$[r_1(\cos\theta_1 + i\sin\theta_1)]\cdot[r_2(\cos\theta_2 + i\sin\theta_2)]$$
$$= r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

and

$$\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)],$$

where $r_2\operatorname{cis}\theta_2 \neq 0$.

If $z_1 = 4(\cos 135° + i\sin 135°)$

and $z_2 = 2(\cos 45° + i\sin 45°)$, then

$$z_1z_2 = 8(\cos 180° + i\sin 180°)$$
$$= 8(-1 + i\cdot 0) = -8,$$

and

$$\frac{z_1}{z_2} = 2(\cos 90° + i\sin 90°)$$
$$= 2(0 + i\cdot 1) = 2i.$$

8.4 De Moivre's Theorem; Powers and Roots of Complex Numbers

De Moivre's Theorem

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

***n*th Root Theorem**

If n is any positive integer, r is a positive real number, and θ is in degrees, then the nonzero complex number $r(\cos\theta + i\sin\theta)$ has exactly n distinct nth roots, given by

$$\sqrt[n]{r}(\cos\alpha + i\sin\alpha), \quad \text{or} \quad \sqrt[n]{r}\operatorname{cis}\alpha,$$

where

$$\alpha = \frac{\theta + 360°\cdot k}{n}, \quad \text{or} \quad \alpha = \frac{\theta}{n} + \frac{360°\cdot k}{n},$$

$k = 0, 1, 2, \ldots, n - 1$.

Let $z = 4(\cos 180° + i\sin 180°)$. Find z^3 and the square roots of z.

$$z^3 = 4^3(\cos 3\cdot 180° + i\sin 3\cdot 180°)$$
$$= 64(\cos 540° + i\sin 540°)$$
$$= 64(-1 + i\cdot 0)$$
$$= -64$$

For the given z, $r = 4$ and $\theta = 180°$. Its square roots are

$$\sqrt{4}\left(\cos\frac{180°}{2} + i\sin\frac{180°}{2}\right) = 2(0 + i\cdot 1) = 2i$$

and $\sqrt{4}\left(\cos\dfrac{180° + 360°}{2} + i\sin\dfrac{180° + 360°}{2}\right)$

$$= 2(0 + i(-1)) = -2i.$$

CONCEPTS	EXAMPLES

8.5 Polar Equations and Graphs

Rectangular and Polar Coordinates

The following relationships hold between the point (x, y) in the rectangular coordinate plane and the same point (r, θ) in the polar coordinate plane.

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}, \quad \text{if } x \neq 0$$

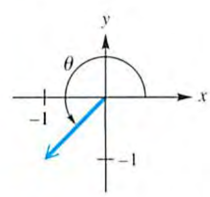
Find the rectangular coordinates for the point $(5, 60°)$ in polar coordinates.

$$x = 5\cos 60° = 5\left(\frac{1}{2}\right) = \frac{5}{2}$$

$$y = 5\sin 60° = 5\left(\frac{\sqrt{3}}{2}\right) = \frac{5\sqrt{3}}{2}$$

The rectangular coordinates are $\left(\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)$.

Find polar coordinates for $(-1, -1)$ in rectangular coordinates.



$$r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$$

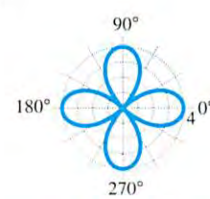
$\tan\theta = 1$ and θ is in quadrant III, so $\theta = 225°$.

One pair of polar coordinates for $(-1, -1)$ is $(\sqrt{2}, 225°)$.

Polar Graphs

Examples of polar graphs include lines, circles, limaçons, lemniscates, roses, and spirals. (See page 400.)

Graph $r = 4\cos 2\theta$.



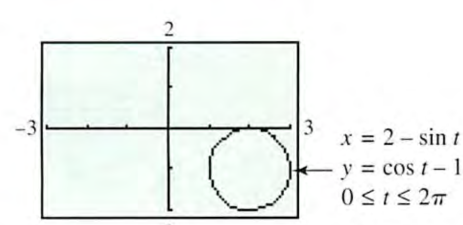
$r = 4\cos 2\theta$

8.6 Parametric Equations, Graphs, and Applications

Plane Curve

A **plane curve** is a set of points (x, y) such that $x = f(t)$, $y = g(t)$, and f and g are both defined on an interval I. The equations $x = f(t)$ and $y = g(t)$ are **parametric equations** with **parameter** t.

Graph $x = 2 - \sin t$, $y = \cos t - 1$, for $0 \le t \le 2\pi$.



(continued)

CONCEPTS	EXAMPLES
Flight of an Object If an object has an initial velocity v, initial height h, and travels so that its initial angle of elevation is θ, then its flight after t seconds is modeled by the parametric equations $\mathbf{x = (v \cos \theta)t}$ and $\mathbf{y = (v \sin \theta)t - 16t^2 + h}$.	Joe kicks a football from the ground at an angle of 45° with a velocity of 48 ft per sec. Give the parametric equations that model the path of the football and the distance it travels before hitting the ground. $x = (48 \cos 45°)t = 24\sqrt{2}t$ $y = (48 \sin 45°)t - 16t^2 = 24\sqrt{2}t - 16t^2$ When the ball hits the ground, $y = 0$. $24\sqrt{2}t - 16t^2 = 0$ $8t(3\sqrt{2} - 2t) = 0$ $t = 0$ or $t = \frac{3\sqrt{2}}{2}$ (Reject) The distance it travels is $x = 24\sqrt{2}\left(\frac{3\sqrt{2}}{2}\right) = 72$ ft.

CHAPTER 8 ▶ Review Exercises

Write as a multiple of i.

1. $\sqrt{-9}$

2. $\sqrt{-12}$

Solve each quadratic equation.

3. $x^2 = -81$

4. $x(2x + 3) = -4$

Perform each operation. Write answers in rectangular form.

5. $(1 - i) - (3 + 4i) + 2i$

6. $(2 - 5i) + (9 - 10i) - 3$

7. $(6 - 5i) + (2 + 7i) - (3 - 2i)$

8. $(4 - 2i) - (6 + 5i) - (3 - i)$

9. $(3 + 5i)(8 - i)$

10. $(4 - i)(5 + 2i)$

11. $(2 + 6i)^2$

12. $(6 - 3i)^2$

13. $(1 - i)^3$

14. $(2 + i)^3$

15. $\dfrac{25 - 19i}{5 + 3i}$

16. $\dfrac{2 - 5i}{1 + i}$

17. $\dfrac{2 + i}{1 - 5i}$

18. $\dfrac{3 + 2i}{i}$

19. i^{53}

20. i^{-41}

Perform each operation. Write answers in rectangular form.

21. $[5(\cos 90° + i \sin 90°)][6(\cos 180° + i \sin 180°)]$

22. $[3 \text{ cis } 135°][2 \text{ cis } 105°]$

23. $\dfrac{2(\cos 60° + i \sin 60°)}{8(\cos 300° + i \sin 300°)}$

24. $\dfrac{4 \text{ cis } 270^\circ}{2 \text{ cis } 90^\circ}$

25. $(\sqrt{3} + i)^3$

26. $(2 - 2i)^5$

27. $(\cos 100^\circ + i \sin 100^\circ)^6$

28. **Concept Check** The vector representing a real number will lie on the ______-axis in the complex plane.

Graph each complex number as a vector.

29. $5i$

30. $-4 + 2i$

31. $3 - 3i\sqrt{3}$

32. Find and graph the resultant of $7 + 3i$ and $-2 + i$.

Perform each conversion, using a calculator to approximate answers as necessary.

	Rectangular Form	Trigonometric Form
33.	$-2 + 2i$	______
34.	______	$3(\cos 90^\circ + i \sin 90^\circ)$
35.	______	$2(\cos 225^\circ + i \sin 225^\circ)$
36.	$-4 + 4i\sqrt{3}$	______
37.	$1 - i$	______
38.	______	$4 \text{ cis } 240^\circ$
39.	$-4i$	______

Concept Check *The complex number z, where $z = x + yi$, can be graphed in the plane as (x, y). Describe the graphs of all complex numbers z satisfying the conditions in Exercises 40 and 41.*

40. The absolute value of z is 2.

41. The imaginary part of z is the negative of the real part of z.

Find all roots as indicated. Express them in trigonometric form.

42. the fifth roots of $-2 + 2i$

43. the cube roots of $1 - i$

44. **Concept Check** How many real fifth roots does -32 have?

45. **Concept Check** How many real sixth roots does -64 have?

Solve each equation. Leave answers in trigonometric form.

46. $x^3 + 125 = 0$

47. $x^4 + 16 = 0$

48. $x^2 + i = 0$

49. Convert $(-1, \sqrt{3})$ to polar coordinates, with $0^\circ \le \theta < 360^\circ$ and $r > 0$.

50. Convert $(5, 315^\circ)$ to rectangular coordinates.

51. **Concept Check** If a point lies on an axis in the rectangular plane, then what kind of angle must θ be if (r, θ) represents the point in polar coordinates?

52. **Concept Check** What will the graph of $r = k$ be, for $k > 0$?

Identify and graph each polar equation for θ in $[0^\circ, 360^\circ)$.

53. $r = 4 \cos \theta$

54. $r = -1 + \cos \theta$

55. $r = 2 \sin 4\theta$

56. $r = \dfrac{2}{2 \cos \theta - \sin \theta}$

Find an equivalent equation in rectangular coordinates.

57. $r = \dfrac{3}{1 + \cos\theta}$ **58.** $r = \sin\theta + \cos\theta$ **59.** $r = 2$

Find an equivalent equation in polar coordinates.

60. $y = x$ **61.** $y = x^2$

In Exercises 62–65, identify the geometric symmetry (A, B, or C) that the graph will possess.

A. *symmetry with respect to the origin*
B. *symmetry with respect to the y-axis*
C. *symmetry with respect to the x-axis*

62. Whenever (r, θ) is on the graph, then so is $(-r, -\theta)$.

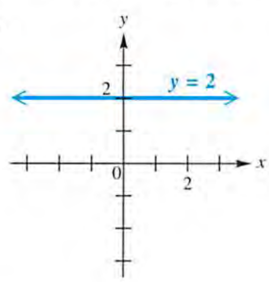
63. Whenever (r, θ) is on the graph, then so is $(-r, \theta)$.

64. Whenever (r, θ) is on the graph, then so is $(r, -\theta)$.

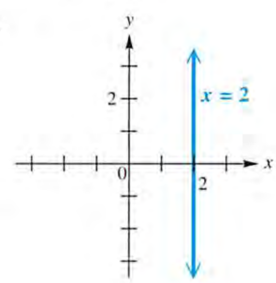
65. Whenever (r, θ) is on the graph, then so is $(r, \pi - \theta)$.

In Exercises 66–69, find a polar equation having the given graph.

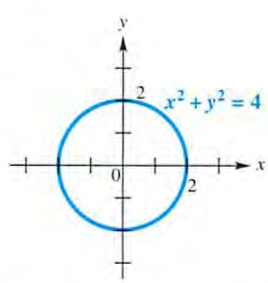
66.



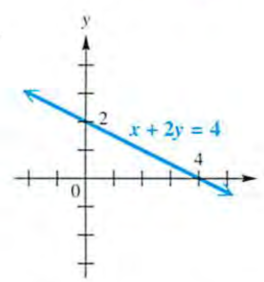
67.



68.



69.



70. Show that the distance between (r_1, θ_1) and (r_2, θ_2) in polar coordinates is given by

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}.$$

71. Graph the plane curve defined by the parametric equations $x = t + \cos t$, $y = \sin t$, for t in $[0, 2\pi]$.

Find a rectangular equation for each plane curve with the given parametric equations.

72. $x = 3t + 2$, $y = t - 1$, for t in $[-5, 5]$

73. $x = \sqrt{t - 1}$, $y = \sqrt{t}$, for t in $[1, \infty)$

74. $x = t^2 + 5$, $y = \dfrac{1}{t^2 + 1}$, for t in $(-\infty, \infty)$

75. $x = 5\tan t$, $y = 3\sec t$, for t in $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

76. $x = \cos 2t$, $y = \sin t$, for t in $(-\pi, \pi)$

77. Find a pair of parametric equations whose graph is the circle having center $(3, 4)$ and passing through the origin.

78. ***Mandelbrot Set*** Follow the steps in Exercise 64 of **Section 8.2** to show that the graph of the Mandelbrot set in Exercise 49 of **Section 8.4** is symmetric with respect to the x-axis.

79. ***Flight of a Baseball*** Albert hits a baseball when it is 3.2 ft above the ground. It leaves the bat with velocity 118 ft per sec at an angle of 27° with respect to the ground.

(a) Determine the parametric equations that model the path of the baseball.

(b) Determine the rectangular equation that models the path of the baseball.

(c) Determine approximately how long the projectile is in flight and the horizontal distance covered.

CHAPTER 8 ▶ Test

1. Multiply or divide as indicated. Simplify each answer.

(a) $\sqrt{-8} \cdot \sqrt{-6}$ **(b)** $\dfrac{\sqrt{-2}}{\sqrt{8}}$ **(c)** $\dfrac{\sqrt{-20}}{\sqrt{-180}}$

2. For the complex numbers $w = 2 - 4i$ and $z = 5 + i$, find each of the following in rectangular form.

(a) $w + z$ (and give a geometric representation) **(b)** $w - z$ **(c)** wz **(d)** $\dfrac{w}{z}$

3. Express each of the following in rectangular form.

(a) i^{15} **(b)** $(1 + i)^2$

4. Solve $2x^2 - x + 4 = 0$ over the complex number system.

5. Write each complex number in trigonometric (polar) form, where $0° \le \theta < 360°$.

(a) $3i$ **(b)** $1 + 2i$ **(c)** $-1 - i\sqrt{3}$

6. Write each complex number in rectangular form.

(a) $3(\cos 30° + i \sin 30°)$ **(b)** $4 \text{ cis } 40°$ **(c)** $3(\cos 90° + i \sin 90°)$

7. For the complex numbers $w = 8(\cos 40° + i \sin 40°)$ and $z = 2(\cos 10° + i \sin 10°)$, find each of the following in the form specified.

(a) wz (trigonometric form) **(b)** $\dfrac{w}{z}$ (rectangular form) **(c)** z^3 (rectangular form)

8. Find the four complex fourth roots of $-16i$. Express them in trigonometric form.

9. Convert the given rectangular coordinates to polar coordinates. Give two pairs of polar coordinates for each point.

(a) $(0, 5)$ **(b)** $(-2, -2)$

10. Convert the given polar coordinates to rectangular coordinates.

(a) $(3, 315°)$ (b) $(-4, 90°)$

Identify and graph each polar equation for θ *in* $[0°, 360°)$.

11. $r = 1 - \cos\theta$ **12.** $r = 3\cos 3\theta$

13. Convert each polar equation to a rectangular equation, and sketch its graph.

(a) $r = \dfrac{4}{2\sin\theta - \cos\theta}$ (b) $r = 6$

Graph each pair of parametric equations.

14. $x = 4t - 3, y = t^2$, for t in $[-3, 4]$

15. $x = 2\cos 2t, y = 2\sin 2t$, for t in $[0, 2\pi]$

16. ***Julia Set*** Consider the complex number $z = -1 + i$. Compute the value of $z^2 - 1$, and show that its absolute value exceeds 2, indicating that $-1 + i$ is not in the Julia set.

CHAPTER 8 ▶ Quantitative Reasoning

How Rugged Is Your Coastline?*

An interesting feature of coastlines is that their ruggedness is independent of the distance from which they are viewed. From an airplane, we see irregularities as bays, peninsulas, river mouths, and so on. On foot, we see each rock outcropping and creek that makes the coastline appear more rugged. An ant sees every pebble as a mountain to be scaled.

An interesting result of this phenomenon is that the total distance you travel along a coastline is dependent on the size of the steps you take. The closer (or smaller) you are, the smaller your steps will be. This means you will have more obstacles in your way, which results in a longer distance to travel. The more rugged the coastline, the longer it will be. In theory, this means that if you could take small enough steps on a rugged enough coastline, the length of the coastline would approach infinity. This is related to the study of fractals.

From a mathematical perspective, we can say that the number of steps needed (y) varies inversely with some power of the size of the steps taken (x):

$$y = \frac{k}{x^n}.$$

Each coastline will have different values for k and n, depending on its ruggedness. For a particular map of Lake Tahoe, these values are $k = 23.5$ and $n = 1.153$. Use the equation to determine how long the coastline would be if you "walked" with the given step sizes. Remember that the equation gives you the number of steps, so multiply that value by the step size to get the total length. Round answers to the nearest tenth of an inch.

1. 6 in. **2.** .1 in. **3.** .01 in.

*This material is based on an idea presented by Lori Lambertson, of the Nueva School and the Exploratorium in San Francisco.