Appendixes Overview

This section contains a review of some basic algebraic skills. (You should read Section P.1 before reading this appendix.) Radical and rational expressions are introduced and radical expressions are simplified algebraically. We add, subtract, and multiply polynomials and factor simple polynomials by a variety of techniques. These factoring techniques are used to add, subtract, multiply, and divide fractional expressions.

A.1 Radicals and Rational Exponents

What you'll learn about

- Radicals
- Simplifying Radical Expressions
- Rationalizing the Denominator
- Rational Exponents

. . . and why

You need to review these basic algebraic skills if you don't remember them.

Radicals

If $b^2 = a$, then b is a **square root** of a. For example, both 2 and -2 are square roots of 4 because $2^2 = (-2)^2 = 4$. Similarly, b is a **cube root** of a if $b^3 = a$. For example, 2 is a cube root of 8 because $2^3 = 8$.

DEFINITION Real *n*th Root of a Real Number

Let n be an integer greater than 1 and a and b real numbers.

1. If $b^n = a$, then b is an ***n*th root** of a.
2. If a has an nth root, the **principal *n*th root** of a is the nth root having the same sign as a.

The principal nth root of a is denoted by the **radical expression** $\sqrt[n]{a}$. The positive integer n is the **index** of the radical and a is the **radicand**.

Every real number has exactly one real *nth root* whenever n is odd. For instance, 2 is the only real cube root of 8. When n is even, positive real numbers have two real *nth roots* and negative real numbers have no real nth roots. For example, the real fourth roots of 16 are ± 2, and -16 has no real fourth roots. The *principal* fourth root of 16 is 2.

When $n = 2$, special notation is used for roots. We omit the index and write $\sqrt{a}$ instead of $\sqrt[2]{a}$. If a is a positive real number and n a positive even integer, its two nth roots are denoted by $\sqrt[n]{a}$ and $-\sqrt[n]{a}$.

EXAMPLE 1 Finding Principal *n*th Roots

(a) $\sqrt{36} = 6$ because $6^2 = 36$.

(b) $\sqrt[3]{\frac{27}{8}} = \frac{3}{2}$ because $\left(\frac{3}{2}\right)^3 = \frac{27}{8}$.

(c) $\sqrt[3]{-\frac{27}{8}} = -\frac{3}{2}$ because $\left(-\frac{3}{2}\right)^3 = -\frac{27}{8}$.

(d) $\sqrt[4]{-625}$ is not a real number because the index 4 is even and the radicand -625 is negative (there is *no* real number whose fourth power is negative).

Now try Exercises 7 and 9.

PRINCIPAL nTH ROOTS AND CALCULATORS

Most calculators have a key for the principal nth root. Use this feature of your calculator to check the computations in Example 1.

Here are some properties of radicals together with examples that help illustrate their meaning.

Properties of Radicals

Let u and v be real numbers, variables, or algebraic expressions, and m and n be positive integers greater than 1. We assume that all of the roots are real numbers and all of the denominators are not zero.

Property	Example
1. $\sqrt[n]{uv} = \sqrt[n]{u} \cdot \sqrt[n]{v}$	$\sqrt{75} = \sqrt{25 \cdot 3}$ $= \sqrt{25} \cdot \sqrt{3} = 5\sqrt{3}$
2. $\sqrt[n]{\dfrac{u}{v}} = \dfrac{\sqrt[n]{u}}{\sqrt[n]{v}}$	$\dfrac{\sqrt[4]{96}}{\sqrt[4]{6}} = \sqrt[4]{\dfrac{96}{6}} = \sqrt[4]{16} = 2$
3. $\sqrt[m]{\sqrt[n]{u}} = \sqrt[m \cdot n]{u}$	$\sqrt{\sqrt[3]{7}} = \sqrt[2 \cdot 3]{7} = \sqrt[6]{7}$
4. $(\sqrt[n]{u})^n = u$	$(\sqrt[4]{5})^4 = 5$
5. $\sqrt[n]{u^m} = (\sqrt[n]{u})^m$	$\sqrt[3]{27^2} = (\sqrt[3]{27})^2 = 3^2 = 9$
6. $\sqrt[n]{u^n} = \begin{cases} \lvert u \rvert & n \text{ even} \\ u & n \text{ odd} \end{cases}$	$\sqrt{(-6)^2} = \lvert -6 \rvert = 6$ $\sqrt[3]{(-6)^3} = -6$

CAUTION

Without the restriction that preceded the list, Property 5 would need special attention. For example,

$$\sqrt{(-3)^2} \neq (\sqrt{-3})^2$$

because $\sqrt{-3}$ on the right is not a real number.

Simplifying Radical Expressions

Many simplifying techniques for roots of real numbers have been rendered obsolete because of calculators. For example, when determining the decimal form of $1/\sqrt{2}$, it was once very common first to change the fraction so that the radical was in the numerator:

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

Using paper and pencil, it was then easier to divide a decimal approximation for $\sqrt{2}$ by 2 than to divide that decimal into 1. Now either form is quickly computed with a calculator. However, these techniques are still valid for radicals involving algebraic expressions and for numerical computations when you need exact answers. Example 2 illustrates the technique of *removing factors from radicands*.

PROPERTIES OF EXPONENTS

Check the Properties of Exponents on page 8 of Section P.1 to see why

$$16 = 2^4 \text{ and } 9x^4 = (3x^2)^2.$$

EXAMPLE 2 Removing Factors from Radicands

(a) $\sqrt[4]{80} = \sqrt[4]{16 \cdot 5}$ — Finding greatest fourth-power factor

$= \sqrt[4]{2^4 \cdot 5}$ — $16 = 2^4$

$= \sqrt[4]{2^4} \cdot \sqrt[4]{5}$ — Property 1

$= 2\sqrt[4]{5}$ — Property 6

(b) $\sqrt{18x^5} = \sqrt{9x^4 \cdot 2x}$ — Finding greatest square factor

$= \sqrt{(3x^2)^2 \cdot 2x}$ — $9x^4 = (3x^2)^2$

$= 3x^2\sqrt{2x}$ — Properties 1 and 6

continued

(c) $\sqrt[4]{x^4y^4} = \sqrt[4]{(xy)^4}$ — Finding greatest fourth-power factor

$= |xy|$ — Property 6

(d) $\sqrt[3]{-24y^6} = \sqrt[3]{(-2y^2)^3 \cdot 3}$ — Finding greatest cube factor

$= -2y^2\sqrt[3]{3}$ — Properties 1 and 6

Now try Exercises 29 and 33.

Rationalizing the Denominator

The process of rewriting fractions containing radicals so that the denominator is free of radicals is **rationalizing the denominator**. When the denominator has the form $\sqrt[n]{u^k}$, multiplying numerator and denominator by $\sqrt[n]{u^{n-k}}$ and using Property 6 will eliminate the radical from the denominator because

$$\sqrt[n]{u^k} \cdot \sqrt[n]{u^{n-k}} = \sqrt[n]{u^{k+n-k}} = \sqrt[n]{u^n}.$$

Example 3 illustrates the process.

EXAMPLE 3 Rationalizing the Denominator

(a) $\sqrt{\dfrac{2}{3}} = \dfrac{\sqrt{2}}{\sqrt{3}} = \dfrac{\sqrt{2}}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{\sqrt{6}}{3}$

(b) $\dfrac{1}{\sqrt[4]{x}} = \dfrac{1}{\sqrt[4]{x}} \cdot \dfrac{\sqrt[4]{x^3}}{\sqrt[4]{x^3}} = \dfrac{\sqrt[4]{x^3}}{\sqrt[4]{x^4}} = \dfrac{\sqrt[4]{x^3}}{|x|}$

(c) $\sqrt[5]{\dfrac{x^2}{y^3}} = \dfrac{\sqrt[5]{x^2}}{\sqrt[5]{y^3}} = \dfrac{\sqrt[5]{x^2}}{\sqrt[5]{y^3}} \cdot \dfrac{\sqrt[5]{y^2}}{\sqrt[5]{y^2}} = \dfrac{\sqrt[5]{x^2y^2}}{\sqrt[5]{y^5}} = \dfrac{\sqrt[5]{x^2y^2}}{y}$

Now try Exercise 37.

Rational Exponents

We know how to handle exponential expressions with integer exponents (see Section P.1). For example, $x^3 \cdot x^4 = x^7$, $(x^3)^2 = x^6$, $x^5/x^2 = x^3$, $x^{-2} = 1/x^2$, and so forth. But exponents can also be rational numbers. How should we define $x^{1/2}$? If we assume that the same rules that apply for integer exponents also apply for rational exponents we get a clue. For example, we want

$$x^{1/2} \cdot x^{1/2} = x^1.$$

This equation suggests that $x^{1/2} = \sqrt{x}$. In general, we have the following definition.

DEFINITION Rational Exponents

Let u be a real number, variable, or algebraic expression, and n an integer greater than 1. Then

$$u^{1/n} = \sqrt[n]{u}.$$

If m is a positive integer, m/n is in reduced form, and all roots are real numbers, then

$$u^{m/n} = (u^{1/n})^m = (\sqrt[n]{u})^m \quad \text{and} \quad u^{m/n} = (u^m)^{1/n} = \sqrt[n]{u^m}.$$

The numerator of a rational exponent is the *power* to which the base is raised, and the denominator is the *root* to be taken. The fraction m/n needs to be in reduced form because, for instance,

$$u^{2/3} = (\sqrt[3]{u})^2$$

is defined for all real numbers u (every real number has a cube root), but

$$u^{4/6} = (\sqrt[6]{u})^4$$

is defined only for $u \geq 0$ (only nonnegative real numbers have sixth roots).

SIMPLIFYING RADICALS

If you also want the radical form in Example 4d to be simplified, then continue as follows:

$$\frac{1}{\sqrt{z^3}} = \frac{1}{\sqrt{z^3}} \cdot \frac{\sqrt{z}}{\sqrt{z}} = \frac{\sqrt{z}}{z^2}$$

EXAMPLE 4 Converting Radicals to Exponentials and Vice Versa

(a) $\sqrt{(x+y)^3} = (x+y)^{3/2}$

(b) $3x\sqrt[5]{x^2} = 3x \cdot x^{2/5} = 3x^{7/5}$

(c) $x^{2/3}y^{1/3} = (x^2y)^{1/3} = \sqrt[3]{x^2y}$

(d) $z^{-3/2} = \dfrac{1}{z^{3/2}} = \dfrac{1}{\sqrt{z^3}}$

Now try Exercises 43 and 47.

An expression involving powers is *simplified* if each factor appears only once and all exponents are positive. Example 5 illustrates.

EXAMPLE 5 Simplifying Exponential Expressions

(a) $(x^2y^9)^{1/3}(xy^2) = (x^{2/3}y^3)(xy^2) = x^{5/3}y^5$

(b) $\left(\dfrac{3x^{2/3}}{y^{1/2}}\right)\left(\dfrac{2x^{-1/2}}{y^{2/5}}\right) = \dfrac{6x^{1/6}}{y^{9/10}}$

Now try Exercise 61.

Example 6 suggests how to simplify a sum or difference of radicals.

EXAMPLE 6 Combining Radicals

(a)
$$\begin{aligned} 2\sqrt{80} - \sqrt{125} &= 2\sqrt{16 \cdot 5} - \sqrt{25 \cdot 5} && \text{Find greatest square factors.} \\ &= 8\sqrt{5} - 5\sqrt{5} && \text{Remove factors from radicands.} \\ &= 3\sqrt{5} && \text{Distributive property} \end{aligned}$$

(b)
$$\begin{aligned} \sqrt{4x^2y} - \sqrt{y^3} &= \sqrt{(2x)^2y} - \sqrt{y^2y} && \text{Find greatest square factors.} \\ &= 2|x|\sqrt{y} - |y|\sqrt{y} && \text{Remove factors from radicands.} \\ &= (2|x| - |y|)\sqrt{y} && \text{Distributive property.} \end{aligned}$$

Now try Exercise 71.

Here's a summary of the procedures we use to *simplify expressions* involving radicals.

Simplifying Radical Expressions

1. Remove factors from the radicand (see Example 2).
2. Eliminate radicals from denominators and denominators from radicands (see Example 3).
3. Combine sums and differences of radicals, if possible (see Example 6).

APPENDIX A.1 EXERCISES

In Exercises 1–6, find the indicated real roots.

1. Square roots of 81 **2.** Fourth roots of 81

3. Cube roots of 64 **4.** Fifth roots of 243

5. Square roots of 16/9 **6.** Cube roots of $-27/8$

In Exercises 7–12, evaluate the expression without using a calculator.

7. $\sqrt{144}$ **8.** $\sqrt{-16}$ **9.** $\sqrt[3]{-216}$

10. $\sqrt[3]{216}$ **11.** $\sqrt[3]{-\frac{64}{27}}$ **12.** $\sqrt{\frac{64}{25}}$

In Exercises 13–22, use a calculator to evaluate the expression.

13. $\sqrt[4]{256}$ **14.** $\sqrt[5]{3125}$

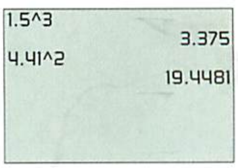
15. $\sqrt[3]{15.625}$ **16.** $\sqrt{12.25}$

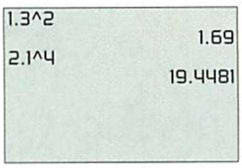
17. $81^{3/2}$ **18.** $16^{5/4}$

19. $32^{-2/5}$ **20.** $27^{-4/3}$

21. $\left(-\frac{1}{8}\right)^{-1/3}$ **22.** $\left(-\frac{125}{64}\right)^{-1/3}$

In Exercises 23–26, use the information from the grapher screens below to evaluate the expression.





23. $\sqrt{1.69}$ **24.** $\sqrt{19.4481}$

25. $\sqrt[4]{19.4481}$ **26.** $\sqrt[3]{3.375}$

In Exercises 27–36, simplify by removing factors from the radicand.

27. $\sqrt{288}$ **28.** $\sqrt[3]{500}$

29. $\sqrt[3]{-250}$ **30.** $\sqrt[4]{192}$

31. $\sqrt{2x^3y^4}$ **32.** $\sqrt[3]{-27x^3y^6}$

33. $\sqrt[4]{3x^8y^6}$ **34.** $\sqrt[3]{8x^6y^4}$

35. $\sqrt[5]{96x^{10}}$ **36.** $\sqrt{108x^4y^9}$

In Exercises 37–42, rationalize the denominator.

37. $\frac{4}{\sqrt[3]{2}}$ **38.** $\frac{1}{\sqrt{5}}$

39. $\frac{1}{\sqrt[5]{x^2}}$ **40.** $\frac{2}{\sqrt[4]{y}}$

41. $\sqrt[3]{\frac{x^2}{y}}$ **42.** $\sqrt[5]{\frac{a^3}{b^2}}$

In Exercises 43–46, convert to exponential form.

43. $\sqrt[3]{(a+2b)^2}$ **44.** $\sqrt[5]{x^2y^3}$

45. $2x\sqrt[3]{x^2y}$ **46.** $xy\sqrt[4]{xy^3}$

In Exercises 47–50, convert to radical form.

47. $a^{3/4}b^{1/4}$ **48.** $x^{2/3}y^{1/3}$

49. $x^{-5/3}$ **50.** $(xy)^{-3/4}$

In Exercises 51–56, write using a single radical.

51. $\sqrt{\sqrt{2x}}$ **52.** $\sqrt{\sqrt[3]{3x^2}}$

53. $\sqrt[4]{\sqrt{xy}}$ **54.** $\sqrt[3]{\sqrt{ab}}$

55. $\frac{\sqrt[5]{a^2}}{\sqrt[3]{a}}$ **56.** $\sqrt{a}\sqrt[3]{a^2}$

In Exercises 57–64, simplify the exponential expression.

57. $\dfrac{a^{3/5}a^{1/3}}{a^{3/2}}$

58. $(x^2y^4)^{1/2}$

59. $(a^{5/3}b^{3/4})(3a^{1/3}b^{5/4})$

60. $\left(\dfrac{x^{1/2}}{y^{2/3}}\right)^6$

61. $\left(\dfrac{-8x^6}{y^{-3}}\right)^{2/3}$

62. $\dfrac{(p^2q^4)^{1/2}}{(27q^3p^6)^{1/3}}$

63. $\dfrac{(x^9y^6)^{-1/3}}{(x^6y^2)^{-1/2}}$

64. $\left(\dfrac{2x^{1/2}}{y^{2/3}}\right)\left(\dfrac{3x^{-2/3}}{y^{1/2}}\right)$

In Exercises 65–74, simplify the radical expression.

65. $\sqrt{9x^{-6}y^4}$

66. $\sqrt{16y^8z^{-2}}$

67. $\sqrt[4]{\dfrac{3x^8y^2}{8x^2}}$

68. $\sqrt[5]{\dfrac{4x^6y}{9x^3}}$

69. $\sqrt[3]{\dfrac{4x^2}{y^2}} \cdot \sqrt[3]{\dfrac{2x^2}{y}}$

70. $\sqrt[5]{9ab^6} \cdot \sqrt[5]{27a^2b^{-1}}$

71. $3\sqrt{48} - 2\sqrt{108}$

72. $2\sqrt{175} - 4\sqrt{28}$

73. $\sqrt{x^3} - \sqrt{4xy^2}$

74. $\sqrt{18x^2y} + \sqrt{2y^3}$

In Exercises 75–82, replace ○ with <, =, or > to make a true statement.

75. $\sqrt{2+6} \bigcirc \sqrt{2} + \sqrt{6}$

76. $\sqrt{4} + \sqrt{9} \bigcirc \sqrt{4+9}$

77. $(3^{-2})^{-1/2} \bigcirc 3$

78. $(2^{-3})^{1/3} \bigcirc 2$

79. $\sqrt[4]{(-2)^4} \bigcirc -2$

80. $\sqrt[3]{(-2)^3} \bigcirc -2$

81. $2^{2/3} \bigcirc 3^{3/4}$

82. $4^{-2/3} \bigcirc 3^{-3/4}$

83. The time t (in seconds) that it takes for a pendulum to complete one period is approximately $t = 1.1\sqrt{L}$, where L is the length (in feet) of the pendulum. How long is the period of a pendulum of length 10 ft?

84. The time t (in seconds) that it takes for a rock to fall a distance d (in meters) is approximately $t = 0.45\sqrt{d}$. How long does it take for the rock to fall a distance of 200 m?

85. **Writing to Learn** Explain why $\sqrt[n]{a}$ and a real nth root of a need not have the same value.

A.2
Polynomials and Factoring

What you'll learn about

- Adding, Subtracting, and Multiplying Polynomials
- Special Products
- Factoring Polynomials Using Special Products
- Factoring Trinomials
- Factoring by Grouping

. . . and why

You need to review these basic algebraic skills if you don't remember them.

Adding, Subtracting, and Multiplying Polynomials

A **polynomial in** x is any expression that can be written in the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where n is a nonnegative integer, and $a_n \neq 0$. The numbers $a_{n-1}, \ldots, a_1, a_0$ are real numbers called **coefficients**. The **degree of the polynomial** is n and the **leading coefficient** is a_n. Polynomials with one, two, or three terms are **monomials**, **binomials**, or **trinomials**, respectively. A polynomial written with powers of x in *descending order* is in **standard form**.

To add or subtract polynomials, we add or subtract *like terms* using the distributive property. Terms of polynomials that have the same variable each raised to the same power are **like terms**.

EXAMPLE 1 Adding and Subtracting Polynomials

(a) $(2x^3 - 3x^2 + 4x - 1) + (x^3 + 2x^2 - 5x + 3)$

(b) $(4x^2 + 3x - 4) - (2x^3 + x^2 - x + 2)$

SOLUTION

(a) We group like terms and then combine them as follows:

$$(2x^3 + x^3) + (-3x^2 + 2x^2) + (4x + (-5x)) + (-1 + 3)$$
$$= 3x^3 - x^2 - x + 2$$

(b) We group like terms and then combine them as follows:

$$(0 - 2x^3) + (4x^2 - x^2) + (3x - (-x)) + (-4 - 2)$$
$$= -2x^3 + 3x^2 + 4x - 6$$

Now try Exercises 9 and 11.

To **expand the product** of two polynomials we use the distributive property. Here is what the procedure looks like when we multiply the binomials $3x + 2$ and $4x - 5$.

$(3x + 2)(4x - 5)$

$= 3x(4x - 5) + 2(4x - 5)$ — Distributive property

$= (3x)(4x) - (3x)(5) + (2)(4x) - (2)(5)$ — Distributive property

$$= \underbrace{12x^2}_{\text{Product of First terms}} - \underbrace{15x}_{\text{Product of Outer terms}} + \underbrace{8x}_{\text{Product of Inner terms}} - \underbrace{10}_{\text{Product of Last terms}}$$

In the above **FOIL method** for products of binomials, the outer (O) and inner (I) terms are like terms and can be added to give

$$(3x + 2)(4x - 5) = 12x^2 - 7x - 10.$$

Multiplying two polynomials requires multiplying each term of one polynomial by every term of the other polynomial. A convenient way to compute a product is to arrange the polynomials in standard form one on top of another so their terms align vertically, as illustrated in Example 2.

EXAMPLE 2 Multiplying Polynomials in Vertical Form

Write $(x^2 - 4x + 3)(x^2 + 4x + 5)$ in standard form.

SOLUTION

$$\begin{array}{rrrrrl} & x^2 & -4x & +3 & & \\ & x^2 & +4x & +5 & & \\ \hline x^4 - 4x^3 + & 3x^2 & & & & = x^2(x^2 - 4x + 3) \\ 4x^3 - & 16x^2 & +12x & & & = 4x(x^2 - 4x + 3) \\ & 5x^2 & -20x & +15 & & = 5(x^2 - 4x + 3) \\ \hline x^4 + 0x^3 - & 8x^2 & -8x & +15 & & \text{Add.} \end{array}$$

Thus,

$$(x^2 - 4x + 3)(x^2 + 4x + 5) = x^4 - 8x^2 - 8x + 15.$$

Now try Exercise 33.

Special Products

Certain products provide patterns that will be useful when we factor polynomials. Here is a list of some special products for binomials.

Special Binomial Products

Let u and v be real numbers, variables, or algebraic expressions.

1. **Product of a sum and a difference:** $(u + v)(u - v) = u^2 - v^2$
2. **Square of a sum:** $(u + v)^2 = u^2 + 2uv + v^2$
3. **Square of a difference:** $(u - v)^2 = u^2 - 2uv + v^2$
4. **Cube of a sum:** $(u + v)^3 = u^3 + 3u^2v + 3uv^2 + v^3$
5. **Cube of a difference:** $(u - v)^3 = u^3 - 3u^2v + 3uv^2 - v^3$

EXAMPLE 3 Using Special Products

Expand the products.

(a) $(3x + 8)(3x - 8) = (3x)^2 - 8^2$ — Product of a sum and a difference

$= 9x^2 - 64$ — Simplify.

(b) $(5y - 4)^2 = (5y)^2 - 2(5y)(4) + 4^2$ — Square of a difference

$= 25y^2 - 40y + 16$ — Simplify.

continued

(c) $(2x - 3y)^3 = (2x)^3 - 3(2x)^2(3y)$

$+ 3(2x)(3y)^2 - (3y)^3$ — Cube of a difference

$= 8x^3 - 36x^2y + 54xy^2 - 27y^3$ — Simplify.

Now try Exercises 23, 25, and 27.

Factoring Polynomials Using Special Products

When we write a polynomial as a product of two or more **polynomial factors** we are **factoring a polynomial**. Unless specified otherwise, we factor polynomials into factors of lesser degree and with integer coefficients in this appendix. A polynomial that cannot be factored using integer coefficients is a **prime polynomial**.

A polynomial is **completely factored** if it is written as a product of its prime factors. For example,

$$2x^2 + 7x - 4 = (2x - 1)(x + 4)$$

and

$$x^3 + x^2 + x + 1 = (x + 1)(x^2 + 1)$$

are completely factored (it can be shown that $x^2 + 1$ is prime). However,

$$x^3 - 9x = x(x^2 - 9)$$

is not completely factored because $(x^2 - 9)$ is *not* prime. In fact, $x^2 - 9 = (x - 3)(x + 3)$ and

$$x^3 - 9x = x(x - 3)(x + 3)$$

is completely factored.

The first step in factoring a polynomial is to remove common factors from its terms using the distributive property as illustrated by Example 4.

EXAMPLE 4 Removing Common Factors

(a) $2x^3 + 2x^2 - 6x = 2x(x^2 + x - 3)$ — $2x$ is the common factor.

(b) $u^3v + uv^3 = uv(u^2 + v^2)$ — uv is the common factor.

Now try Exercise 43.

Recognizing the expanded form of the five special binomial products will help us factor them. The special form that is easiest to identify is the difference of two squares. The two binomial factors have opposite signs:

Two squares — Square roots

$$u^2 - v^2 = (u + v)(u - v).$$

Difference — Opposite signs

EXAMPLE 5 Factoring the Difference of Two Squares

(a) $25x^2 - 36 = (5x)^2 - 6^2$ — Difference of two squares

$= (5x + 6)(5x - 6)$ — Factors are prime.

(b) $4x^2 - (y + 3)^2 = (2x)^2 - (y + 3)^2$ — Difference of two squares

$= [2x + (y + 3)][2x - (y + 3)]$ — Factors are prime.

$= (2x + y + 3)(2x - y - 3)$ — Simplify.

Now try Exercise 45.

A perfect square trinomial is the square of a binomial and has one of the two forms shown here. The first and last terms are squares of u and v, and the middle term is twice the product of u and v. The operation signs before the middle term and in the binomial factor are the same.

Perfect square (sum): $u^2 + 2uv + v^2 = (u + v)^2$ — Same signs

Perfect square (difference): $u^2 - 2uv + v^2 = (u - v)^2$ — Same signs

EXAMPLE 6 Factoring Perfect Square Trinomials

(a) $9x^2 + 6x + 1 = (3x)^2 + 2(3x)(1) + 1^2$ — $u = 3x,\ v = 1$

$= (3x + 1)^2$

(b) $4x^2 - 12xy + 9y^2 = (2x)^2 - 2(2x)(3y) + (3y)^2$ — $u = 2x,\ v = 3y$

$= (2x - 3y)^2$

Now try Exercise 49.

In the sum and difference of two cubes, notice the pattern of the signs.

$u^3 + v^3 = (u + v)(u^2 - uv + v^2)$ — Same signs; Opposite signs

$u^3 - v^3 = (u - v)(u^2 + uv + v^2)$ — Same signs; Opposite signs

EXAMPLE 7 Factoring the Sum and Difference of Two Cubes

(a) $x^3 - 64 = x^3 - 4^3$ — Difference of two cubes

$= (x - 4)(x^2 + 4x + 16)$ — Factors are prime.

(b) $8x^3 + 27 = (2x)^3 + 3^3$ — Sum of two cubes

$= (2x + 3)(4x^2 - 6x + 9)$ — Factors are prime.

Now try Exercise 55.

Factoring Trinomials

Factoring the trinomial $ax^2 + bx + c$ into a product of binomials with integer coefficients requires factoring the integers a and c.

Factors of a

$$ax^2 + bx + c = (\square x + \square)(\square x + \square)$$

Factors of c

Because the number of integer factors of a and c is finite, we can list all possible binomial factors. Then we begin checking each possibility until we find a pair that works. (If no pair works, then the trinomial is prime.) Example 8 illustrates.

EXAMPLE 8 Factoring a Trinomial with Leading Coefficient = 1

Factor $x^2 + 5x - 14$.

SOLUTION The only factor pair of the leading coefficient is 1 and 1. The factor pairs of 14 are 1 and 14, and 2 and 7. Here are the four possible factorizations of the trinomial:

$$(x + 1)(x - 14) \qquad (x - 1)(x + 14)$$
$$(x + 2)(x - 7) \qquad (x - 2)(x + 7)$$

If you check the middle term from each factorization you will find that

$x^2 + 5x - 14 = (x - 2)(x + 7)$. *Now try Exercise 59.*

With practice you will find that it usually is not necessary to list all possible binomial factors. Often you can test the possibilities mentally.

EXAMPLE 9 Factoring a Trinomial with Leading Coefficient ≠ 1

Factor $35x^2 - x - 12$.

SOLUTION The factor pairs of the leading coefficient are 1 and 35, and 5 and 7. The factor pairs of 12 are 1 and 12, 2 and 6, and 3 and 4. The possible factorizations must be of the form

$$(x - *)(35x + ?), \qquad (x + *)(35x - ?),$$
$$(5x - *)(7x + ?), \qquad (5x + *)(7x - ?),$$

where * and ? are one of the factor pairs of 12. Because the two binomial factors have opposite signs, there are 6 possibilities for each of the four forms—a total of 24 possibilities in all. If you try them, mentally and systematically, you should find that

$$35x^2 - x - 12 = (5x - 3)(7x + 4).$$

Now try Exercise 63.

We can extend the technique of Examples 8 and 9 to trinomials in two variables as illustrated in Example 10.

EXAMPLE 10 Factoring Trinomials in *x* and *y*

Factor $3x^2 - 7xy + 2y^2$.

SOLUTION The only way to get $-7xy$ as the middle term is with $3x^2 - 7xy + 2y^2 = (3x - ?y)(x - ?y)$.

The signs in the binomials must be negative because the coefficient of y^2 is positive *and* the coefficient of the middle term is negative. Checking the two possibilities, $(3x - y)(x - 2y)$ and $(3x - 2y)(x - y)$, shows that

$$3x^2 - 7xy + 2y^2 = (3x - y)(x - 2y).$$

Now try Exercise 67.

Factoring by Grouping

Notice that $(a + b)(c + d) = ac + ad + bc + bd$. If a polynomial with four terms is the product of two binomials, we can group terms to factor. There are only three ways to group the terms and two of them work. So, if two of the possibilities fail, then it is not factorable.

EXAMPLE 11 Factoring by Grouping

(a) $3x^3 + x^2 - 6x - 2$

$= (3x^3 + x^2) - (6x + 2)$	Group terms.
$= x^2(3x + 1) - 2(3x + 1)$	Factor each group.
$= (3x + 1)(x^2 - 2)$	Distributive property

(b) $2ac - 2ad + bc - bd$

$= (2ac - 2ad) + (bc - bd)$	Group terms.
$= 2a(c - d) + b(c - d)$	Factor each group.
$= (c - d)(2a + b)$	Distributive property

Now try Exercise 69.

Here is a checklist for factoring polynomials.

Factoring Polynomials

1. Look for common factors.
2. Look for special polynomial forms.
3. Use factor pairs.
4. If there are four terms, try grouping.

APPENDIX A.2 EXERCISES

In Exercises 1–4, write the polynomial in standard form and state its degree.

1. $2x - 1 + 3x^2$
2. $x^2 - 2x - 2x^3 + 1$
3. $1 - x^7$
4. $x^2 - x^4 + x - 3$

In Exercises 5–8, state whether the expression is a polynomial.

5. $x^3 - 2x^2 + x^{-1}$
6. $\dfrac{2x - 4}{x}$
7. $(x^2 + x + 1)^2$
8. $1 - 3x + x^4$

In Exercises 9–18, simplify the expression. Write your answer in standard form.

9. $(x^2 - 3x + 7) + (3x^2 + 5x - 3)$
10. $(-3x^2 - 5) - (x^2 + 7x + 12)$
11. $(4x^3 - x^2 + 3x) - (x^3 + 12x - 3)$
12. $-(y^2 + 2y - 3) + (5y^2 + 3y + 4)$
13. $2x(x^2 - x + 3)$
14. $y^2(2y^2 + 3y - 4)$
15. $-3u(4u - 1)$
16. $-4v(2 - 3v^3)$
17. $(2 - x - 3x^2)(5x)$
18. $(1 - x^2 + x^4)(2x)$

In Exercises 19–40, expand the product. Use vertical alignment in Exercises 33 and 34.

19. $(x - 2)(x + 5)$
20. $(2x + 3)(4x + 1)$
21. $(3x - 5)(x + 2)$
22. $(2x - 3)(2x + 3)$
23. $(3x - y)(3x + y)$
24. $(3 - 5x)^2$
25. $(3x + 4y)^2$
26. $(x - 1)^3$
27. $(2u - v)^3$
28. $(u + 3v)^3$
29. $(2x^3 - 3y)(2x^3 + 3y)$
30. $(5x^3 - 1)^2$
31. $(x^2 - 2x + 3)(x + 4)$
32. $(x^2 + 3x - 2)(x - 3)$
33. $(x^2 + x - 3)(x^2 + x + 1)$
34. $(2x^2 - 3x + 1)(x^2 - x + 2)$
35. $(x - \sqrt{2})(x + \sqrt{2})$
36. $(x^{1/2} - y^{1/2})(x^{1/2} + y^{1/2})$
37. $(\sqrt{u} + \sqrt{v})(\sqrt{u} - \sqrt{v})$
38. $(x^2 - \sqrt{3})(x^2 + \sqrt{3})$
39. $(x - 2)(x^2 + 2x + 4)$
40. $(x + 1)(x^2 - x + 1)$

In Exercises 41–44, factor out the common factor.

41. $5x - 15$
42. $5x^3 - 20x$
43. $yz^3 - 3yz^2 + 2yz$
44. $2x(x + 3) - 5(x + 3)$

In Exercises 45–48, factor the difference of two squares.

45. $z^2 - 49$
46. $9y^2 - 16$
47. $64 - 25y^2$
48. $16 - (x + 2)^2$

In Exercises 49–52, factor the perfect square trinomial.

49. $y^2 + 8y + 16$
50. $36y^2 + 12y + 1$
51. $4z^2 - 4z + 1$
52. $9z^2 - 24z + 16$

In Exercises 53–58, factor the sum or difference of two cubes.

53. $y^3 - 8$
54. $z^3 + 64$
55. $27y^3 - 8$
56. $64z^3 + 27$
57. $1 - x^3$
58. $27 - y^3$

In Exercises 59–68, factor the trinomial.

59. $x^2 + 9x + 14$
60. $y^2 - 11y + 30$
61. $z^2 - 5z - 24$
62. $6t^2 + 5t + 1$
63. $14u^2 - 33u - 5$
64. $10v^2 + 23v + 12$
65. $12x^2 + 11x - 15$
66. $2x^2 - 3xy + y^2$
67. $6x^2 + 11xy - 10y^2$
68. $15x^2 + 29xy - 14y^2$

In Exercises 69–74, factor by grouping.

69. $x^3 - 4x^2 + 5x - 20$
70. $2x^3 - 3x^2 + 2x - 3$
71. $x^6 - 3x^4 + x^2 - 3$
72. $x^6 + 2x^4 + x^2 + 2$
73. $2ac + 6ad - bc - 3bd$
74. $3uw + 12uz - 2vw - 8vz$

In Exercises 75–90, factor completely.

75. $x^3 + x$
76. $4y^3 - 20y^2 + 25y$
77. $18y^3 + 48y^2 + 32y$
78. $2x^3 - 16x^2 + 14x$
79. $16y - y^3$
80. $3x^4 + 24x$
81. $5y + 3y^2 - 2y^3$
82. $z - 8z^4$
83. $2(5x + 1)^2 - 18$
84. $5(2x - 3)^2 - 20$
85. $12x^2 + 22x - 20$
86. $3x^2 + 13xy - 10y^2$
87. $2ac - 2bd + 4ad - bc$
88. $6ac - 2bd + 4bc - 3ad$
89. $x^3 - 3x^2 - 4x + 12$
90. $x^4 - 4x^3 - x^2 + 4x$

91. Writing to Learn Show that the grouping

$$(2ac + bc) - (2ad + bd)$$

leads to the same factorization as in Example 11b. Explain why the third possibility,

$$(2ac - bd) + (-2ad + bc)$$

does not lead to a factorization.

A.3 Fractional Expressions

What you'll learn about

- Domain of an Algebraic Expression
- Reducing Rational Expressions
- Operations with Rational Expressions
- Compound Rational Expressions

. . . and why

You need to review these basic algebraic skills if you don't remember them.

Domain of an Algebraic Expression

A quotient of two algebraic expressions, besides being another algebraic expression, is a **fractional expression**, or simply a fraction. If the quotient can be written as the ratio of two polynomials, the fractional expression is a **rational expression**. Here are examples of each.

$$\frac{x^2 - 5x + 2}{\sqrt{x^2 + 1}} \qquad \frac{2x^3 - x^2 + 1}{5x^2 - x - 3}$$

The one on the left is a fractional expression but not a rational expression. The other is both a fractional expression and a rational expression.

Unlike polynomials, which are defined for all real numbers, some algebraic expressions are not defined for some real numbers. The set of real numbers for which an algebraic expression is defined is the **domain of the algebraic expression**.

EXAMPLE 1 Finding Domains of Algebraic Expressions

(a) $3x^2 - x + 5$ **(b)** $\sqrt{x - 1}$ **(c)** $\dfrac{x}{x - 2}$

SOLUTION

(a) The domain of $3x^2 - x + 5$, like that of any polynomial, is the set of all real numbers.

(b) Because only nonnegative numbers have square roots, $x - 1 \geq 0$, or $x \geq 1$. In interval notation, the domain is $[1, \infty)$.

(c) Because division by zero is undefined, $x - 2 \neq 0$, or $x \neq 2$. The domain is the set of all real numbers except 2. *Now try Exercises 11 and 13.*

Reducing Rational Expressions

Let u, v, and z be real numbers, variables, or algebraic expressions. We can write rational expressions in simpler form using

$$\frac{uz}{vz} = \frac{u}{v}$$

provided $z \neq 0$. This requires that we first factor the numerator and denominator into prime factors. When all factors common to numerator and denominator have been removed, the rational expression (or rational number) is in **reduced form**.

EXAMPLE 2 Reducing Rational Expressions

Write $(x^2 - 3x)/(x^2 - 9)$ in reduced form.

SOLUTION

$$\frac{x^2 - 3x}{x^2 - 9} = \frac{x(x - 3)}{(x + 3)(x - 3)} \qquad \text{Factor completely.}$$

$$= \frac{x}{x + 3}, \quad x \neq 3 \qquad \text{Remove common factors.}$$

continued

We include $x \neq 3$ as part of the reduced form because 3 is not in the domain of the original rational expression and thus should not be in the domain of the final rational expression. *Now try Exercise 35.*

Two rational expressions are **equivalent** if they have the same domain and have the same value for all numbers in the domain. The reduced form of a rational expression must have the same domain as the original rational expression. This is why we attached the restriction $x \neq 3$ to the reduced form in Example 2.

Operations with Rational Expressions

Two fractions are **equal**, $u/v = z/w$, if and only if $uw = vz$. Here is how we operate with fractions.

Operations with Fractions

Let u, v, w, and z be real numbers, variables, or algebraic expressions. All of the denominators are assumed to be different from zero.

Operation	Example
1. $\frac{u}{v} + \frac{w}{v} = \frac{u+w}{v}$	$\frac{2}{3} + \frac{5}{3} = \frac{2+5}{3} = \frac{7}{3}$
2. $\frac{u}{v} + \frac{w}{z} = \frac{uz+vw}{vz}$	$\frac{2}{3} + \frac{4}{5} = \frac{2 \cdot 5 + 3 \cdot 4}{3 \cdot 5} = \frac{22}{15}$
3. $\frac{u}{v} \cdot \frac{w}{z} = \frac{uw}{vz}$	$\frac{2}{3} \cdot \frac{4}{5} = \frac{2 \cdot 4}{3 \cdot 5} = \frac{8}{15}$
4. $\frac{u}{v} \div \frac{w}{z} = \frac{u}{v} \cdot \frac{z}{w} = \frac{uz}{vw}$	$\frac{2}{3} \div \frac{4}{5} = \frac{2}{3} \cdot \frac{5}{4} = \frac{10}{12} = \frac{5}{6}$

5. For subtraction, replace "+" by "−" in 1 and 2.

INVERT AND MULTIPLY

The division step shown in 4 is often referred to as invert the divisor (the fraction following the division symbol) and multiply the result times the numerator (the first fraction).

EXAMPLE 3 Multiplying and Dividing Rational Expressions

(a) $\frac{2x^2 + 11x - 21}{x^3 + 2x^2 + 4x} \cdot \frac{x^3 - 8}{x^2 + 5x - 14}$

$= \frac{(2x-3)(x+7)}{x(x^2+2x+4)} \cdot \frac{(x-2)(x^2+2x+4)}{(x-2)(x+7)}$ Factor completely.

$= \frac{2x-3}{x}, \quad x \neq 2, \quad x \neq -7$ Remove common factors.

(b) $\frac{x^3 + 1}{x^2 - x - 2} \div \frac{x^2 - x + 1}{x^2 - 4x + 4}$

$= \frac{(x^3+1)(x^2-4x+4)}{(x^2-x-2)(x^2-x+1)}$ Invert and multiply.

$= \frac{(x+1)(x^2-x+1)(x-2)^{2}}{(x+1)(x-2)(x^2-x+1)}$ Factor completely.

$= x - 2, \quad x \neq -1, \quad x \neq 2$ Remove common factors.

Now try Exercises 49 and 55.

NOTE ON EXAMPLE

The numerator, $x^2 + 4x - 6$, of the final expression in Example 4 is a prime polynomial. Thus, there are no common factors.

EXAMPLE 4 Adding Rational Expressions

$$\frac{x}{3x-2} + \frac{3}{x-5} = \frac{x(x-5) + 3(3x-2)}{(3x-2)(x-5)} \quad \text{Definition of addition}$$

$$= \frac{x^2 - 5x + 9x - 6}{(3x-2)(x-5)} \quad \text{Distributive property}$$

$$= \frac{x^2 + 4x - 6}{(3x-2)(x-5)} \quad \text{Combine like terms.}$$

Now try Exercise 59.

If the denominators of fractions have common factors, then it is often more efficient to find the LCD before adding or subtracting the fractions. The **LCD (least common denominator)** is the product of all the prime factors in the denominators, where each factor is raised to the greatest power found in any one denominator for that factor.

EXAMPLE 5 Using the LCD

Write the following expression as a fraction in reduced form.

$$\frac{2}{x^2 - 2x} + \frac{1}{x} - \frac{3}{x^2 - 4}$$

SOLUTION The factored denominators are $x(x-2)$, x, and $(x-2)(x+2)$, respectively. The LCD is $x(x-2)(x+2)$.

$$\frac{2}{x^2 - 2x} + \frac{1}{x} - \frac{3}{x^2 - 4}$$

$$= \frac{2}{x(x-2)} + \frac{1}{x} - \frac{3}{(x-2)(x+2)} \quad \text{Factor.}$$

$$= \frac{2(x+2)}{x(x-2)(x+2)} + \frac{(x-2)(x+2)}{x(x-2)(x+2)} - \frac{3x}{x(x-2)(x+2)} \quad \text{Equivalent fractions}$$

$$= \frac{2(x+2) + (x-2)(x+2) - 3x}{x(x-2)(x+2)} \quad \text{Combine numerators.}$$

$$= \frac{2x + 4 + x^2 - 4 - 3x}{x(x-2)(x+2)} \quad \text{Expand terms.}$$

$$= \frac{x^2 - x}{x(x-2)(x+2)} \quad \text{Simplify.}$$

$$= \frac{x(x-1)}{x(x-2)(x+2)} \quad \text{Factor.}$$

$$= \frac{x-1}{(x-2)(x+2)}, \quad x \neq 0 \quad \text{Reduce.}$$

Now try Exercise 61.

Compound Rational Expressions

Sometimes a complicated algebraic expression needs to be changed to a more familiar form before we can work on it. A **compound fraction** (sometimes called a **complex fraction**), in which the numerators and denominators may themselves contain fractions, is such an example. One way to simplify a compound fraction is to write both the numerator and denominator as single fractions and then invert and multiply. If the fraction then takes the form of a rational expression, then we write the expression in reduced or simplest form.

EXAMPLE 6 Simplifying a Compound Fraction

$$\frac{3 - \dfrac{7}{x+2}}{1 - \dfrac{1}{x-3}} = \frac{\dfrac{3(x+2) - 7}{x+2}}{\dfrac{(x-3) - 1}{x-3}} \qquad \text{Combine fractions.}$$

$$= \frac{\dfrac{3x-1}{x+2}}{\dfrac{x-4}{x-3}} \qquad \text{Simplify.}$$

$$= \frac{(3x-1)(x-3)}{(x+2)(x-4)}, \quad x \neq 3 \qquad \text{Invert and multiply.}$$

Now try Exercise 63.

A second way to simplify a compound fraction is to multiply the numerator and denominator by the LCD of all fractions in the numerator and denominator as illustrated in Example 7.

EXAMPLE 7 Simplifying Another Compound Fraction

Use the LCD to simplify the compound fraction

$$\frac{\dfrac{1}{a^2} - \dfrac{1}{b^2}}{\dfrac{1}{a} - \dfrac{1}{b}}.$$

SOLUTION The LCD of the four fractions in the numerator and denominator is a^2b^2.

$$\frac{\dfrac{1}{a^2} - \dfrac{1}{b^2}}{\dfrac{1}{a} - \dfrac{1}{b}} = \frac{\left(\dfrac{1}{a^2} - \dfrac{1}{b^2}\right)a^2b^2}{\left(\dfrac{1}{a} - \dfrac{1}{b}\right)a^2b^2} \qquad \text{Multiply numerator and}$$

$$= \frac{b^2 - a^2}{ab^2 - a^2b} \qquad \text{Simplify.}$$

$$= \frac{(b+a)(b-a)}{ab(b-a)} \qquad \text{Factor.}$$

$$= \frac{b+a}{ab}, \quad a \neq b \qquad \text{Reduce.}$$

Now try Exercise 69.

APPENDIX A.3 EXERCISES

In Exercises 1–8, rewrite as a single fraction.

1. $\frac{5}{9}+\frac{10}{9}$
2. $\frac{17}{32}-\frac{9}{32}$
3. $\frac{20}{21}\cdot\frac{9}{22}$
4. $\frac{33}{25}\cdot\frac{20}{77}$
5. $\frac{2}{3}\div\frac{4}{5}$
6. $\frac{9}{4}\div\frac{15}{10}$
7. $\frac{1}{14}+\frac{4}{15}-\frac{5}{21}$
8. $\frac{1}{6}+\frac{6}{35}-\frac{4}{15}$

In Exercises 9–18, find the domain of the algebraic expression.

9. $5x^2-3x-7$
10. $2x-5$
11. $\sqrt{x-4}$
12. $\frac{2}{\sqrt{x+3}}$
13. $\frac{2x+1}{x^2+3x}$
14. $\frac{x^2-2}{x^2-4}$
15. $\frac{x}{x-1},\quad x\neq 2$
16. $\frac{3x-1}{x-2},\quad x\neq 0$
17. x^2+x^{-1}
18. $x(x+1)^{-2}$

In Exercises 19–26, find the missing numerator or denominator so that the two rational expressions are equal.

19. $\frac{2}{3x}=\frac{?}{12x^3}$
20. $\frac{5}{2y}=\frac{15y}{?}$
21. $\frac{x-4}{x}=\frac{x^2-4x}{?}$
22. $\frac{x}{x+2}=\frac{?}{x^2-4}$
23. $\frac{x+3}{x-2}=\frac{?}{x^2+2x-8}$
24. $\frac{x-4}{x+5}=\frac{x^2-x-12}{?}$
25. $\frac{x^2-3x}{?}=\frac{x-3}{x^2+2x}$
26. $\frac{?}{x^2-9}=\frac{x^2+x-6}{x-3}$

In Exercises 27–32, consider the original fraction and its reduced form from the specified example. Explain why the given restriction is needed on the reduced form.

27. Example 3a, $x\neq 2, x\neq -7$
28. Example 3b, $x\neq -1, x\neq 2$
29. Example 4, none
30. Example 5, $x\neq 0$
31. Example 6, $x\neq 3$
32. Example 7, $a\neq b$

In Exercises 33–44, write the expression in reduced form.

33. $\frac{18x^3}{15x}$
34. $\frac{75y^2}{9y^4}$
35. $\frac{x^3}{x^2-2x}$
36. $\frac{2y^2+6y}{4y+12}$
37. $\frac{z^2-3z}{9-z^2}$
38. $\frac{x^2+6x+9}{x^2-x-12}$
39. $\frac{y^2-y-30}{y^2-3y-18}$
40. $\frac{y^3+4y^2-21y}{y^2-49}$
41. $\frac{8z^3-1}{2z^2+5z-3}$
42. $\frac{2z^3+6z^2+18z}{z^3-27}$
43. $\frac{x^3+2x^2-3x-6}{x^3+2x^2}$
44. $\frac{y^2+3y}{y^3+3y^2-5y-15}$

In Exercises 45–62, simplify.

45. $\frac{3}{x-1}\cdot\frac{x^2-1}{9}$
46. $\frac{x+3}{7}\cdot\frac{14}{2x+6}$
47. $\frac{x+3}{x-1}\cdot\frac{1-x}{x^2-9}$
48. $\frac{18x^2-3x}{3xy}\cdot\frac{12y^2}{6x-1}$
49. $\frac{x^3-1}{2x^2}\cdot\frac{4x}{x^2+x+1}$
50. $\frac{y^3+2y^2+4y}{y^3+2y^2}\cdot\frac{y^2-4}{y^3-8}$
51. $\frac{2y^2+9y-5}{y^2-25}\cdot\frac{y-5}{2y^2-y}$
52. $\frac{y^2+8y+16}{3y^2-y-2}\cdot\frac{3y^2+2y}{y+4}$
53. $\frac{1}{2x}\div\frac{1}{4}$
54. $\frac{4x}{y}\div\frac{8y}{x}$
55. $\frac{x^2-3x}{14y}\div\frac{2xy}{3y^2}$
56. $\frac{7x-7y}{4y}\div\frac{14x-14y}{3y}$
57. $\dfrac{\frac{2x^2y}{(x-3)^2}}{\frac{8xy}{x-3}}$
58. $\dfrac{\frac{x^2-y^2}{2xy}}{\frac{y^2-x^2}{4x^2y}}$
59. $\frac{2x+1}{x+5}-\frac{3}{x+5}$
60. $\frac{3}{x-2}+\frac{x+1}{x-2}$
61. $\frac{3}{x^2+3x}-\frac{1}{x}-\frac{6}{x^2-9}$
62. $\frac{5}{x^2+x-6}-\frac{2}{x-2}+\frac{4}{x^2-4}$

In Exercises 63–70, simplify the compound fraction.

63. $\dfrac{\frac{x}{y^2}-\frac{y}{x^2}}{\frac{1}{y^2}-\frac{1}{x^2}}$
64. $\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x^2}-\frac{1}{y^2}}$
65. $\dfrac{2x+\frac{13x-3}{x-4}}{2x+\frac{x+3}{x-4}}$
66. $\dfrac{2-\frac{13}{x+5}}{2+\frac{3}{x-3}}$
67. $\dfrac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$
68. $\dfrac{\frac{x+h}{x+h+2}-\frac{x}{x+2}}{h}$
69. $\dfrac{\frac{b}{a}-\frac{a}{b}}{\frac{1}{a}-\frac{1}{b}}$
70. $\dfrac{\frac{1}{a}+\frac{1}{b}}{\frac{b}{a}-\frac{a}{b}}$

In Exercises 71–74, write with positive exponents and simplify.

71. $\left(\frac{1}{x}+\frac{1}{y}\right)(x+y)^{-1}$
72. $\frac{(x+y)^{-1}}{(x-y)^{-1}}$
73. $x^{-1}+y^{-1}$
74. $(x^{-1}+y^{-1})^{-1}$

B
Key Formulas

B.1 Formulas from Algebra

Exponents

If all bases are nonzero:

$u^m u^n = u^{m+n}$ $\qquad \dfrac{u^m}{u^n} = u^{m-n}$

$u^0 = 1$ $\qquad u^{-n} = \dfrac{1}{u^n}$

$(uv)^m = u^m v^m$ $\qquad (u^m)^n = u^{mn}$

$\left(\dfrac{u}{v}\right)^m = \dfrac{u^m}{v^m}$

Radicals and Rational Exponents

If all roots are real numbers:

$\sqrt[n]{uv} = \sqrt[n]{u} \cdot \sqrt[n]{v}$ $\qquad \sqrt[n]{\dfrac{u}{v}} = \dfrac{\sqrt[n]{u}}{\sqrt[n]{v}}\ (v \neq 0)$

$\sqrt[m]{\sqrt[n]{u}} = \sqrt[mn]{u}$ $\qquad (\sqrt[n]{u})^n = u$

$\sqrt[n]{u^m} = (\sqrt[n]{u})^m$ $\qquad \sqrt[n]{u^n} = \begin{cases} |u| & n \text{ even} \\ u & n \text{ odd} \end{cases}$

$u^{1/n} = \sqrt[n]{u}$ $\qquad u^{m/n} = (u^{1/n})^m = (\sqrt[n]{u})^m$

$u^{m/n} = (u^m)^{1/n} = \sqrt[n]{u^m}$

Special Products

$(u + v)(u - v) = u^2 - v^2$

$(u + v)^2 = u^2 + 2uv + v^2$

$(u - v)^2 = u^2 - 2uv + v^2$

$(u + v)^3 = u^3 + 3u^2v + 3uv^2 + v^3$

$(u - v)^3 = u^3 - 3u^2v + 3uv^2 - v^3$

Factoring Polynomials

$u^2 - v^2 = (u + v)(u - v)$

$u^2 + 2uv + v^2 = (u + v)^2$

$u^2 - 2uv + v^2 = (u - v)^2$

$u^3 + v^3 = (u + v)(u^2 - uv + v^2)$

$u^3 - v^3 = (u - v)(u^2 + uv + v^2)$

Inequalities

If $u < v$ and $v < w$, then $u < w$.

If $u < v$, then $u + w < v + w$.

If $u < v$ and $c > 0$, then $uc < vc$.

If $u < v$ and $c < 0$, then $uc > vc$.

If $c > 0$, $|u| < c$ is equivalent to $-c < u < c$.

If $c > 0$, $|u| > c$ is equivalent to $u < -c$ or $u > c$.

Quadratic Formula

If $a \neq 0$, the solutions of the equation $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Logarithms

If $0 < b \neq 1, 0 < a \neq 1, x, R, S, > 0$

$y = \log_b x$ if and only if $b^y = x$

$\log_b 1 = 0$ $\qquad \log_b b = 1$

$\log_b b^y = y$ $\qquad b^{\log_b x} = x$

$\log_b RS = \log_b R + \log_b S$ $\qquad \log_b \dfrac{R}{S} = \log_b R - \log_b S$

$\log_b R^c = c \log_b R$ $\qquad \log_b x = \dfrac{\log_a x}{\log_a b}$

Determinants

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Arithmetic Sequences and Series

$a_n = a_1 + (n - 1)d$

$S_n = n\left(\dfrac{a_1 + a_n}{2}\right)$ or $S_n = \dfrac{n}{2}[2a_1 + (n - 1)d]$

Geometric Sequences and Series

$a_n = a_1 \cdot r^{n-1}$

$S_n = \dfrac{a_1(1 - r^n)}{1 - r}\ (r \neq 1)$

$S = \dfrac{a_1}{1 - r}$ $(|r| < 1)$ infinite geometric series

Factorial

$n! = n \cdot (n - 1) \cdot (n - 2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$

$n \cdot (n - 1)! = n!, 0! = 1$

Binomial Coefficient

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \text{ (integers } n \text{ and } r,\ n \geq r \geq 0)$$

Binomial Theorem

If n is a positive integer

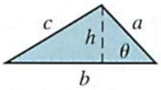
$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n}b^n$$

B.2 Formulas from Geometry

Triangle

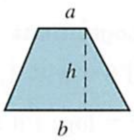
$h = a \sin \theta$

Area $= \dfrac{1}{2}bh$



Trapezoid

Area $= \dfrac{h}{2}(a+b)$



Circle

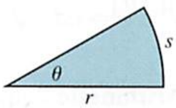
Area $= \pi r^2$

Circumference $= 2\pi r$

Sector of Circle

Area $= \dfrac{\theta r^2}{2}$ (θ in radians)

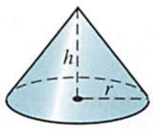
$s = r\theta$ (θ in radians)



Right Circular Cone

Volume $= \dfrac{\pi r^2 h}{3}$

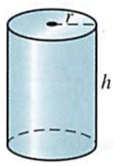
Lateral surface area $= \pi r\sqrt{r^2+h^2}$



Right Circular Cylinder

Volume $= \pi r^2 h$

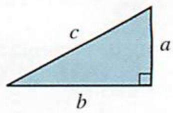
Lateral surface area $= 2\pi rh$



Right Triangle

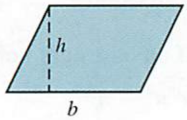
Pythagorean Theorem:

$c^2 = a^2 + b^2$



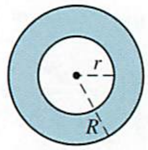
Parallelogram

Area $= bh$



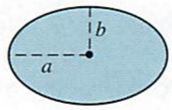
Circular Ring

Area $= \pi(R^2 - r^2)$



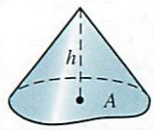
Ellipse

Area $= \pi ab$



Cone

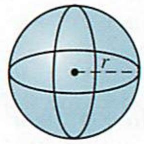
Volume $= \dfrac{Ah}{3}$ (A = Area of base)



Sphere

Volume $= \dfrac{4}{3}\pi r^3$

Surface area $= 4\pi r^2$



B.3 Formulas from Trigonometry

Angular Measure

π radians $= 180°$

So, 1 radian $= \dfrac{180}{\pi}$ degrees,

and 1 degree $= \dfrac{\pi}{180}$ radians.

Reciprocal Identities

$\sin x = \dfrac{1}{\csc x}$ $\qquad$ $\csc x = \dfrac{1}{\sin x}$

$\cos x = \dfrac{1}{\sec x}$ $\qquad$ $\sec x = \dfrac{1}{\cos x}$

$\tan x = \dfrac{1}{\cot x}$ $\qquad$ $\cot x = \dfrac{1}{\tan x}$

Quotient Identities

$\tan x = \dfrac{\sin x}{\cos x}$ $\qquad$ $\cot x = \dfrac{\cos x}{\sin x}$

Pythagorean Identities

$\sin^2 x + \cos^2 x = 1$

$1 + \tan^2 x = \sec^2 x$

$1 + \cot^2 x = \csc^2 x$

Odd-Even Identities

$\sin(-x) = -\sin x$ $\qquad$ $\csc(-x) = -\csc x$

$\cos(-x) = \cos x$ $\qquad$ $\sec(-x) = \sec x$

$\tan(-x) = -\tan x$ $\qquad$ $\cot(-x) = -\cot x$

Sum and Difference Identities

$\sin(u + v) = \sin u \cos v + \cos u \sin v$

$\sin(u - v) = \sin u \cos v - \cos u \sin v$

$\cos(u + v) = \cos u \cos v - \sin u \sin v$

$\cos(u - v) = \cos u \cos v + \sin u \sin v$

$$\tan(u + v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$$

$$\tan(u - v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$$

Cofunction Identities

$\cos\left(\dfrac{\pi}{2} - u\right) = \sin u$

$\sin\left(\dfrac{\pi}{2} - u\right) = \cos u$

$\tan\left(\dfrac{\pi}{2} - u\right) = \cot u$

$\cot\left(\dfrac{\pi}{2} - u\right) = \tan u$

$\sec\left(\dfrac{\pi}{2} - u\right) = \csc u$

$\csc\left(\dfrac{\pi}{2} - u\right) = \sec u$

Double-Angle Identities

$\sin 2u = 2 \sin u \cos u$

$$\begin{aligned}\cos 2u &= \cos^2 u - \sin^2 u \\ &= 2\cos^2 u - 1 \\ &= 1 - 2\sin^2 u\end{aligned}$$

$$\tan 2u = \frac{2 \tan u}{1 - \tan^2 u}$$

Power-Reducing Identities

$$\sin^2 u = \frac{1 - \cos 2u}{2}$$

$$\cos^2 u = \frac{1 + \cos 2u}{2}$$

$$\tan^2 u = \frac{1 - \cos 2u}{1 + \cos 2u}$$

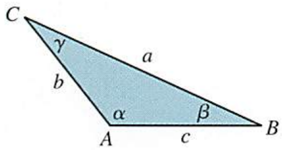
Half-Angle Identities

$$\sin \frac{u}{2} = \pm\sqrt{\frac{1 - \cos u}{2}}$$

$$\cos \frac{u}{2} = \pm\sqrt{\frac{1 + \cos u}{2}}$$

$$\tan \frac{u}{2} = \pm\sqrt{\frac{1 - \cos u}{1 + \cos u}} = \frac{1 - \cos u}{\sin u} = \frac{\sin u}{1 + \cos u}$$

Triangles



Law of sines:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Law of cosines:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Area:

$$\text{Area} = \frac{1}{2} bc \sin A$$

$$= \frac{1}{2} ac \sin B = \frac{1}{2} ab \sin C$$

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a + b + c)$

Trigonometric Form of a Complex Number

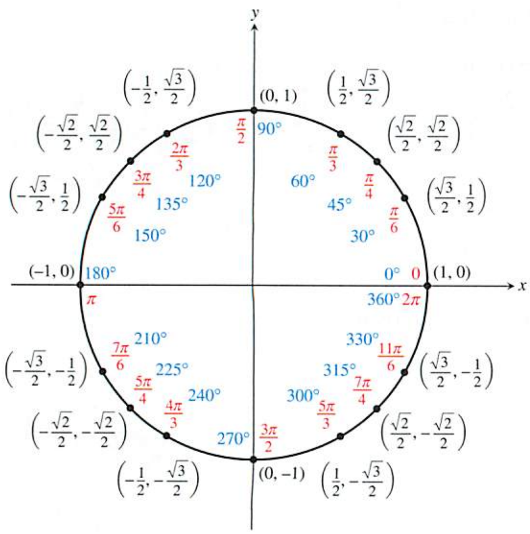
$$z = a + bi = (r \cos \theta) + (r \sin \theta)i$$

$$= r(\cos \theta + i \sin \theta)$$

De Moivre's Theorem

$$z^n = [r(\cos \theta + i \sin \theta)]^n$$

$$= r^n(\cos n\theta + i \sin n\theta)$$



B.4 Formulas from Analytic Geometry

Basic Formulas

Distance d between points $P(x_1, y_1)$ and $Q(x_2, y_2)$:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Midpoint: $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Slope of a line: $m = \frac{y_2 - y_1}{x_2 - x_1}$

Condition for parallel lines: $m_1 = m_2$

Condition for perpendicular lines: $m_2 = \frac{-1}{m_1}$

Equations of a Line

The point-slope form, slope m and through (x_1, y_1):

$$y - y_1 = m(x - x_1)$$

The slope-intercept form, slope m and y-intercept b:

$$y = mx + b$$

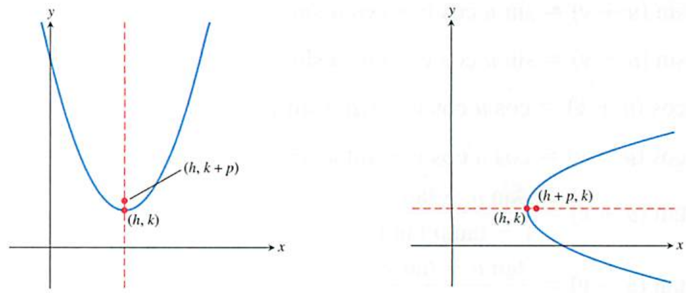
Equation of a Circle

The circle with center (h, k) and radius r:

$$(x - h)^2 + (y - k)^2 = r^2$$

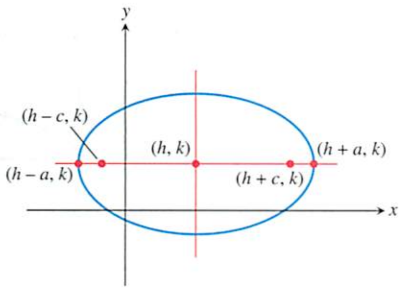
Parabolas with Vertex (h, k)

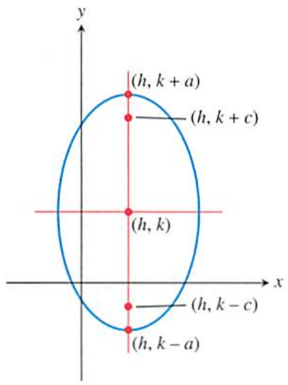
Standard equation	$(x - h)^2 = 4p(y - k)$	$(y - k)^2 = 4p(x - h)$
Opens	Upward or downward	To the right or to the left
Focus	$(h, k + p)$	$(h + p, k)$
Directrix	$y = k - p$	$x = h - p$
Axis	$x = h$	$y = k$



Ellipses with Center (h, k) and $a > b > 0$

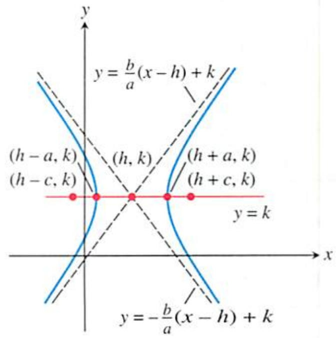
Standard equation	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$	$\frac{(y-k)^2}{a^2}+\frac{(x-h)^2}{b^2}=1$
Focal axis	$y = k$	$x = h$
Foci	$(h \pm c, k)$	$(h, k \pm c)$
Vertices	$(h \pm a, k)$	$(h, k \pm a)$
Pythagorean relation	$a^2 = b^2 + c^2$	$a^2 = b^2 + c^2$

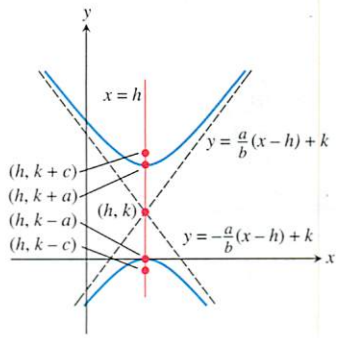




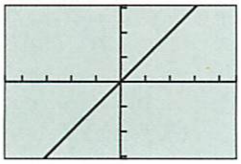
Hyperbolas with Center (h, k)

Standard equation	$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$	$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$
Focal axis	$y = k$	$x = h$
Foci	$(h \pm c, k)$	$(h, k \pm c)$
Vertices	$(h \pm a, k)$	$(h, k \pm a)$
Pythagorean relation	$c^2 = a^2 + b^2$	$c^2 = a^2 + b^2$
Asymptotes	$y = \pm\frac{b}{a}(x-h)+k$	$y = \pm\frac{a}{b}(x-h)+k$



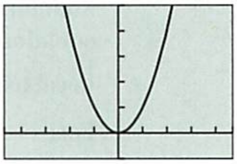


B.5 Gallery of Basic Functions



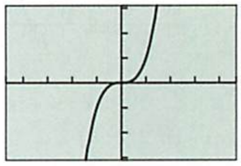
[–4.7, 4.7] by [–3.1, 3.1]

Identity Function
$f(x) = x$
Domain $= (-\infty, \infty)$
Range $= (-\infty, \infty)$



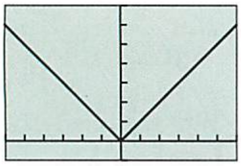
[–4.7, 4.7] by [–1, 5]

Squaring Function
$f(x) = x^2$
Domain $= (-\infty, \infty)$
Range $= [0, \infty)$



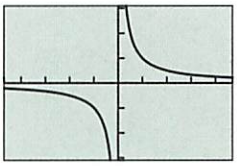
[–4.7, 4.7] by [–3.1, 3.1]

Cubing Function
$f(x) = x^3$
Domain $= (-\infty, \infty)$
Range $= (-\infty, \infty)$



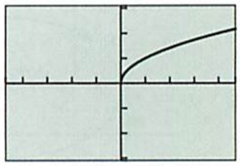
[–6, 6] by [–1, 7]

Absolute Value Function
$f(x) = |x| = \text{abs}(x)$
Domain $= (-\infty, \infty)$
Range $= [0, \infty)$



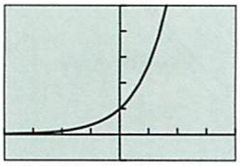
[–4.7, 4.7] by [–3.1, 3.1]

Reciprocal Function
$f(x) = \frac{1}{x}$
Domain $= (-\infty, 0) \cup (0, \infty)$
Range $= (-\infty, 0) \cup (0, \infty)$



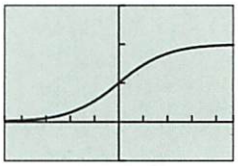
[–4.7, 4.7] by [–3.1, 3.1]

Square Root Function
$f(x) = \sqrt{x}$
Domain $= [0, \infty)$
Range $= [0, \infty)$



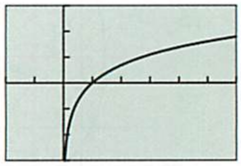
[–4, 4] by [–1, 5]

Exponential Function
$f(x) = e^x$
Domain $= (-\infty, \infty)$
Range $= (0, \infty)$



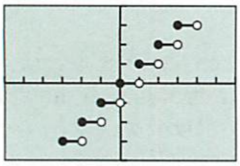
[–4.7, 4.7] by [–0.5, 1.5]

Logistic Function
$f(x) = \frac{1}{1 + e^{-x}}$
Domain $= (-\infty, \infty)$
Range $= (0, 1)$



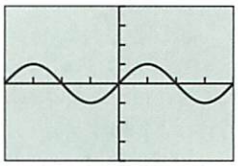
[–2, 6] by [–3, 3]

Natural Logarithmic Function
$f(x) = \ln x$
Domain $= (0, \infty)$
Range $= (-\infty, \infty)$



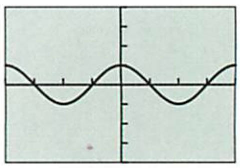
[–6, 6] by [–4, 4]

Greatest Integer Function
$f(x) = \text{int}(x)$
Domain $= (-\infty, \infty)$
Range = all integers



[-2π, 2π] by [–4, 4]

Sine Function
$f(x) = \sin(x)$
Domain $= (-\infty, \infty)$
Range $= [-1, 1]$



[-2π, 2π] by [–4, 4]

Cosine Function
$f(x) = \cos(x)$
Domain $= (-\infty, \infty)$
Range $= [-1, 1]$

C.1 Logic: An Introduction

What you'll learn about
- Statements
- Compound Statements

. . . and why

These topics are important in the study of logic.

Statements

Logic is a tool used in mathematical thinking and problem solving. In logic, a **statement** *is a sentence that is either true or false, but not both.*

The following expressions are not statements because their truth values cannot be determined without more information.

1. She has blue eyes.
2. $x + 7 = 18$
3. $2y + 7 > 1$

The expressions above become statements if, for (1), "she" is identified, and for (2) and (3), values are assigned to x and y, respectively. However, an expression involving *he* or *she* or x or y may already be a statement. For example, "If he is over 210 cm tall, then he is over 2 m tall," and "$2(x + y) = 2x + 2y$" are both statements because they are true no matter who he is or what the numerical values of x and y are.

From a given statement, it is possible to create a new statement by forming a **negation**. The negation of a statement is a statement with the opposite truth value of the given statement. If a statement is true, its negation is false, and if a statement is false, its negation is true. Consider the statement "It is snowing." The negation of this statement may be stated simply as "It is not snowing."

EXAMPLE 1 Negation of Statements

Negate each of the following statements:

(a) $2 + 3 = 5$

(b) A hexagon has six sides.

(c) Today is not Monday.

SOLUTION

(a) $2 + 3 \neq 5$

(b) A hexagon does not have six sides.

(c) Today is Monday.

Now try Exercise 5, parts (a), (b), and (c).

The statements "The shirt is blue" and "The shirt is green" are not negations of each other. A statement and its negation must have opposite truth values. If the shirt is actually red, then both of the above statements are false and, hence, cannot be negations of each other. However, the statements "The shirt is blue" and "The shirt is not blue" are negations of each other because they have opposite truth values no matter what color the shirt really is.

Some statements involve **quantifiers** and are more complicated to negate. Quantifiers include words such as *all, some, every,* and *there exists.*

The quantifiers *all, every,* and *no* refer to each and every element in a set and are **universal quantifiers** . The quantifiers *some* and *there exists at least one* refer to one or more, or possibly all, of the elements in a set. *Some* and *there exists* are called **existential quantifiers**. Examples with universal and existential quantifiers follow:

1. All roses are red. [universal]
2. Every student is important. [universal]
3. For each counting number x, $x + 0 = x$. [universal]
4. Some roses are red. [existential]
5. There exists at least one even counting number less than 3. [existential]
6. There are women who are taller than 200 cm. [existential]

Venn diagrams can be used to picture statements involving quantifiers. For example, Figures C.1a and C.1b picture statements (1) and (4). The x in Figure C.1b is used to show that there must be at least one element of the set of roses that is red.

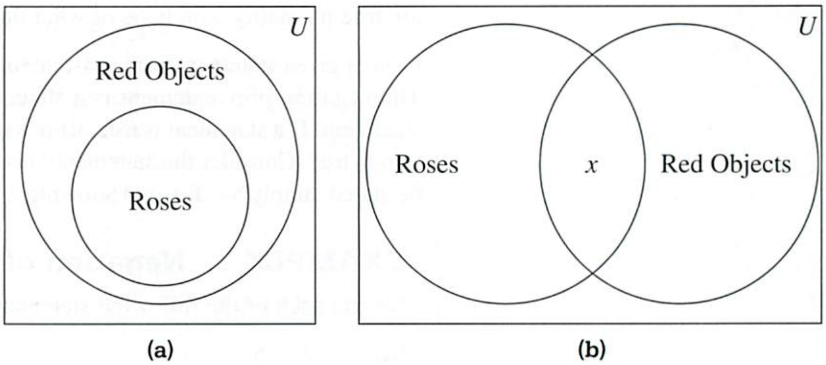


FIGURE C.1 **(a)** All roses are red. **(b)** Some roses are red.

Consider the following statement involving the existential quantifier *some*. "Some professors at Paxson University have blue eyes." This means that at least one professor at Paxson University has blue eyes. It does not rule out the possibilities that all the Paxson professors have blue eyes or that some of the Paxson professors do not have blue eyes. Because the negation of a true statement is false, neither "Some professors at Paxson University do not have blue eyes" nor "All professors at Paxson have blue eyes" are negations of the original statement. One possible negation of the original statement is "No professors at Paxson University have blue eyes."

Statement	Negation
Some *a* are *b*.	No *a* is *b*.
Some *a* are not *b*.	All *a* are *b*.
All *a* are *b*.	Some *a* are not *b*.
No *a* is *b*.	Some *a* are *b*.

EXAMPLE 2 Negation with Quantifiers

Negate each of the following statements:

(a) All students like hamburgers.

(b) Some people like mathematics.

(c) There exists a counting number x such that $3x = 6$.

(d) For all counting numbers x, $3x = 3x$.

SOLUTION

(a) Some students do not like hamburgers.

(b) No people like mathematics.

(c) For all counting numbers x, $3x \neq 6$.

(d) There exists a counting number x such that $3x \neq 3x$.

Now try Exercise 5, parts (e) and (f).

Table C.1 Negation

p	$\sim p$
T	F
F	T

There is a symbolic system defined to help in the study of logic. If p represents a statement, the negation of the statement p is denoted by $\sim p$. **Truth tables** are often used to show all possible true-false patterns for statements. Table C.1 summarizes the truth tables for p and $\sim p$.

Observe that p and $\sim p$ are analogous to sets P and $\overline{P}$. If x is an element of P, then x is not an element of $\overline{P}$.

Compound Statements

From two given statements, it is possible to create a new, **compound statement** by using a connective such as *and*. For example, "It is snowing" and "the ski run is open" together with *and* give "It is snowing and the ski run is open." Other compound statements can be obtained by using the connective *or*. For example, "It is snowing or the ski run is open."

The symbols $\wedge$ and $\vee$ are used to represent the connectives *and* and *or*, respectively. For example, if p represents "It is snowing," and if q represents "The ski run is open," then "It is snowing and the ski run is open" is denoted by $p \wedge q$. Similarly, "It is snowing or the ski run is open" is denoted by $p \vee q$.

Table C.2 Conjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

The truth value of any compound statement, such as $p \wedge q$, is defined using the truth table of each of the simple statements. Because each of the statements p and q may be either true or false, there are four distinct possibilities for the truth values of p and q, as shown in Table C.2. The compound statement $p \wedge q$, is the **conjunction** of p and q and is defined to be true if, and only if, both p and q are true. Otherwise, it is false.

Table C.3 Disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The compound statement $p \vee q$—that is, p or q—is a **disjunction**. In everyday language, *or* is not always interpreted in the same way. In logic, we use an *inclusive or*. The statement "I will go to a movie or I will read a book" means that I will either go to a movie, or read a book, or do both. Hence, in logic, p or q, symbolized as $p \vee q$, is defined to be false if both p and q are false and true in all other cases. This is summarized in Table C.3.

EXAMPLE 3 Conjunction and Disjunction

Given the following statements, classify each of the conjunctions and disjunctions as true or false:

p: $2 + 3 = 5$ r: $5 + 3 = 9$
q: $2 \cdot 3 = 6$ s: $2 \cdot 4 = 9$

(a) $p \wedge q$ **(b)** $p \wedge r$ **(c)** $s \wedge q$ **(d)** $r \wedge s$

(e) $\sim p \wedge q$ **(f)** $\sim(p \wedge q)$ **(g)** $p \vee q$ **(h)** $p \vee r$

(i) $s \vee q$ **(j)** $r \vee s$ **(k)** $\sim p \vee q$ **(l)** $\sim(p \vee q)$

SOLUTION

(a) p is true and q is true, so $p \wedge q$ is true.

(b) p is true and r is false, so $p \wedge r$ is false.

(c) s is false and q is true, so $s \wedge q$ is false.

(d) r is false and s is false, so $r \wedge s$ is false.

(e) $\sim p$ is false and q is true, so $\sim p \wedge q$ is false.

(f) $p \wedge q$ is true [part (a)], so $\sim(p \wedge q)$ is false.

(g) p is true and q is true, so $p \vee q$ is true.

(h) p is true and r is false, so $p \vee r$ is true.

(i) s is false and q is true, so $s \vee q$ is true.

(j) r is false and s is false, so $r \vee s$ is false.

(k) $\sim p$ is false and q is true, so $\sim p \vee q$ is true.

(l) $p \vee q$ is true [part (g)], so $\sim(p \vee q)$ is false.

Now try Exercise 7, parts (a) and (f).

There is an analogy between the connectives $\wedge$ and $\vee$ and the set operations of intersection ($\cap$) and union ($\cup$). Just as the statement $p \wedge q$ is true only when p and q are both true, so an element x belongs to the set $P \cap Q$ only when x belongs to both P and Q. Similarly, the statement $p \vee q$ is true when either p or q is true, and an element x belongs to the set $P \cup Q$ when x belongs to either P or Q.

EXAMPLE 4 Statements and Sets

Use set operations to construct a set that corresponds, by analogy, to each of the following statements:

(a) $p \wedge r$ **(b)** $\sim r \vee q$ **(c)** $\sim(p \wedge q)$ **(d)** $\sim(p \vee \sim r)$

SOLUTION

(a) $P \cap R$ **(b)** $\overline{R} \cup Q$ **(c)** $\overline{P \cap Q}$ **(d)** $\overline{P \cup R}$

Now try Exercise 9.

Not only are truth tables used to summarize the truth values of compound statements, they also are used to determine if two statements are logically equivalent. Two statements are **logically equivalent** if, and only if, they have the same truth values. For example, we could show that $p \wedge q$ is logically equivalent to $q \wedge p$ by using a truth table as in Table C.4.

Table C.4

p	q	$p \wedge q$	$q \wedge p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

EXAMPLE 5 Logical Equivalence

Use a truth table to determine if $\sim p \vee \sim q$ and $\sim(p \wedge q)$ are logically equivalent.

SOLUTION Table C.5 shows headings and the four distinct possibilities for p and q. In the column headed $\sim p$, we write the negations of the p column. In the $\sim q$ column, we write the negations of the q column. Next, we use the values in the $\sim p$ and the $\sim q$ columns to construct the $\sim p \vee \sim q$ column. To find the truth values for $\sim(p \wedge q)$, we use the p and q columns to find the truth values for $p \wedge q$ and then negate $p \wedge q$.

Table C.5

p	q	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$p \wedge q$	$\sim(p \wedge q)$
T	T	F	F	F	T	F
T	F	F	T	T	F	T
F	T	T	F	T	F	T
F	F	T	T	T	F	T

Since the values in the columns for $\sim p \vee \sim q$ and $\sim(p \wedge q)$ are identical, the statements are equivalent. *Now try Exercise 4, parts (b) and (d).*

APPENDIX C.1 EXERCISES

1. Determine which of the following are statements, and then classify each statement as true or false:

(a) $2 + 4 = 8$ **(b)** Shut the window.
(c) Los Angeles is a state. **(d)** He is in town.
(e) What time is it? **(f)** $5x = 15$
(g) $3 \cdot 2 = 6$ **(h)** $2x^2 > x$
(i) This statement is false. **(j)** Stay put!

2. Use quantifiers to make each of the following true where x is a natural number:

(a) $x + 8 = 11$ **(b)** $x + 0 = x$
(c) $x^2 = 4$ **(d)** $x + 1 = x + 2$

3. Use quantifiers to make each equation in Exercise 2 false.

4. Complete each of the following truth tables:

(a)

p	$\sim p$	$\sim(\sim p)$
T		
F		

(b)

p	$\sim p$	$p \vee \sim p$	$p \wedge \sim p$
T			
F			

(c) Based on part (a), is p logically equivalent to $\sim(\sim p)$?
(d) Based on part (b), is $p \vee \sim p$ logically equivalent to $p \wedge \sim p$?

5. Write the negation for each of the following statements:

(a) The book has 500 pages.

(b) Six is less than eight.

(c) $3 \cdot 5 = 15$

(d) Some people have blond hair.

(e) All dogs have four legs.

(f) Some cats do not have nine lives.

(g) All squares are rectangles.

(h) Not all rectangles are squares.

(i) For all natural numbers x, $x + 3 = 3 + x$.

(j) There exists a natural number x such that $3 \cdot (x + 2) = 12$.

(k) Every counting number is divisible by itself and 1.

(l) Not all natural numbers are divisible by 2.

(m) For all natural numbers x, $5x + 4x = 9x$.

6. If q stands for "This course is easy" and r stands for "Lazy students do not study," write each of the following in symbolic form:

(a) This course is easy and lazy students do not study.

(b) Lazy students do not study or this course is not easy.

(c) It is false that both this course is easy and lazy students do not study.

(d) This course is not easy.

7. If p is false and q is true, find the truth values for each of the following:

(a) $p \wedge q$ **(b)** $p \vee q$

(c) $\sim p$ **(d)** $\sim q$

(e) $\sim(\sim p)$ **(f)** $\sim p \vee q$

(g) $p \wedge \sim q$ **(h)** $\sim(p \vee q)$

(i) $\sim(\sim p \wedge q)$ **(j)** $\sim q \wedge \sim p$

8. Find the truth value for each statement in Exercise 7 if p is false and q is false.

9. Use set operations to construct a set that corresponds, by analogy, to each of the following statements.

(a) $r \vee s$ **(b)** $q \wedge \sim q$

(c) $\sim(r \vee q)$ **(d)** $p \wedge (r \vee s)$

10. For each of the following, is the pair of statements logically equivalent?

(a) $\sim(p \vee q)$ and $\sim p \vee \sim q$

(b) $\sim(p \vee q)$ and $\sim p \wedge \sim q$

(c) $\sim(p \wedge q)$ and $\sim p \wedge \sim q$

(d) $\sim(p \wedge q)$ and $\sim p \vee \sim q$

11. (a) Write two logical equivalences discovered in parts 10(a)–(d). These equivalences are called DeMorgan's Laws for "*and*" and "*or*."

(b) Write an explanation of the analogy between DeMorgan's laws for sets and those found in part (a).

12. Complete the following truth table:

p	q	$\sim p$	$\sim q$	$\sim p \vee q$
T	T			
T	F			
F	T			
F	F			

13. Restate the following in a logically equivalent form:

(a) It is not true that both today is Wednesday and the month is June.

(b) It is not true that yesterday I both ate breakfast and watched television.

(c) It is not raining or it is not July.

C.2 Conditionals and Biconditionals

What you'll learn about

- Forms of Statements
- Valid Reasoning

. . . and why

These topics are important in the study of logic.

Forms of Statements

Statements expressed in the form "if p, then q" are called **conditionals**, or **implications**, and are denoted by $p \to q$. Such statements also can be read "p implies q." The "if" part of a conditional is called the **hypothesis** of the implication and the "then" part is called the **conclusion**.

Many types of statements can be put in "if-then" form; an example follows:

Statement: All first graders are 6 years old.
If-then form: If a child is a first grader, then the child is 6 years old.

An implication may also be thought of as a promise. Suppose Betty makes the promise, "If I get a raise, then I will take you to dinner." If Betty keeps her promise, the implication is true; if Betty breaks her promise, the implication is false. Consider the following four possibilities:

	p	q	
(1)	T	T	Betty gets the raise; she takes you to dinner.
(2)	T	F	Betty gets the raise; she does not take you to dinner.
(3)	F	T	Betty does not get the raise; she takes you to dinner.
(4)	F	F	Betty does not get the raise; she does not take you to dinner.

The only case in which Betty breaks her promise is when she gets her raise and fails to take you to dinner, case (2). If she does not get the raise, she can either take you to dinner or not without breaking her promise. The definition of implication is summarized in Table C.6. Observe that the only cause for which the implication is false is when p is true and q is false.

Table C.6 Implication

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

An implication may be worded in several equivalent ways, as follows:

1. If the sun shines, then the swimming pool is open. (If p, then q.)
2. If the sun shines, the swimming pool is open. (If p, q.)
3. The swimming pool is open if the sun shines. (q if p.)
4. The sun shines implies the swimming pool is open. (p implies q.)
5. The sun is shining only if the pool is open. (p only if q.)
6. The sun's shining is a sufficient condition for the swimming pool to be open. (p is a sufficient condition for q.)
7. The swimming pool's being open is a necessary condition for the sun to be shining. (q is a necessary condition for p.)

Any implication $p \to q$ has three related implication statements, as follows:

Statement:	If p, then q.	$p \to q$
Converse:	If q, then p.	$q \to p$
Inverse:	If not p, then not q.	$\sim p \to \sim q$
Contrapositive:	If not q, then not p.	$\sim q \to \sim p$

EXAMPLE 1 Converse, Inverse, Contrapositive

Write the converse, the inverse, and the contrapositive for each of the following statements:

(a) If $2x = 6$, then $x = 3$.

(b) If I am in San Francisco, then I am in California.

SOLUTION

(a) *Converse:* If $x = 3$, then $2x = 6$.
Inverse: If $2x \neq 6$, then $x \neq 3$.
Contrapositive: If $x \neq 3$, then $2x \neq 6$.

(b) *Converse:* If I am in California, then I am in San Francisco.
Inverse: If I am not in San Francisco, then I am not in California.
Contrapositive: If I am not in California, then I am not in San Francisco.

Now try Exercise 3, parts (a) and (b).

Table C.7 shows that an implication and its converse do not always have the same truth value. However, an implication and its contrapositive always have the same truth value. Also, the converse and inverse of a conditional statement are logically equivalent.

Table C.7 Converse, Inverse, Contrapositive

p	q	$\sim p$	$\sim q$	Implication $p \to q$	Converse $q \to p$	Inverse $\sim p \to \sim q$	Contra-positive $\sim q \to \sim p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Connecting a statement and its converse with the connective *and* gives $(p \to q) \wedge (q \to p)$. This compound statement can be written as $p \leftrightarrow q$ and usually is read "p if and only if q." The statement "p if and only if q" is a **biconditional**. A truth table for $p \leftrightarrow q$ is given in Table C.8. Observe that $p \leftrightarrow q$ is true if and only if both statements are true or both are false.

Table C.8 Biconditional

p	q	$p \to q$	$q \to p$	Biconditional $(p \to q) \wedge (q \to p)$ or $p \leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

EXAMPLE 2 Biconditionals

Given the following statements, classify each of the biconditionals as true or false:

p: $2 = 2$ r: $2 = 1$

q: $2 \neq 1$ s: $2 + 3 = 1 + 3$

(a) $p \leftrightarrow q$ **(b)** $p \leftrightarrow r$

(c) $s \leftrightarrow q$ **(d)** $r \leftrightarrow s$

SOLUTION

(a) $p \to q$ is true and $q \to p$ is true, so $p \leftrightarrow q$ is true.

(b) $p \to r$ is false and $r \to p$ is true, so $p \leftrightarrow r$ is false.

(c) $s \to q$ is true and $q \to s$ is false, so $s \leftrightarrow q$ is false.

(d) $r \to s$ is true and $s \to r$ is true, so $r \leftrightarrow s$ is true.

Now try Exercise 5, parts (a) and (f).

Now consider the following statement:

It is raining or it is not raining.

This statement, which can be modeled as $p \vee (\sim p)$, is always true, as shown in Table C.9. A statement that is always true is called a **tautology**. One way to make a tautology is to take two logically equivalent statements such as $p \to q$ and $\sim q \to \sim p$ (from Table C.7) and form them into a biconditional as follows:

$$p \to q \leftrightarrow (\sim q \to \sim p)$$

Because $p \to q$ and $\sim q \to \sim p$ have the same truth values, $(p \to q) \leftrightarrow (\sim q \to \sim p)$ is a tautology.

Table C.9 A Tautology

p	$\sim p$	$p \vee (\sim p)$
T	F	T
F	T	T

Valid Reasoning

In problem solving, the reasoning used is said to be **valid** if the conclusion follows unavoidably from the hypotheses. Consider the following example:

Hypotheses: All roses are red.
This flower is a rose.
Conclusion: Therefore, this flower is red.

The statement "All roses are red" can be written as the implication, "If a flower is a rose, then it is red" and pictured with the Venn diagram is Figure C.2a.

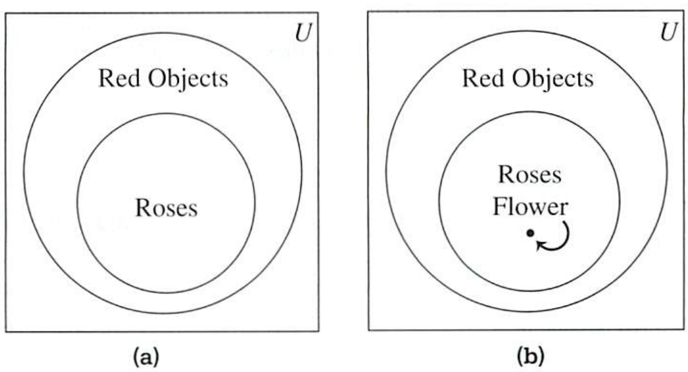


FIGURE C.2 **(a)** All roses are red. **(b)** This flower is a rose.

The information "This flower is a rose" implies that this flower must belong to the circle containing roses, as pictured in Figure C.2b. This flower also must belong to the circle containing red objects. Thus the reasoning is valid because it is impossible to draw a picture satisfying the hypotheses and contradicting the conclusion.

Consider the following argument:

Hypotheses:	All elementary school teachers are mathematically literate. Some mathematically literate people are not children.
Conclusion:	Therefore, no elementary school teacher is a child.

Let E be the set of elementary school teachers, M be the set of mathematically literate people, and C be the set of children. Then the statement "All elementary school teachers are mathematically literate" can be pictured as in Figure C.3a. The statement "Some mathematically literate people are not children" can be pictured in several ways. Three of these are illustrated in Figure C.3b–d.

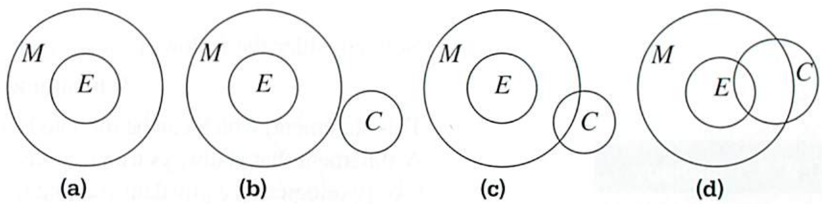


FIGURE C.3 **(a)** All elementary school teachers are mathematically literate. **(b)–(d)** Some mathematically literate people are not children.

According to Figure C.3d, it is possible that some elementary school teachers are children, and yet the given statements are satisfied. Therefore, the conclusion that "No elementary school teacher is a child" does not follow from the given hypotheses. Hence, the reasoning is not valid.

If a single picture can be drawn to satisfy the hypotheses of an argument and contradict the conclusion, the argument is not valid. However, to show that an argument is valid, all possible pictures must be considered to show that there are no contradictions. There must be no way to satisfy the hypotheses and contradict the conclusion if the argument is valid.

EXAMPLE 3 Argument Validity

Determine if the following argument is valid:

Hypotheses:	In Washington, D.C., all senators wear power ties. No one in Washington, D.C., over 6 ft tall wears a power tie.
Conclusion:	Persons over 6 ft tall are not senators in Washington, D.C.

SOLUTION

If S represents the set of senators and P represents the set of people who wear power ties, the first hypothesis is pictured as shown in Figure C.4a. If T represents the set of people in Washington, D.C., over 6 ft tall, the second hypothesis is pictured in Figure C.4b. Because people over 6 ft tall are outside the circle representing power tie wearers and senators are inside the circle P, the conclusion is valid and no person over 6 ft tall can be a senator in Washington, D.C. ***Now try Exercise 14(a).***

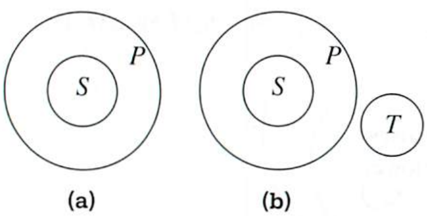


FIGURE C.4 **(a)** In Washington, D.C., all senators wear power ties. **(b)** No one in Washington, D.C., over 6 ft tall wears a power tie.

A different method for determining if an argument is valid uses **direct reasoning** and a form of argument called the Law of Detachment (or **Modus Ponens**). For example, consider the following true statements:

If the sun is shining, then we shall take a trip.
The sun is shining.

Using these two statements, we can conclude that we shall take a trip. In general, the **Law of Detachment** is stated as follows:

If a statement in the form "if p, then q" is true, and p is true, then q must also be true.

The Law of Detachment is sometimes described schematically as follows, where all statements above the horizontal line are true and the statement below the horizontal line is the conclusion.

$$\begin{array}{l} p \rightarrow q \\ \underline{p \qquad\;\;} \\ \qquad\;\; q \end{array}$$

The Law of Detachment follows from the truth table for $p \rightarrow q$ given in Table C.6. The only case in which both p and $p \rightarrow q$ are true is when q is true (line 1 in the table).

EXAMPLE 4 Applications of the Law of Detachment

Determine if each of the following arguments is valid:

Hypotheses:	If you eat spinach, then you will be strong. You eat spinach.
Conclusion:	Therefore, you will be strong.
Hypotheses:	If Claude goes skiing, he will break his leg. If Claude breaks his leg, he cannot enter the dance contest. Claude goes skiing.
Conclusion:	Therefore, Claude cannot enter the dance contest.

SOLUTION

(a) Using the Law of Detachment, we see that the conclusion is valid.

(b) By using the Law of Detachment twice, we see that the conclusion is valid.

Now try Exercise 14(d).

A different type of reasoning, **indirect reasoning**, uses a form of argument called **Modus Tollens**. For example, consider the following true statements:

If Chicken Little had been hit by a jumping frog, he would have thought Earth was rising.

Chicken Little did not think Earth was rising.

What is the conclusion? The conclusion is that Chicken Little did not get hit by a jumping frog. This leads us to the general form of Modus Tollens:

If we have a conditional accepted as true, and we know the conclusion is false, then the hypothesis must be false.

Modus Tollens is sometimes schematically described as follows:

$$\begin{array}{r} p \to q \\ \underline{\sim q \quad\;\;} \\ \sim p \end{array}$$

The validity of Modus Tollens also follows from the truth table for $p \to q$ given in Table C.6. The only case in which both $p \to q$ is true and q is false is when p is false (line 4 in the table). The validity of Modus Tollens also can be established from the fact that an implication and its contrapositive are equivalent.

EXAMPLE 5 Applications of Modus Tollens

Determine conclusions for each of the following sets of true statements:

(a) If an old woman lives in a shoe, then she does not know what to do.
Mrs. Pumpkin Eater, an old woman, knows what to do.

(b) If Jack is nimble, he will not get burned. Jack was burned.

SOLUTION

(a) Mrs. Pumpkin Eater does not live in a shoe.

(b) Jack was not nimble.

Now try Exercise 13(a).

People often make invalid conclusions based on advertising or other information. Consider, for example, the statement "Healthy people eat Super-Bran cereal." Are the following conclusions valid?

If a person eats Super-Bran cereal, then the person is healthy.
If a person is not healthy, the person does not eat Super-Bran cereal.

If the original statement is denoted by $p \to q$, where p is "a person is healthy" and q is "a person eats Super-Bran cereal," then the first conclusion is the converse of $p \to q$—that is, $q \to p$—and the second conclusion is the inverse of $p \to q$—that is, $\sim p \to \sim q$. Table C.7 points out that neither the converse nor the inverse are logically equivalent to the original statement, and consequently the conclusions are not necessarily true.

The final reasoning argument to be considered here involves the **Chain Rule**. Consider the following statements:

If my wife works, I will retire early.
If I retire early, I will become lazy.

What is the conclusion? The conclusion is that if my wife works, I will become lazy. In general, the Chain Rule can be stated as follows:

If "if p, then q," and "if q, then r" are true, then "if p, then r" is true.

The Chain Rule is sometimes schematically described as follows:

$$\begin{array}{c} p \to q \\ \underline{q \to r} \\ p \to r \end{array}$$

Notice that the chain rule shows that implication is a transitive relation.

REMARK

Note that in Example 6, the Chain Rule can be extended to contain several implications.

EXAMPLE 6 Applications of the Chain Rule

Determine conclusions for each of the following sets of true statements:

(a) If Alice follows the White Rabbit, she falls into a hole. If she falls into a hole, she goes to a tea party.

(b) If Chicken Little is hit by an acorn, we think the sky is falling. If we think the sky is falling, we will go to a fallout shelter. If we go to a fallout shelter, we will stay there a month.

SOLUTION

(a) If Alice follows the White Rabbit, she goes to a tea party.

(b) If Chicken Little is hit by an acorn, we will stay in a fallout shelter for a month.

Now try Exercise 13(c).

APPENDIX C.2 EXERCISES

1. Write each of the following in symbolic form if p is the statement "It is raining" and q is the statement "The grass is wet."

(a) If it is raining, then the grass is wet.

(b) If it is not raining, then the grass is wet.

(c) If it is raining, then the grass is not wet.

(d) The grass is wet if it is raining.

(e) The grass is not wet implies that it is not raining.

(f) The grass is wet if, and only if, it is raining.

2. Construct a truth table for each of the following:

(a) $p \to (p \vee q)$ **(b)** $(p \wedge q) \to q$

(c) $p \leftrightarrow \sim(\sim p)$ **(d)** $\sim(p \to q)$

3. For each of the following implications, state the converse, inverse, and contrapositive.

(a) If you eat Meaties, then you are good in sports.

(b) If you do not like this book, then you do not like mathematics.

(c) If you do not use Ultra Brush toothpaste, then you have cavities.

(d) If you are good at logic, then your grades are high.

4. Can an implication and its converse both be false? Explain your answer.

5. If p is true and q is false, find the truth values for each of the following:

(a) $\sim p \to \sim q$ **(b)** $\sim(p \to q)$

(c) $(p \vee q) \to (p \wedge q)$ **(d)** $p \to \sim p$

(e) $(p \vee \sim p) \to p$ **(f)** $(p \vee q) \leftrightarrow (p \wedge q)$

6. If p is false and q is false, find the truth values for each of the statements in Exercise 5.

7. Iris makes the true statement, "If it rains, then I am going to the movies." Does it follow logically that if it does not rain, then Iris does not go to the movies?

8. Consider the statement "If every digit of a number is 6, then the number is divisible by 3." Determine whether each of the following is logically equivalent to the statement.

(a) If every digit of a number is not 6, then the number is not divisible by 3.

(b) If a number is not divisible by 3, then some digit of the number is not 6.

(c) If a number is divisible by 3, then every digit of the number is 6.

9. Write a statement logically equivalent to the statement "If a number is a multiple of 8, then it is a multiple of 4."

10. Use truth tables to prove that the following are tautologies:

(a) $(p \to q) \to [(p \wedge r) \to q]$ Law of Added Hypothesis

(b) $[(p \to q) \wedge p] \to q$ Law of Detachment

(c) $[(p \to q) \wedge \sim q] \to \sim p$ Modus Tollens

(d) $[(p \to q) \wedge (q \to r)] \to (p \to r)$ Chain Rule

11. **(a)** Suppose that $p \to q$, $q \to r$, and $r \to s$ are all true, but s is false. What can you conclude about the truth value of p?

(b) Suppose that $(p \wedge q) \to r$ is true, r is false, and q is true. What can you conclude about the truth value of p?

(c) Suppose that $p \to q$ is true and $q \to p$ is false. Can q be true? Why or why not?

12. Translate the following statements into symbolic form. Give the meanings of the symbols that you use.

(a) If Mary's little lamb follows her to school, then its appearance there will break the rules and Mary will be sent home.

(b) If it is not the case that Jack is nimble and quick, then Jack will not make it over the candlestick.

(c) If the apple had not hit Isaac Newton on the head, then the laws of gravity would not have been discovered.

13. For each of the following, form a conclusion that follows logically from the given statements:

(a) All college students are poor.
Helen is a college student.

(b) Some freshmen like mathematics.
All people who like mathematics are intelligent.

(c) If I study for the final, then I will pass the final.
If I pass the final, then I will pass the course.
If I pass the course, then I will look for a teaching job.

(d) Every equilateral triangle is isosceles.
There exist triangles that are equilateral.

14. Investigate the validity of each of the following arguments:

(a) All women are mortal.
Hypatia was a woman.
Therefore, Hypatia was mortal.

(b) All squares are quadrilaterals.
All quadrilaterals are polygons.
Therefore, all squares are polygons.

(c) All teachers are intelligent.
Some teachers are rich.
Therefore, some intelligent people are rich.

(d) If a student is a freshman, then she takes mathematics.
Jane is a sophomore.
Therefore, Jane does not take mathematics.

15. Write the following in if-then form:

(a) Every figure that is a square is a rectangle.

(b) All integers are rational numbers.

(c) Figures with exactly three sides may be triangles.

(d) It rains only if it is cloudy.