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Don't Take This Out of Context
Analyzing Polynomial Functions

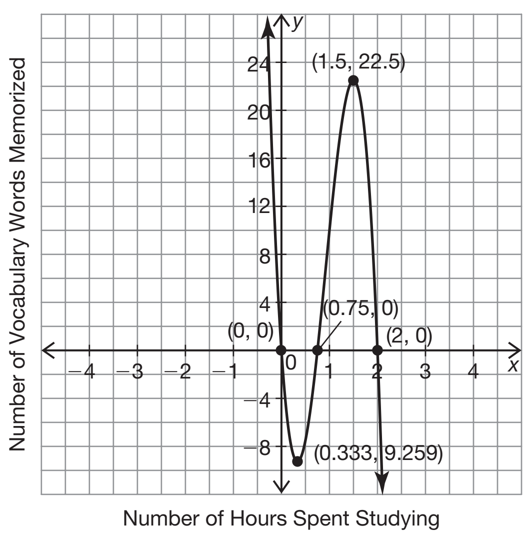
Vocabulary

Write a definition for the term in your own words.

1. average rate of change
   The average rate of change of a function is the ratio of the independent variable to the dependent variable over a specific interval. The formula for average rate of change is $\frac{f(b) - f(a)}{b - a}$ for an interval (a, b).

Problem Set

The graph shows the number of vocabulary words a student is able to memorize based on the amount of time spent studying. Use the graph to answer the questions.



1. How many vocabulary words does the student know at the start of the study? Where is this information located on the graph?
   The student knows 0 vocabulary words at the start of the study. This event is represented by the y-intercept at the origin.

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2. Describe the relative minimum in terms of the problem situation.

After studying for about 20 minutes, the student knows 10 less vocabulary words than they knew at the start of the study.

3. What is the minimum amount of time the student studies before they begin to remember the vocabulary? Where is this information located on the graph?

The minimum amount of time the student studies before they begin to remember the vocabulary is 0.75 hours. This event is represented by the *x*-intercept (0.75, 0).

4. How long did the student need to study in order to remember 22 vocabulary words? Where is this information located on the graph?

The student needs to study about 1.5 hours to remember 22 vocabulary words. This event is represented just before the relative maximum at (1.5, 22.5).

5. The graph has an *x*-intercept at (2, 0). Describe the activity of the student at this point in terms of the problem situation.

After two hours of studying, the student did not remember any vocabulary words.

6. Does the graph accurately describe the problem situation? Explain your reasoning.

Answers will vary.

Parts of the graph describe the problem situation. It is conceivable that the student has no knowledge when they begin studying and that their knowledge does not increase until they study for 45 minutes, which corresponds to the *x*-intercepts at (0, 0) and (0.75, 0).

The relative minimum at (0.333, 9.259) does not make sense in the problem situation because a student should not lose knowledge that they never had, which is suggested by this point. The *x*-intercept at (2, 0) does not make sense in the problem situation because a student should not lose all of their vocabulary knowledge if they study longer than 1.5 hours, which is what is suggested by the *x*-intercept.

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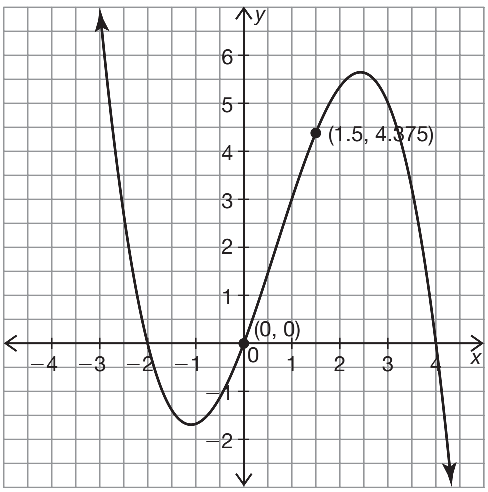
Determine the average rate of change for the given interval for each polynomial function.

7. (0, 1.5)



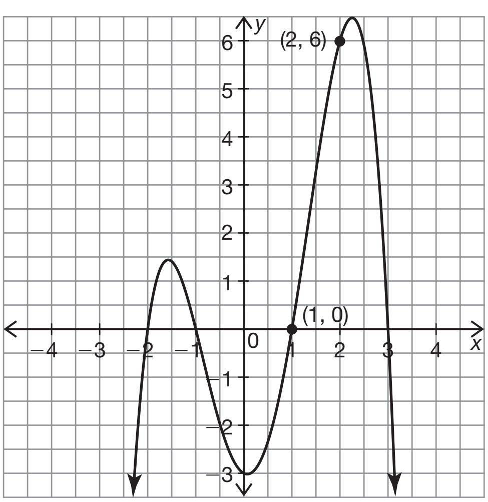
$$\frac{f(b) - f(a)}{b - a} = \frac{f(1.5) - f(0)}{1.5 - 0}$$

$$= \frac{4.375 - 0}{1.5 - 0}$$

$$= \frac{4.375}{1.5}$$

$$\approx 2.92$$

8. (1, 2)



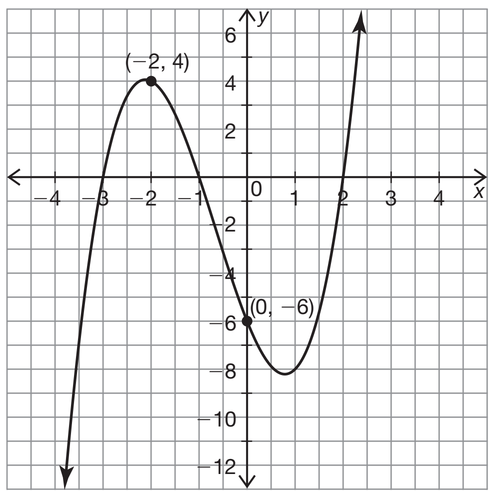
$$\frac{f(b) - f(a)}{b - a} = \frac{f(2) - f(1)}{2 - 1}$$

$$= \frac{6 - 0}{1}$$

$$= 6$$

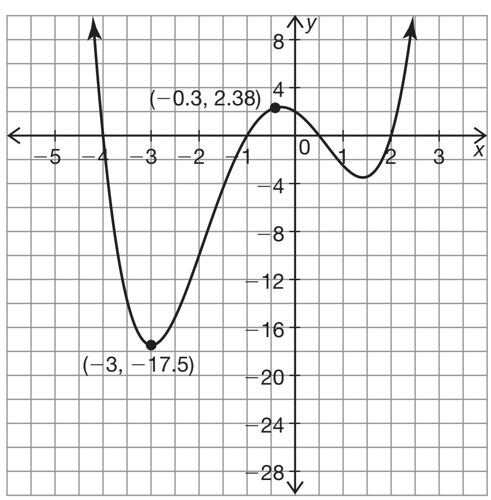
6

9. (−2, 0)



$$\frac{f(b) - f(a)}{b - a} = \frac{f(0) - f(-2)}{0 - (-2)}$$
$$= \frac{-6 - 4}{2}$$
$$= \frac{-10}{2}$$
$$= -5$$

10. (−3, −0.3)



$$\frac{f(b) - f(a)}{b - a} = \frac{f(-0.3) - f(-3)}{-0.3 - (-3)}$$
$$= \frac{2.38 - (-17.5)}{-0.3 - (-3)}$$
$$= \frac{19.88}{2.7}$$
$$\approx 7.36$$

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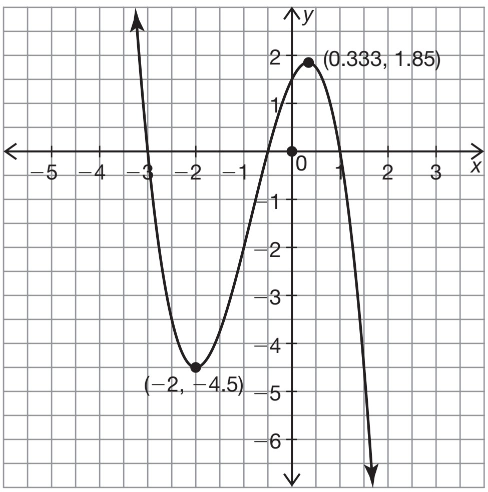
11. (−2, 0.333)



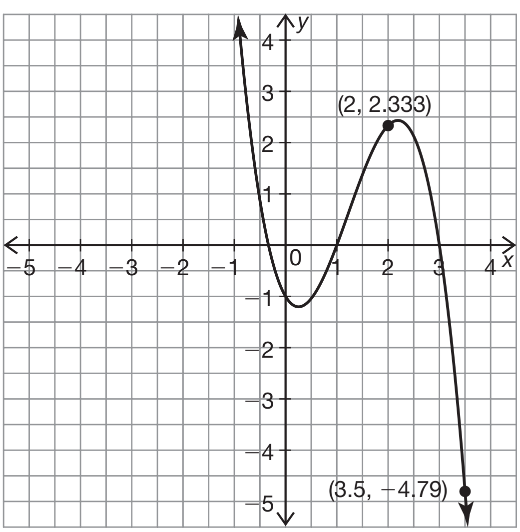
$$\frac{f(b) - f(a)}{b - a} = \frac{f(0.333) - f(-2)}{0.333 - (-2)}$$

$$= \frac{1.85 - (-4.5)}{0.333 + 2}$$

$$= \frac{6.35}{2.333}$$

$$\approx 2.72$$

12. (2, 3.5)



$$\frac{f(b) - f(a)}{b - a} = \frac{f(3.5) - f(2)}{3.5 - 2}$$

$$= \frac{-4.79 - 2.333}{1.5}$$

$$= \frac{-7.123}{1.5}$$

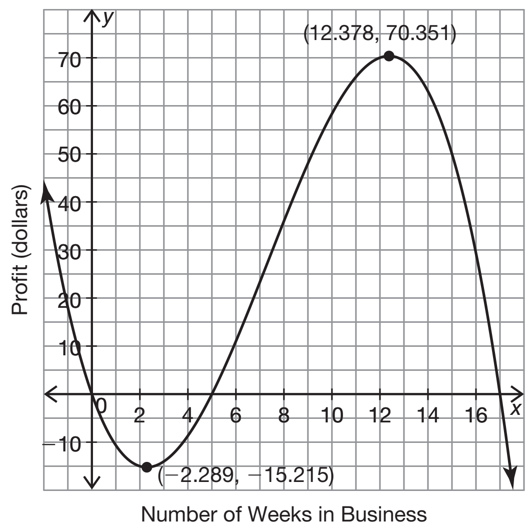
$$\approx -4.75$$

6

Solve each equation using the information found in the graph.

13. The graph models the profit a group of students earns running a tutoring business. After how many weeks did it take the group to earn a profit? Where is this information located on the graph?

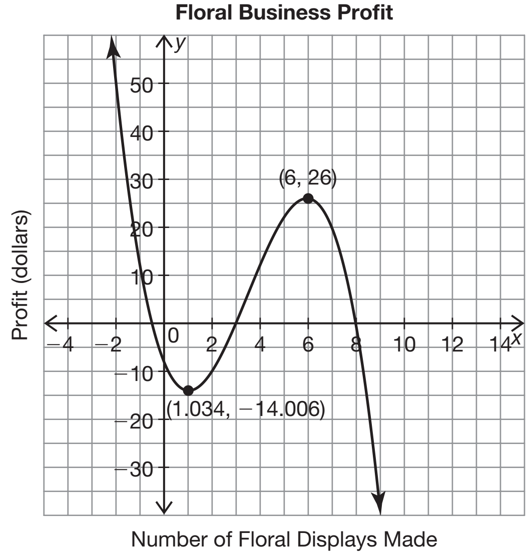
Tutoring Business Profit



The group earns a profit after 5 weeks in business. The information is located at the *x*-intercept of (5, 0).

14. The graph models the amount of money a company makes producing floral displays. What is the maximum number of floral displays that the company can create and continue to increase their profit? Where is this information located on the graph?

Floral Business Profit



The maximum number of floral displays that the company can create and continue to increase their profit is 6. The information is located at the relative maximum.

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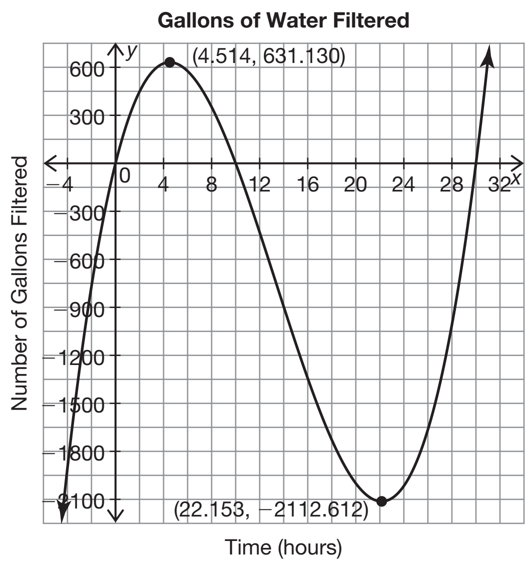
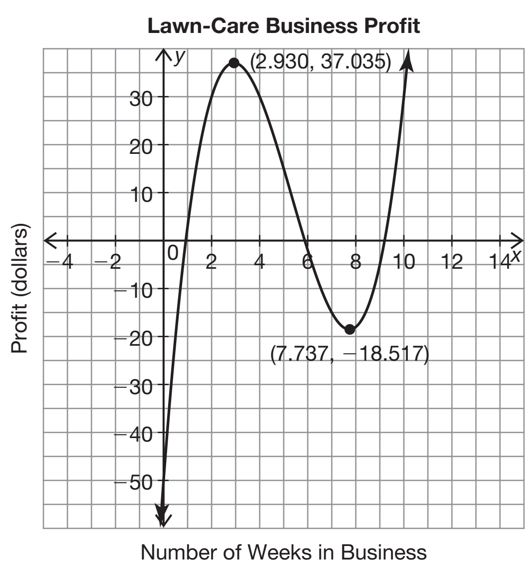
15. The graph models the number of gallons of water that are filtered at a filtration plant hourly. How many gallons of water has the plant filtered after running for about 4.5 hours? Where is this information located on the graph?



The plant has filtered about 625 gallons of water after running for about 4.5 hours. This information is located close to the relative maximum.

16. The graph models the amount of profit Emilio earns from his own lawn-care business. How much did Emilio initially invest to start his business? Where is this information located on the graph?



Emilio invested $50 to start his business. This information is located at the *y*-intercept.

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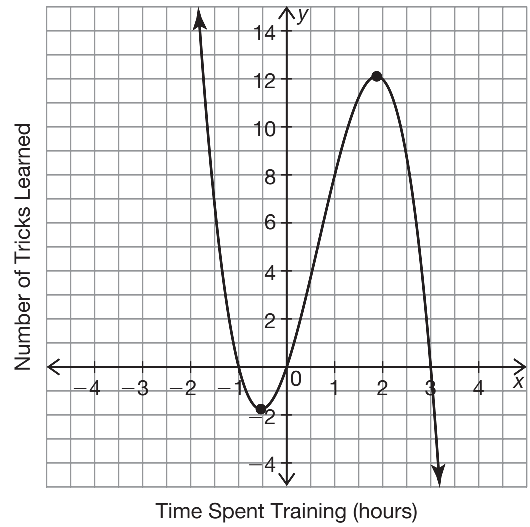
17. The graph models the number of tricks that a dog can perform based on the number of hours it is trained. Estimate how long it takes the dog to learn 8 tricks. Where is this information located on the graph?

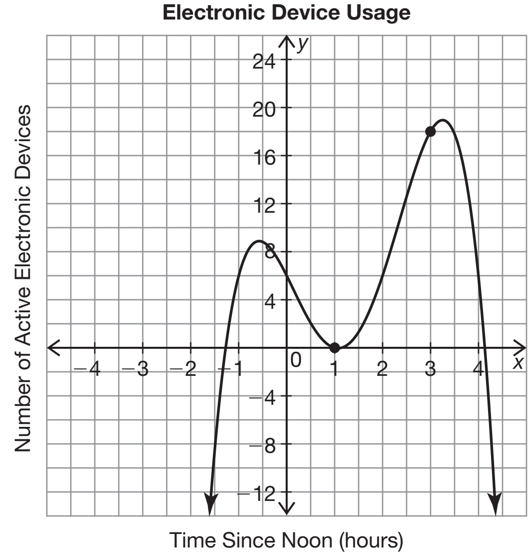
Dog Tricks Learned in Time Trained



The dog can perform 8 tricks after about 1 hour of training. The information is found at the point (1, 8).

18. The graph models the number of electronic devices that are being use in a home during the hours of noon and 4:00 pm. Estimate the time when the greatest number of electronic devices are being used. Where is this information located on the graph?

Electronic Device Usage



There are approximately 19 electronic devices being used at 3:15 pm. This information is located at the absolute maximum.

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The Great Polynomial Divide
Polynomial Division

Vocabulary

Write an example for each term. Write the dividend as the product of the divisor and the quotient plus the remainder.

1. Polynomial long division

Example:

$$\begin{array}{r} x - 3 \\ x+1\overline{)x^2 - 2x + 4} \\ \underline{x^2 + x} \\ -3x + 4 \\ \underline{-3x - 3} \\ 7 \end{array}$$

$$(x + 1)\left((x - 3) + \frac{7}{x + 1}\right)$$

2. Synthetic division

Example:

$$\begin{array}{r|rrr} -1 & 1 & -2 & 4 \\ & & -1 & 3 \\ \hline & 1 & -3 & \boxed{7} \end{array}$$

$$(x + 1)\left((x - 3) + \frac{7}{x + 1}\right)$$

Problem Set

Write the zero that corresponds to each factor.

1. $x + 5$

   $x = -5$

2. $x - 12$

   $x = 12$

3. $2x + 1$

   $x = -\frac{1}{2}$

4. $10x - 9$

   $x = \frac{9}{10}$

5. $x - 13$

   $x = 13$

6. $3x + 4$

   $x = -\frac{4}{3}$

6

Write the factor that corresponds to each zero.

7. $x = 2$

$x - 2$

8. $x = -7$

$x + 7$

9. $x = -75$

$x + 75$

10. $x = \frac{2}{3}$

$3x - 2$

11. $x = -\frac{3}{8}$

$8x + 3$

12. $x = \frac{5}{4}$

$4x - 5$

Determine if the given factor is a factor of each polynomial. Explain your reasoning.

13. Is $x - 1$ a factor of $x^4 - 3x^3 + 6x^2 - 12x + 8$?

$$\begin{array}{r} x^3 - 2x^2 + 4x - 8 \\ x - 1\overline{\big)\,x^4 - 3x^3 + 6x^2 - 12x + 8} \\ \underline{x^4 - x^3 \qquad\qquad\qquad\quad} \\ -2x^3 + 6x^2 \qquad\qquad \\ \underline{-2x^3 + 2x^2 \qquad\qquad} \\ 4x^2 - 12x \qquad \\ \underline{4x^2 - 4x \qquad} \\ -8x + 8 \\ \underline{-8x + 8} \\ 0 \end{array}$$

Yes, $x - 1$ is a factor of $x^4 - 3x^3 + 6x^2 - 12x + 8$ because it divides into the polynomial without a remainder.

14. Is $x - 1$ a factor of $x^4 + 6x^3 - 12x^2 - 38x - 21$?

1	1	6	−12	−38	−21
		1	7	−5	−43
	1	7	−5	−43	−64

No, $x - 1$ is not a factor of $x^4 + 6x^3 - 12x^2 - 38x - 21$ because the remainder is not 0.

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15. Is $3x + 2$ a factor of $3x^5 + 20x^4 + 9x^3 - 92x^2 - 60x$?

$$\begin{array}{r} x^4 + 6x^3 - x^2 - 30x \\ 3x+2\overline{)3x^5 + 20x^4 + 9x^3 - 92x^2 - 60x} \\ \underline{3x^5 + 2x^4} \qquad\qquad\qquad\qquad\qquad \\ 18x^4 + 9x^3 \qquad\qquad\qquad \\ \underline{18x^4 + 12x^3} \qquad\qquad\qquad \\ -3x^3 - 92x^2 \qquad\quad \\ \underline{-3x^3 - 2x^2} \qquad\quad \\ -90x^2 - 60x \\ \underline{-90x^2 - 60x} \\ 0 \end{array}$$

Yes, $3x + 2$ is a factor of $3x^5 + 20x^4 + 9x^3 - 92x^2 - 60x$ because it divides into the polynomial without a remainder.

16. Is $x - 3$ a factor of $x^3 + 12x^2 + 17x - 30$?

3	1	12	17	−30
		3	45	186
	1	15	62	156

No, $x - 3$ is not a factor of $x^3 + 12x^2 + 17x - 30$ because the remainder is not 0.

6

17. Is $x + 4$ a factor of $2x^3 + 7x^2 - 10x - 24$?

$$\begin{array}{r|rrrr} -4 & 2 & 7 & -10 & -24 \\ & & -8 & 4 & 24 \\ \hline & 2 & -1 & -6 & \boxed{0} \end{array}$$

Yes, $x + 4$ is a factor of $2x^3 + 7x^2 - 10x - 24$ because it divides into the polynomial without a remainder.

18. Is $x + 2$ a factor of $x^4 - 2x^3 - x^2 - 4x - 6$?

$$\begin{array}{r} x^3 - 4x^2 + 7x - 18 \\ x + 2\overline{)x^4 - 2x^3 - x^2 - 4x - 6} \\ \underline{x^4 + 2x^3} \qquad\qquad\qquad \\ -4x^3 - x^2 \qquad\qquad \\ \underline{-4x^3 - 8x^2} \qquad\qquad \\ 7x^2 - 4x \qquad \\ \underline{7x^2 + 14x} \qquad \\ -18x - 6 \\ \underline{-18x - 36} \\ 30 \end{array}$$

No, $x + 2$ is not a factor of $x^4 - 2x^3 - x^2 - 4x - 6$ because the remainder is not 0.

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Determine each quotient using polynomial long division. Write the dividend as the product of the divisor and the quotient plus the remainder.

19. $x - 4\overline{)2x^3 - 7x^2 - 19x + 60}$

$$\begin{array}{r} 2x^2 + x - 15 \\ x - 4\overline{)2x^3 - 7x^2 - 19x + 60} \\ \underline{2x^3 - 8x^2} \qquad\qquad \\ x^2 - 19x \qquad \\ \underline{x^2 - 4x} \qquad \\ -15x + 60 \\ \underline{-15x + 60} \\ 0 \end{array}$$

$2x^3 - 7x^2 - 19x + 60 = (x - 4)(2x^2 + x - 15)$

20. $x - 2\overline{)2x^3 - x^2 - 13x - 6}$

$$\begin{array}{r} 2x^2 + 3x - 7 \\ x - 2\overline{)2x^3 - x^2 - 13x - 6} \\ \underline{2x^3 + 4x^2} \qquad\qquad \\ 3x^2 - 13x \qquad \\ \underline{3x^2 - 6x} \qquad \\ -7x - 6 \\ \underline{-7x + 14} \\ -20 \end{array}$$

$2x^3 - x^2 - 13x - 6 = (x - 2)\left((2x^2 + 3x - 7) + \frac{-20}{x - 2}\right)$

6

21. $x + 3\overline{)x^3 + 8x^2 + 7x + 5}$

$$\begin{array}{r} x^2 + 5x - 8 \\ x + 3\overline{)x^3 + 8x^2 + 7x + 5} \\ \underline{x^3 + 3x^2} \qquad\qquad \\ 5x^2 + 7x \qquad \\ \underline{5x^2 + 15x} \qquad \\ -8x + 5 \\ \underline{-8x - 24} \\ 29 \end{array}$$

$$x^3 + 8x^2 + 7x + 5 = (x + 3)\left((x^2 + 5x - 8) + \frac{29}{x + 3}\right)$$

22. $x + 2\overline{)3x^3 + 5x^2 - 2x}$

$$\begin{array}{r} 3x^2 - x \\ x + 2\overline{)3x^3 + 5x^2 - 2x} \\ \underline{3x^3 + 6x^2} \qquad \\ -x^2 - 2x \\ \underline{-x^2 - 2x} \\ 0 \end{array}$$

$$3x^3 + 5x^2 - 2x = (x + 2)(3x^2 - x)$$

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23. $x + 1\overline{)4x^4 + 9x^3 - 82x^2 - 57x + 18}$

$$
\begin{array}{r}
4x^3 + 5x^2 - 87x + 30 \\
x + 1\overline{)4x^4 + 9x^3 - 82x^2 - 57x + 18} \\
\underline{4x^4 + 4x^3} \qquad\qquad\qquad\qquad\quad \\
5x^3 - 82x^2 \qquad\qquad\quad \\
\underline{5x^3 + 5x^2} \qquad\qquad\quad \\
-87x^2 - 57x \qquad \\
\underline{-87x^2 - 87x} \qquad \\
30x + 18 \\
\underline{30x + 30} \\
-12
\end{array}
$$

$$4x^4 + 9x^3 - 82x^2 - 57x + 18 = (x + 1)\left((4x^3 + 5x^2 - 87x + 30) + \frac{-12}{x + 1}\right)$$

24. $x - 3\overline{)x^4 + 5x^3 - 33x^2 + 27x}$

$$
\begin{array}{r}
x^3 + 8x^2 - 9x \\
x - 3\overline{)x^4 + 5x^3 - 33x^2 + 27x} \\
\underline{x^4 - 3x^3} \qquad\qquad\qquad\quad \\
8x^3 - 33x^2 \qquad\quad \\
\underline{8x^3 - 24x^2} \qquad\quad \\
-9x^2 + 27x \\
\underline{-9x^2 + 27x} \\
0
\end{array}
$$

$$x^4 + 5x^3 - 33x^2 + 27x = (x - 3)(x^3 + 8x^2 - 9x)$$

Determine each quotient using synthetic division. Write the dividend as the product of the divisor and the quotient plus the remainder.

25. $(x^4 + 8x^3 - 3x^2 - 24x) \div (x - 3)$

3	1	8	−3	−24	0
		3	33	90	198
	1	11	30	66	198

$x^4 + 8x^3 - 3x^2 - 24x = (x - 3)\left((x^3 + 11x^2 + 30x + 66) + \frac{198}{x - 3}\right)$

26. $(x^4 - 3x^3 + 6x^2 - 12x + 8) \div (x - 1)$

1	1	−3	6	−12	8
		1	−2	4	−8
	1	−2	4	−8	0

$x^4 - 3x^3 + 6x^2 - 12x + 8 = (x - 1)(x^3 - 2x^2 + 4x - 8)$

27. $(2x^3 + 21x^2 + 22x - 45) \div (2x + 5)$

$-\frac{5}{2}$	2	21	22	−45
		−5	−40	45
	2	16	−18	0

$2x^3 + 21x^2 + 22x - 45 = (2x + 5)(2x^2 + 16x - 18)$

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28. $(x^3 + x^2 - 16x - 16) \div (x + 2)$

−2	1	1	−16	−16
		−2	2	28
	1	−1	−14	12

$x^3 + x^2 - 16x - 16 = (x + 2)\left((x^2 - x - 14) + \frac{12}{x + 2}\right)$

29. $(x^4 - 6x^3 - 19x^2 + 24x) \div (x + 3)$

−3	1	−6	−19	24	0
		−3	27	−24	0
	1	−9	8	0	0

$x^4 - 6x^3 - 19x^2 + 24x = (x + 3)(x^3 - 9x^2 + 8x)$

30. $(x^4 + 5x^3 - 33x^2 + 27x) \div (x - 9)$

9	1	5	−33	27	0
		9	126	837	7776
	1	14	93	864	7776

$x^4 + 5x^3 - 33x^2 + 27x = (x - 9)\left((x^3 + 14x^2 + 93x + 864) + \frac{7776}{x - 9}\right)$

6

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The Factors of Life

The Factor Theorem and Remainder Theorem

Vocabulary

Remainder Theorem	Factor Theorem

Choose the term from the box that best completes each statement.

1. The **Factor Theorem** states that a linear polynomial $(x - r)$ is a factor of a polynomial $p(x)$ if and only if $p(r) = 0$ and $\frac{p(x)}{(x - r)}$ has a remainder of zero.

2. The **Remainder Theorem** states that when any polynomial equation or function is divided by a linear factor $(x - r)$, the remainder is the value of the equation or function when $x = r$.

Problem Set

Determine each function value using the Remainder Theorem. Explain your reasoning.

1. Determine $p(3)$ if $p(x) = 2x^3 - 6x^2 - 36x - 36$.

$$\begin{array}{r} 2x^2 + 0x^2 - 36 \\ x - 3\overline{)2x^3 - 6x^2 - 36x - 36} \\ \underline{2x^3 - 6x^2} \qquad\qquad\quad \\ 0 \;\; -36x - 36 \\ \underline{-36x + 108} \\ -144 \end{array}$$

When $p(x)$ is divided by $x - 3$, the remainder is -144. So, by the Remainder Theorem $p(3) = -144$.

2. Determine $p(-2)$ if $p(x) = x^4 - 10x^3 + 8x^2 + 106x - 105$.

−2	1	−10	8	106	−105
		−2	24	−64	−84
	1	−12	32	42	−189

When $p(x)$ is divided by $x + 2$, the remainder is -189. So, by the Remainder Theorem $p(-2) = -189$.

6

3. Determine $p(-3)$ if $p(x) = 2x^4 + 5x^3 + 8x^2 + 15x + 6$.

$$\begin{array}{r|rrrrr} -3 & 2 & 5 & 8 & 15 & 6 \\ & & -6 & 3 & -33 & 54 \\ \hline & 2 & -1 & 11 & -18 & \boxed{60} \end{array}$$

When $p(x)$ is divided by $x + 3$, the remainder is 60. So, by the Remainder Theorem $p(-3) = 60$.

4. Determine $p(1)$ if $p(x) = x^4 + 3x^3 - 6x^2 - 8x$.

$$\begin{array}{r} x^3 + 4x^2 - 2x - 10 \\ x - 1 \overline{) x^4 + 3x^3 - 6x^2 - 8x + 0} \\ \underline{x^4 - x^3} \qquad\qquad\qquad \\ 4x^3 - 6x^2 \qquad\qquad \\ \underline{4x^3 - 4x^2} \qquad\qquad \\ -2x^2 - 8x \qquad \\ \underline{-2x^2 + 2x} \qquad \\ -10x + 0 \\ \underline{-10x + 10} \\ -10 \end{array}$$

When $p(x)$ is divided by $x - 1$, the remainder is -10. So, by the Remainder Theorem $p(1) = -10$.

5. Determine $p(10)$ if $p(x) = 6x^3 + 11x^2 - 3x - 2$.

$$\begin{array}{r|rrrr} 10 & 6 & 11 & -3 & -2 \\ & & 60 & 710 & 7070 \\ \hline & 6 & 71 & 707 & \boxed{7068} \end{array}$$

When $p(x)$ is divided by $x - 10$, the remainder is 7068. So, by the Remainder Theorem $p(10) = 7068$.

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6. Determine $p\left(\frac{1}{3}\right)$ if $p(x) = x^4 - x^3 + 7x^2 - 9x - 18$.

$\frac{1}{3}$	1	-1	7	-9	-18
		$\frac{1}{3}$	$\frac{-2}{9}$	$\frac{61}{27}$	$\frac{-182}{81}$
	1	$\frac{-2}{3}$	$\frac{61}{9}$	$\frac{-182}{27}$	$\frac{-1640}{81}$

When $p(x)$ is divided by $3x - 1$, the remainder is $-\frac{1640}{81}$. So, by the Remainder Theorem $p\left(\frac{1}{3}\right) = -\frac{1640}{81}$.

Use the Factor Theorem to determine whether the given expression is a factor of each polynomial. Explain your reasoning.

7. Is $x - 2$ a factor of $f(x) = x^3 + 8x^2 - 31x + 22$?

If $x - 2$ is a factor of $f(x)$, then by the Factor Theorem $f(2) = 0$.

$f(2) = (2)^3 + 8(2)^2 - 31(2) + 22$

$f(2) = 8 + 32 - 62 + 22$

$f(2) = 0$

When $f(x)$ is evaluated at 2, the result is 0. According to the Factor Theorem $x - 2$ is a factor of $f(x)$.

8. Is $x - 3$ a factor of $f(x) = 4x^4 - x^3 - 52x^2 - 35x + 12$?

If $x - 3$ is a factor of $f(x)$, then by the Factor Theorem $f(3) = 0$.

$f(3) = 4(3)^4 - (3)^3 - 52(3)^2 - 35(3) + 12$

$f(3) = 324 - 27 - 468 - 105 + 12$

$f(3) = -264$

When $f(x)$ is evaluated at 3, the result is -264. According to the Factor Theorem $x - 3$ is not a factor of $f(x)$.

9. Is $x - 12$ a factor of $f(x) = x^4 - 12x^3 + x^2 - 12x$?

If $x - 12$ is a factor of $f(x)$, then by the Factor Theorem $f(12) = 0$.

$f(12) = (12)^4 - 12(12)^3 + (12)^2 - 12(12)$

$f(12) = 20{,}736 - 20{,}736 + 144 - 144$

$f(12) = 0$

When $f(x)$ is evaluated at 12, the result is 0. According to the Factor Theorem $x - 12$ is a factor of $f(x)$.

10. Is $x - 8$ a factor of $f(x) = x^3 - 7x^2 - 14x + 48$?

If $x - 8$ is a factor of $f(x)$, then by the Factor Theorem $f(8) = 0$.

$f(8) = (8)^3 - 7(8)^2 - 14(8) + 48$

$f(8) = 512 - 448 - 112 + 48$

$f(8) = 0$

When $f(x)$ is evaluated at 8, the result is 0. According to the Factor Theorem $x - 8$ is a factor of $f(x)$.

11. Is $x - 5$ a factor of $f(x) = x^3 + 5x^2 - x - 5$?

If $x - 5$ is a factor of $f(x)$, then by the Factor Theorem $f(5) = 0$.

$f(5) = (5)^3 + 5(5)^2 - (5) - 5$

$f(5) = 125 + 125 - 5 - 5$

$f(5) = 240$

When $f(x)$ is evaluated at 5, the result is 240. According to the Factor Theorem $x - 5$ is not a factor of $f(x)$.

12. Is $3x + 4$ a factor of $f(x) = 3x^3 + 13x^2 + 18x + 8$?

If $3x + 4$ is a factor of $f(x)$, then by the Factor Theorem $f\left(-\frac{4}{3}\right) = 0$.

$f\left(-\frac{4}{3}\right) = 3\left(-\frac{4}{3}\right)^3 + 13\left(-\frac{4}{3}\right)^2 + 18\left(-\frac{4}{3}\right) + 8$

$f\left(-\frac{4}{3}\right) = -\frac{64}{9} + \frac{208}{9} - 24 + 8$

$f\left(-\frac{4}{3}\right) = 0$

When $f(x)$ is evaluated at $-\frac{4}{3}$, the result is 0. According to the Factor Theorem $3x + 4$ is a factor of $f(x)$.

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Use the Factor Theorem to determine whether $g(x)$ is the factored form of $f(x)$. Explain your reasoning.

13. Is $g(x) = (x + 8)(x - 1)(x + 2)$ the factored form of $f(x) = x^3 - 7x^2 - 10x + 16$?

$f(-8) = (-8)^3 - 7(-8)^2 - 10(-8) + 16$
$f(-8) = 512 - 448 + 80 + 16$
$f(-8) = 160$

$f(-2) = (-2)^3 - 7(-2)^2 - 10(-2) + 16$
$f(-2) = -8 - 28 + 20 + 16$
$f(-2) = 0$

$f(1) = (1)^3 - 7(1)^2 - 10(1) + 16$
$f(1) = 1 - 7 - 10 + 16$
$f(1) = 0$

No, the function $g(x)$ is not the factored form of $f(x)$. Since $f(-8) = 160$, $x + 8$ is not a factor of $f(x)$ by the Factor Theorem.

14. Is $g(x) = (x - 3)(x + 5)(x + 2)(x - 1)$ the factored form of $f(x) = x^4 + 3x^3 - 15x^2 - 19x + 30$?

$f(3) = (3)^4 + 3(3)^3 - 15(3)^2 - 19(3) + 30$
$f(3) = 81 + 81 - 135 - 57 + 30$
$f(3) = 0$

$f(1) = (1)^4 + 3(1)^3 - 15(1)^2 - 19(1) + 30$
$f(1) = 1 + 3 - 15 - 19 + 30$
$f(0) = 0$

$f(-5) = (-5)^4 + 3(-5)^3 - 15(-5)^2 - 19(-5) + 30$
$f(-5) = 625 - 375 - 375 + 95 + 30$
$f(-5) = 0$

$f(-2) = (-2)^4 + 3(-2)^3 - 15(-2)^2 - 19(-2) + 30$
$f(-2) = 16 - 24 - 60 + 38 + 30$
$f(-2) = 0$

Yes, the function $g(x)$ is the factored form of $f(x)$. When each of the zeros of the factors of $g(x)$ is evaluated in $f(x)$, the result is 0.

15. Is $g(x) = (x - 2)(x + 9)(x + 1)$ the factored form of $f(x) = x^3 + 8x^2 - 11x - 18$?

$f(2) = (2)^3 + 8(2)^2 - 11(2) - 18$
$f(2) = 8 + 32 - 22 - 18$
$f(2) = 0$

$f(-1) = (-1)^3 + 8(-1)^2 - 11(-1) - 18$
$f(-1) = -1 + 8 + 11 - 18$
$f(-1) = 0$

$f(-9) = (-9)^3 + 8(-9)^2 - 11(-9) - 18$
$f(-9) = -729 + 648 + 99 - 18$
$f(-9) = 0$

Yes, the function $g(x)$ is the factored form of $f(x)$. When each of the zeros of the factors of $g(x)$ is evaluated in $f(x)$, the result is 0.

16. Is $g(x) = x(x - 4)(x - i\sqrt{7})(x + i\sqrt{7})$ the factored form of $f(x) = x^4 + 4x^3 + 7x^2 + 28x$?

$f(0) = (0)^4 + 4(0)^3 + 7(0)^2 + 28(0)$
$f(0) = 0 + 0 + 0 + 0$
$f(0) = 0$

$f(4) = (4)^4 + 4(4)^3 + 7(4)^2 + 28(4)$
$f(4) = 256 + 256 + 112 + 112$
$f(4) = 736$

$f(i\sqrt{7}) = (i\sqrt{7})^4 + 4(i\sqrt{7})^3 + (i\sqrt{7})^2 + 28(i\sqrt{7})$
$f(i\sqrt{7}) = 49 - 28i\sqrt{7} - 49 + 28i\sqrt{7}$
$f(i\sqrt{7}) = 0$

$f(-i\sqrt{7}) = (-i\sqrt{7})^4 + 4(-i\sqrt{7})^3 + 7(-i\sqrt{7})^2 + 28(-i\sqrt{7})$
$f(i\sqrt{7}) = 49 + 28i\sqrt{7} - 49 - 28i\sqrt{7}$
$f(i\sqrt{7}) = 0$

No, the function $g(x)$ is not the factored form of $f(x)$. Since $f(4) = 736$, $x - 4$ is not a factor of $f(x)$ by the Factor Theorem.

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17. Is $g(x) = (x + 1)(x - 2)(4x + 7)$ the factored form of $f(x) = 4x^3 - 11x^2 - x + 14$?

$f(-1) = 4(-1)^3 - 11(-1)^2 - (-1) + 14$

$f(-1) = -4 - 11 + 1 + 14$

$f(-1) = 0$

$f(2) = 4(2)^3 - 11(2)^2 - (2) + 14$

$f(2) = 32 - 44 - 2 + 14$

$f(2) = 0$

$f\left(-\frac{7}{4}\right) = 4\left(-\frac{7}{4}\right)^3 - 11\left(-\frac{7}{4}\right)^2 - \left(-\frac{7}{4}\right) + 14$

$f\left(-\frac{7}{4}\right) = -\frac{343}{16} - \frac{539}{16} + \frac{7}{4} + 14$

$f\left(-\frac{7}{4}\right) = -\frac{630}{8}$

No, the function $g(x)$ is not the factored form of $f(x)$. Since $f\left(-\frac{7}{4}\right) = -\frac{630}{8}$, $4x + 7$ is not a factor of $f(x)$ by the Factor Theorem.

18. Is $g(x) = (x - 1)(x + 1)(x - 3i)(x + 3i)$ the factored form of $f(x) = x^4 + 8x^2 - 9$?

$f(1) = (1)^4 + 8(1)^2 - 9$

$f(1) = 1 + 8 - 9$

$f(1) = 0$

$f(-1) = (-1)^4 + 8(-1)^2 - 9$

$f(-1) = 1 + 8 - 9$

$f(-1) = 0$

$f(3i) = (3i)^4 + 8(3i)^2 - 9$

$f(3i) = 81 - 72 - 9$

$f(3i) = 0$

$f(-3i) = (-3i)^4 + 8(-3i)^2 - 9$

$f(-3i) = 81 - 72 - 9$

$f(-3i) = 0$

Yes, the function $g(x)$ is the factored form of $f(x)$. When each of the zeros of the factors of $g(x)$ is evaluated in $f(x)$, the result is 0.

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Use the Factor Theorem to determine the unknown coefficient so that the given linear expression is a factor of the function.

19. Determine a if $x + 3$ is a factor of $f(x) = x^3 + 9x^2 + ax + 15$.

If $x + 3$ is a factor, then by the Factor Theorem $f(-3) = 0$.

$f(-3) = (-3)^3 + 9(-3)^2 + a(-3) + 15$

$= -27 + 81 - 3a + 15$

$= 69 - 3a$

By the Transitive Property, $69 - 3a = 0$.

$69 - 3a = 0$

$69 = 3a$

$a = 23$

20. Determine a if $x - 4$ is a factor of $f(x) = x^3 + ax^2 - 20x - 48$.

If $x - 4$ is a factor, then by the Factor Theorem $f(4) = 0$.

$f(4) = (4)^3 + a(4)^2 - 20(4) - 48$

$= 64 + 16a - 80 - 48$

$= -64 + 16a$

By the Transitive Property, $-64 + 16a = 0$.

$-64 + 16a = 0$

$16a = 64$

$a = 4$

21. Determine a if $x - 1$ is a factor of $f(x) = ax^3 - 10x^2 - 13x + 20$.

If $x - 1$ is a factor, then by the Factor Theorem $f(1) = 0$.

$f(1) = a(1)^3 - 10(1)^2 - 13(1) + 20$

$= a - 10 - 13 + 20$

$= -3 + a$

By the Transitive Property, $-3 + a = 0$.

$-3 + a = 0$

$a = 3$

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22. Determine a if $x - 7$ is a factor of $f(x) = x^4 - 4x^3 + ax^2 - 8x - 42$.

If $x - 7$ is a factor, then by the Factor Theorem $f(7) = 0$.

$f(7) = (7)^4 - 4(7)^3 + a(7)^2 - 8(7) - 42$

$= 2401 - 1372 + 49a - 56 - 42$

$= 931 + 49a$

By the Transitive Property, $931 + 49a = 0$.

$931 + 49a = 0$

$49a = -931$

$a = -19$

23. Determine a if $x + 2$ is a factor of $f(x) = x^3 - x^2 + ax - 36$.

If $x + 2$ is a factor, then by the Factor Theorem $f(-2) = 0$.

$f(-2) = (-2)^3 - (-2)^2 + a(-2) - 36$

$= -8 - 4 - 2a - 36$

$= -48 - 2a$

By the Transitive Property, $-48 - 2a = 0$.

$-48 - 2a = 0$

$-2a = 48$

$a = -24$

24. Determine a if $x - 8$ is a factor of $f(x) = x^4 + ax^3 - 5x^2 - 21x - 24$.

If $x - 8$ is a factor, then by the Factor Theorem $f(8) = 0$.

$f(8) = (8)^4 + a(8)^3 - 5(8)^2 - 21(8) - 24$

$= 4096 + 512a - 320 - 168 - 24$

$= 3584 + 512a$

By the Transitive Property, $3584 + 512a = 0$.

$3584 + 512a = 0$

$512a = -3584$

$a = -7$

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Break It Down
Factoring Higher Order Polynomials

Problem Set

Factor each expression completely.

1. $x^2 + 12x - 13$

   $x^2 + 12x - 13 = (x + 13)(x - 1)$

2. $x^2 + 6x + 8$

   $x^2 + 6x + 8 = (x + 2)(x + 4)$

3. $x^2 - 12x - 28$

   $x^2 - 12x - 28 = (x - 14)(x + 2)$

4. $x^2 + 30x + 81$

   $x^2 + 30x + 81 = (x + 27)(x + 3)$

5. $x^2 - 5x - 14$

   $x^2 - 5x - 14 = (x - 7)(x + 2)$

6. $x^2 - 16x - 36$

   $x^2 - 16x - 36 = (x + 2)(x - 18)$

Factor each expression by factoring out the greatest common factor.

7. $2x^5 - 8x^4 + 10x^3$

   $2x^5 - 8x^4 + 10x^3 = 2x^3(x^2 - 4x + 10)$

8. $-9x^4 + 45x^3 - 9x^2$

   $-9x^4 + 45x^3 - 9x^2 = -9x^2(x^2 - 5x + 1)$

9. $105x^3 - 147x$

   $105x^3 - 147x = 21x(5x^2 - 7)$

10. $-\frac{3}{5}x^4 + \frac{3}{5}x^3 - \frac{27}{5}x^2$

    $-\frac{3}{5}x^4 + \frac{3}{5}x^3 - \frac{27}{5}x^2 = -\frac{3}{5}x^2(x^2 - x + 9)$

11. $\frac{1}{3}x^4 - \frac{8}{3}x^3 + \frac{1}{3}x^2 - \frac{11}{3}x$

    $\frac{1}{3}x^4 - \frac{8}{3}x^3 + \frac{1}{3}x^2 - \frac{11}{3}x = \frac{1}{3}x(x^3 - 8x^2 + x - 11)$

12. $8x^4 - 16x^3 + 56x^2 - 24x$

    $8x^4 - 16x^3 + 56x^2 - 24x = 8x(x^3 - 2x^2 + 7x - 3)$

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Factor each expression completely using the chunking method.

13. $4x^2 + 8x + 3$

$4x^2 + 8x + 3 = (2x)^2 + 4(2x) + 3$

Let $z = 2x$

$= z^2 + 4z + 3$

$= (z + 1)(z + 3)$

$= (2x + 1)(2x + 3)$

14. $25x^2 - 35x + 12$

$25x^2 - 35x + 12 = (5x)^2 - 7(5x) + 12$

Let $z = 5x$

$= z^2 - 7z + 12$

$= (z - 3)(z - 4)$

$= (5x - 3)(5x - 4)$

15. $121x^2 - 44x - 12$

$121x^2 - 44x - 12 = (11x)^2 - 4(11x) - 12$

Let $z = 11x$

$= z^2 - 4z - 12$

$= (z + 2)(z - 6)$

$= (11x + 2)(11x - 6)$

16. $49x^2 + 63x + 18$

$49x^2 + 63x + 18 = (7x)^2 + 9(7x) + 18$

Let $z = 7x$

$= z^2 + 9z + 18$

$= (z + 3)(z + 6)$

$= (7x + 3)(7x + 6)$

17. $9x^2 + 30x - 11$

$9x^2 + 30x - 11 = (3x)^2 + 10(3x) - 11$

Let $z = 3x$

$= z^2 + 10z - 11$

$= (z - 1)(z + 11)$

$= (3x - 1)(3x + 11)$

18. $169x^2 - 130x + 24$

$169x^2 - 130x + 24 = (13x)^2 - 10(13x) + 24$

Let $z = 13x$

$= z^2 - 10z + 24$

$= (z - 4)(z - 6)$

$= (13x - 4)(13x - 6)$

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Factor each expression completely using the factor by grouping method.

19. $x^3 - 2x^2 + 3x - 6$

$$\begin{aligned} x^3 - 2x^2 + 3x - 6 &= x^2(x - 2) + 3(x - 2) \\ &= (x^2 + 3)(x - 2) \\ &= (x + i\sqrt{3})(x - i\sqrt{3})(x - 2) \end{aligned}$$

20. $x^3 + x^2 - 4x - 4$

$$\begin{aligned} x^3 + x^2 - 4x - 4 &= x^2(x + 1) - 4(x + 1) \\ &= (x^2 - 4)(x + 1) \\ &= (x + 2)(x - 2)(x + 1) \end{aligned}$$

21. $x^3 - 6x^2 - 9x + 54$

$$\begin{aligned} x^3 - 6x^2 - 9x + 54 &= x^2(x - 6) - 9(x - 6) \\ &= (x^2 - 9)(x - 6) \\ &= (x + 3)(x - 3)(x - 6) \end{aligned}$$

22. $x^4 - 3x^3 - x^2 - 3x$

$$\begin{aligned} x^4 - 3x^3 - x^2 - 3x &= x^3(x - 3) - x(x - 3) \\ &= (x^3 - x)(x - 3) \\ &= x(x^2 - 1)(x - 3) \\ &= x(x - 1)(x + 1)(x - 3) \end{aligned}$$

23. $-x^3 + 5x^2 + 16x - 80$

$$\begin{aligned} -x^3 + 5x^2 + 16x - 80 &= -x^2(x - 5) + 16(x - 5) \\ &= (-x^2 + 16)(x - 5) \\ &= (-x + 4)(x + 4)(x - 5) \end{aligned}$$

24. $x^3 - 3x^2 - 4x + 12$

$$\begin{aligned} x^3 - 3x^2 - 4x + 12 &= x^3 - 4x - 3x^2 + 12 \\ &= x(x^2 - 4) - 3(x^2 - 4) \\ &= (x - 3)(x^2 - 4) \\ &= (x - 3)(x - 2)(x + 2) \end{aligned}$$

Factor each quartic expression completely using the quadratic form method.

25. $x^4 - 13x^2 + 36$

$x^4 - 13x^2 + 36 = (x^2 - 4)(x^2 - 9)$

$= (x - 2)(x + 2)(x - 3)(x + 3)$

26. $x^4 - 50x^2 + 49$

$x^4 - 50x^2 + 49 = (x^2 - 1)(x^2 - 49)$

$= (x - 1)(x + 1)(x - 7)(x + 7)$

27. $x^4 - 29x^2 + 100$

$x^4 - 29x^2 + 100 = (x^2 - 25)(x^2 - 4)$

$= (x - 5)(x + 5)(x - 2)(x + 2)$

28. $x^4 - 25x^2 + 144$

$x^4 - 25x^2 + 144 = (x^2 - 9)(x^2 - 16)$

$= (x - 3)(x + 3)(x - 4)(x + 4)$

29. $x^4 - 164x^2 + 6400$

$x^4 - 164x^2 + 6400 = (x^2 - 64)(x^2 - 100)$

$= (x - 8)(x + 8)(x - 10)(x + 10)$

30. $x^4 - 61x^2 + 900$

$x^4 - 61x^2 + 900 = (x^2 - 25)(x^2 - 36)$

$= (x - 5)(x + 5)(x - 6)(x + 6)$

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Factor each binomial using the sum or difference of perfect cubes formula.

31. $x^3 + 27$

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

$x^3 + 27 = (x)^3 + (3)^3$

$= (x + 3)(x^2 - 3x + 9)$

32. $x^3 - 8y^3$

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

$x^3 - 8y^3 = (x)^3 - (2y)^3$

$= (x - 2y)(x^2 + 2xy + 4y^2)$

33. $8x^3 - 125$

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

$8x^3 - 125 = (2x)^3 - (5)^3$

$= (2x - 5)(4x^2 + 10x + 25)$

34. $x^3 + 64y^3$

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

$x^3 + 64y^3 = (x)^3 + (4y)^3$

$= (x + 4y)(x^2 + 4xy + 16y^2)$

35. $343x^3 - 1$

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

$343x^3 - 1 = (7x)^3 - (1)^3$

$= (7x - 1)(49x^2 + 7x + 1)$

36. $216x^3 + 125y^3$

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

$216x^3 + 125y^3 = (6x)^3 + (5y)^3$

$= (6x + 5y)(36x^2 + 30xy + 25y^2)$

Factor each binomial completely over the set of real numbers using the difference of squares method.

37. $x^2 - 100$

$a^2 - b^2 = (a + b)(a - b)$

$x^2 - 100 = (x + 10)(x - 10)$

38. $x^4 - 36$

$a^2 - b^2 = (a + b)(a - b)$

$x^4 - 36 = (x^2)^2 - (6)^2$

$= (x^2 + 6)(x^2 - 6)$

$= (x^2 + 6)(x - \sqrt{6})(x + \sqrt{6})$

39. $49x^2 - 4y^2$

$a^2 - b^2 = (a + b)(a - b)$

$49x^2 - 4y^2 = (7x)^2 - (2y)^2$

$= (7x - 2y)(7x + 2y)$

40. $x^{10} - 81$

$a^2 - b^2 = (a + b)(a - b)$

$x^{10} - 81 = (x^5)^2 - (9)^2$

$= (x^5 - 9)(x^5 + 9)$

41. $9x^4 - 121y^2$

$a^2 - b^2 = (a + b)(a - b)$

$9x^4 - 121y^2 = (3x^2)^2 - (11y)^2$

$= (3x^2 - 11y)(3x^2 + 11y)$

42. $4x^{14} - 9y^8$

$a^2 - b^2 = (a + b)(a - b)$

$4x^{14} - 9y^8 = (2x^7)^2 - (3y^4)^2$

$= (2x^7 - 3y^4)(2x^7 + 3y^4)$

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Factor each perfect square trinomial.

43. $4x^2 + 12x + 9$

$a^2 + 2ab + b^2 = (a + b)^2$

$4x^2 + 12x + 9 = (2x)^2 + 2(2x)(3) + (3)^2$

$= (2x + 3)^2$

44. $x^2 - 12xy + 36y^2$

$a^2 - 2ab + b^2 = (a - b)^2$

$x^2 - 12xy + 36y^2 = (x)^2 - 2(x)(6y) + (6y)^2$

$= (x - 6y)^2$

45. $16x^2 + 104x + 169$

$a^2 + 2ab + b^2 = (a + b)^2$

$16x^2 + 104x + 169 = (4x)^2 + 2(4x)(13) + (13)^2$

$= (4x + 13)^2$

46. $25x^2 + 80x + 64$

$a^2 + 2ab + b^2 = (a + b)^2$

$25x^2 + 80x + 64 = (5x)^2 + 2(5x)(8) + (8)^2$

$= (5x + 8)^2$

47. $9x^4 + 42x^2y + 49y^2$

$a^2 + 2ab + b^2 = (a + b)^2$

$9x^4 + 42x^2y + 49y^2 = (3x^2)^2 + 2(3x^2)(7y) + (7y)^2$

$= (3x^2 + 7y)^2$

48. $64x^2 + 16xy^2 + y^4$

$a^2 + 2ab + b^2 = (a + b)^2$

$64x^2 + 16xy^2 + y^4 = (8x)^2 + 2(8x)(y^2) + (y^2)^2$

$= (8x + y^2)^2$

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Getting to the Root of It All
Rational Root Theorem

Vocabulary

Write a definition for the term in your own words.

1. Rational Root Theorem

The Rational Root Theorem states that a rational root of a polynomial $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0x^0$ with integer coefficients must be of the form $\frac{p}{q}$ where p is a factor of the constant term, a_0, and q is a factor of the leading coefficient, a_n.

Problem Set

Determine the possible rational roots of each polynomial using the Rational Root Theorem.

1. $x^3 - 4x^2 + 6x - 8 = 0$

$p = \pm1, \pm2, \pm4, \pm8$

$q = \pm1$

$\frac{p}{q} = \pm1, \pm2, \pm4, \pm8$

2. $2x^4 - 4x^2 + 15 = 0$

$p = \pm1, \pm3, \pm5, \pm15$

$q = \pm1, \pm2$

$\frac{p}{q} = \pm1, \pm3, \pm5, \pm15, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{15}{2}$

3. $-2x^3 + 5x + 18 = 0$

$p = \pm1, \pm2, \pm3, \pm6, \pm9, \pm18$

$q = \pm1, \pm2$

$\frac{p}{q} = \pm1, \pm2, \pm3, \pm6, \pm9, \pm18, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{9}{2}$

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4. $x^3 + 12x^2 - 21x + 32 = 0$

$p = \pm1, \pm2, \pm4, \pm8, \pm16, \pm32$

$q = \pm1$

$\frac{p}{q} = \pm1, \pm2, \pm4, \pm8, \pm16, \pm32$

5. $5x^4 - 7x^3 + 5x - 30 = 0$

$p = \pm1, \pm2, \pm3, \pm5, \pm6, \pm10, \pm15, \pm30$

$q = \pm1, \pm5$

$\frac{p}{q} = \pm1, \pm2, \pm3, \pm5, \pm6, \pm10, \pm15, \pm30, \pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{3}{5}, \pm\frac{6}{5}$

6. $12x^4 - 15x^3 + 24x^2 + 11x - 2 = 0$

$p = \pm1, \pm2$

$q = \pm1, \pm2, \pm3, \pm4, \pm6, \pm12$

$\frac{p}{q} = \pm1, \pm2, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{1}{4}, \pm\frac{1}{6}, \pm\frac{1}{12}$

Use the Rational Root Theorem to determine the possible rational roots for each polynomial equation. Then, solve completely. Use the graph, if given, to identify possible zeros.

7. $x^3 + 3x^2 - 18x - 40 = 0$

- Possible rational roots:

$p = \pm1, \pm2, \pm4, \pm5, \pm8, \pm10, \pm20, \pm40$

$q = \pm1$

$\frac{p}{q} = \pm1, \pm2, \pm4, \pm5, \pm8, \pm10, \pm20, \pm40$

- Solve completely:

4	1	3	−18	−40
		4	28	40
	1	7	10	0

$x^3 + 3x^2 - 18x - 40 = (x - 4)(x^2 + 7x + 10)$

$= (x - 4)(x + 2)(x + 5)$

$x = 4, -2, -5$

6

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8. $6x^3 + 35x^2 - 52x - 21 = 0$

- Possible rational roots:

$p = \pm1, \pm3, \pm7, \pm21$

$q = \pm1, \pm2, \pm3, \pm6$

$\frac{p}{q} = \pm1, \pm3, \pm7, \pm21, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{7}{2}, \pm\frac{21}{2}, \pm\frac{1}{3}, \pm\frac{7}{3}, \pm\frac{1}{6}, \pm\frac{7}{6}$

- Solve completely:

$$\begin{array}{r|rrrr} -7 & 6 & 35 & -52 & -21 \\ & & -42 & 49 & 21 \\ \hline & 6 & -7 & -3 & \boxed{0} \end{array}$$

$6x^3 + 35x^2 - 52x - 21 = (x + 7)(6x^2 - 7x - 3)$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(6)(-3)}}{2(6)}$$

$$= \frac{7 \pm \sqrt{121}}{12}$$

$$= \frac{7 + 11}{12}, \frac{7 - 11}{12}$$

$$= \frac{3}{2}, -\frac{1}{3}$$

$6x^3 + 35x^2 - 52x - 21 = (x + 7)(6x^2 - 7x - 3)$

$= (x + 7)(2x - 3)(3x + 1)$

$x = -7, \frac{3}{2}, -\frac{1}{3}$

6

9. $x^4 + 4x^3 - 21x^2 - 36x + 108 = 0$

- Possible rational roots:

$p = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \pm 12, \pm 18, \pm 27, \pm 36, \pm 54, \pm 108$

$q = \pm 1$

$\frac{p}{q} = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \pm 12, \pm 18, \pm 27, \pm 36, \pm 54, \pm 108$

- Solve completely:

$$\begin{array}{r|rrrrr} 2 & 1 & 4 & -21 & -36 & 108 \\ & & 2 & 12 & -18 & -108 \\ \hline & 1 & 6 & -9 & -54 & 0 \end{array}$$

$$\begin{array}{r|rrrr} -6 & 1 & 6 & -9 & -54 \\ & & -6 & 0 & 54 \\ \hline & 1 & 0 & -9 & 0 \end{array}$$

$$\begin{aligned} x^4 + 4x^3 - 21x^2 - 36x + 108 &= (x - 2)(x^3 + 6x^2 - 9x - 54) \\ &= (x - 2)(x + 6)(x^2 - 9) \\ &= (x - 2)(x + 6)(x + 3)(x - 3) \\ x &= 2, -6, -3, 3 \end{aligned}$$

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10. $x^2 - 8x + 17 = 0$

- Possible rational roots:

  $p = \pm1, \pm17$

  $q = \pm1$

  $\frac{p}{q} = \pm1, \pm17$

- Solve completely:

  There are no rational roots.

  $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(17)}}{2(1)}$$

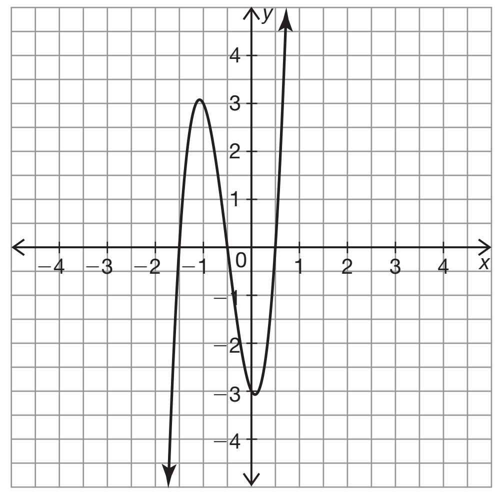
  $$= \frac{8 \pm \sqrt{-4}}{2}$$

  $$= 4 + i, 4 - i$$

  $$x = 4 + i, 4 - i$$

6

11. $8x^3 + 12x^2 - 2x - 3 = 0$



- Possible rational roots:

  $p = \pm 1, \pm 3$

  $q = \pm 1, \pm 2, \pm 4, \pm 8$

  $\frac{p}{q} = \pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{4}, \pm\frac{3}{4}, \pm\frac{1}{8}, \pm\frac{3}{8}$

- Solve completely:

  From the graph, I can determine that one of the zeros is $-\frac{3}{2}$.

  $$\begin{array}{r|rrrr} -\frac{3}{2} & 8 & 12 & -2 & -3 \\ & & -12 & 0 & 3 \\ \hline & 8 & 0 & -2 & \boxed{0} \end{array}$$

  $$8x^3 + 12x^2 - 2x - 3 = \left(x + \frac{3}{2}\right)(8x^2 - 2)$$

  $$= \left(x + \frac{3}{2}\right)\left(x^2 - \frac{1}{4}\right)(8)$$

  $$= \left(x + \frac{3}{2}\right)\left(x + \frac{1}{2}\right)\left(x - \frac{1}{2}\right)(8)$$

  $$x = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}$$

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12. $x^4 - 4x^3 + 5x^2 - 4x + 4 = 0$

- Possible rational roots:

$p = \pm 1, \pm 2, \pm 4$

$q = \pm 1$

$\frac{p}{q} = \pm 1, \pm 2, \pm 4$

- Solve completely:

$$\begin{array}{r|rrrrr} 2 & 1 & -4 & 5 & -4 & 4 \\ & & 2 & -4 & 2 & -4 \\ \hline & 1 & -2 & 1 & -2 & \boxed{0} \end{array}$$

$x^4 - 4x^3 + 5x^2 - 4x + 4 = (x - 2)(x^3 - 2x^2 + x - 2)$

$p = \pm 1, \pm 2$

$q = \pm 1$

$\frac{p}{q} = \pm 1, \pm 2$

$$\begin{array}{r|rrrr} 2 & 1 & -2 & 1 & -2 \\ & & 2 & 0 & 2 \\ \hline & 1 & 0 & 1 & \boxed{0} \end{array}$$

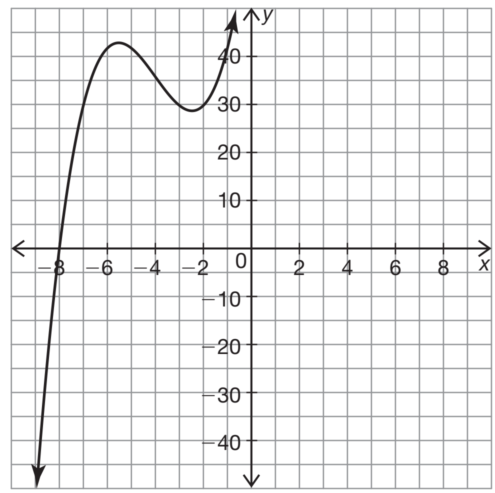
$$\begin{aligned} x^4 - 4x^3 + 5x^2 - 4x + 4 &= (x - 2)(x^3 - 2x^2 + x - 2) \\ &= (x - 2)(x - 2)(x^2 + 1) \end{aligned}$$

$$\begin{aligned} x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} &= \frac{-(0) \pm \sqrt{(0)^2 - 4(1)(1)}}{2(1)} \\ &= \frac{\pm\sqrt{-4}}{2} \\ &= \frac{\pm 2i}{2} \\ &= \pm i \end{aligned}$$

$x = i, -i, 2$ (double root)

6

13. $x^3 + 12x^2 + 41x + 72 = 0$



- Possible rational roots:

  $p = \pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm9, \pm12, \pm18, \pm24, \pm36, \pm72$

  $q = \pm1$

  $\frac{p}{q} = \pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm9, \pm12, \pm18, \pm24, \pm36, \pm72$

- Solve completely:

From the graph, I can determine that one of the zeros is −8.

−8	1	12	41	72
		−8	−32	−72
	1	4	9	0

$x^3 + 12x^2 + 41x + 72 = (x + 8)(x^2 + 4x + 9)$

$$x = \frac{-(4) \pm \sqrt{(4)^2 - 4(1)(9)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{-20}}{2}$$

$$= \frac{-4 \pm 2i\sqrt{5}}{2}$$

$$= -2 \pm i\sqrt{5}$$

$x = -2 + i\sqrt{5}, -2 - i\sqrt{5}, -8$

6

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14. $x^3 - 1.5x^2 - 1.5x + 1 = 0$

- Possible rational roots:

$p = \pm 1$

$q = \pm 1$

$\frac{p}{q} = \pm 1$

- Solve completely:

$$\begin{array}{r|rrrr} -1 & 1 & -1.5 & -1.5 & 1 \\ & & -1 & 2.5 & -1 \\ \hline & 1 & -2.5 & 1 & \boxed{0} \end{array}$$

$x^3 - 1.5x^2 - 1.5x + 1 = (x + 1)(x^2 - 2.5x + 1)$

$$x = \frac{-(-2.5) \pm \sqrt{(-2.5)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{2.5 \pm \sqrt{2.25}}{2}$$

$$= \frac{2.5 \pm 1.5}{2}$$

$= 2, 0.5$

$x = 2, 0.5, -1$

6

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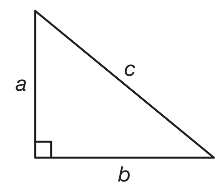
Identity Theft
Exploring Polynomial Identities

Vocabulary

1. Given positive integers r and s, where $r > s$, write the terms in Euclid's Formula that correspond to each side length in a right triangle, a, b, and c.

The values of the Pythagorean triple, a, b, and c, generated by any two positive integers r and s, are $a = r^2 - s^2$, $b = 2rs$, and $c = r^2 + s^2$.



Problem Set

Use polynomial identities and number properties to perform each calculation.

1. 109^2

Answers will vary.

$$\begin{aligned} 109^2 &= (100 + 9)^2 \\ &= 100^2 + 2(100)(9) + 9^2 \\ &= 10{,}000 + 1800 + 81 \\ &= 11{,}881 \end{aligned}$$

2. 54^3

Answers will vary.

$$\begin{aligned} 54^3 &= (50 + 4)^3 \\ &= (50 + 4)(50^2 + 2(50)(4) + 4^2) \\ &= (50 + 4)(2500 + 400 + 16) \\ &= 50(2500) + 50(400) + 50(16) + 4(2500) + 4(400) + 4(16) \\ &= 125{,}000 + 20{,}000 + 800 + 10{,}000 + 1600 + 64 \\ &= 157{,}464 \end{aligned}$$

6

3. 38^3

Answers will vary.

$38^3 = (40 - 2)^3$

$= (40 - 2)(40^2 - 2(40)(2) + 2^2)$

$= (40 - 2)(1600 - 160 + 4)$

$= 40(1600) - 40(160) + 40(4) - 2(1600) + 2(160) - 2(4)$

$= 64{,}000 - 6400 + 160 - 3200 + 320 - 8$

$= 54{,}872$

4. 99^2

Answers will vary.

$99^2 = (100 - 1)^2$

$= 100^2 - 2(100)(1) + 1^2$

$= 10{,}000 - 200 + 1$

$= 9801$

5. 127^2

Answers will vary.

$127^2 = (120 + 7)^2$

$= 120^2 + 2(120)(7) + 7^2$

$= 14{,}400 + 1680 + 49$

$= 16{,}129$

6. 75^3

Answers will vary.

$75^3 = (70 + 5)^3$

$= (70 + 5)(70^2 + 2(70)(5) + 5^2)$

$= (70 + 5)(4900 + 700 + 25)$

$= 70(4900) + 70(700) + 70(25) + 5(4900) + 5(700) + 5(25)$

$= 343{,}000 + 49{,}000 + 1750 + 24{,}500 + 3500 + 125$

$= 421{,}875$

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Determine whether each set of numbers is a Pythagorean triple. Explain your reasoning.

7. 5, 12, 13

$5^2 + 12^2 \stackrel{?}{=} 13^2$

$25 + 144 \stackrel{?}{=} 169$

$169 = 169$

These numbers are a Pythagorean triple because $5^2 + 12^2 = 13^2$.

8. 60, 61, 11

$60^2 + 11^2 \stackrel{?}{=} 61^2$

$3600 + 121 \stackrel{?}{=} 3721$

$3721 = 3721$

These numbers are a Pythagorean triple because $60^2 + 11^2 = 61^2$.

9. 8, 15, 16

$8^2 + 15^2 \stackrel{?}{=} 16^2$

$64 + 225 \stackrel{?}{=} 256$

$289 \neq 256$

These numbers are not a Pythagorean triple because $8^2 + 15^2 \neq 16^2$.

10. 4, 8, 12

$4^2 + 8^2 \stackrel{?}{=} 12^2$

$16 + 64 \stackrel{?}{=} 144$

$80 \neq 144$

These numbers are not a Pythagorean triple because $4^2 + 8^2 \neq 12^2$.

6

11. 10, 24, 26

$10^2 + 24^2 \stackrel{?}{=} 26^2$

$100 + 576 \stackrel{?}{=} 676$

$676 = 676$

These numbers are a Pythagorean triple because $10^2 + 24^2 = 26^2$.

12. 1, 2, $\sqrt{5}$

These numbers are not a Pythagorean triple because $\sqrt{5}$ is not an integer.

Generate a Pythagorean triple using each pair of given numbers and Euclid's Formula.

13. 3 and 8

$(8^2 + 3^2)^2 = (8^2 - 3^2)^2 + (2(8)(3))^2$

$(64 + 9)^2 = (64 - 9)^2 + (6(8))^2$

$73^2 = 55^2 + 48^2$

$5329 = 5329$

The Pythagorean triple is 48, 55, 73.

14. 4 and 12

$(12^2 + 4^2)^2 = (12^2 - 4^2)^2 + (2(12)(4))^2$

$(144 + 16)^2 = (144 - 16)^2 + (8(12)^2$

$160^2 = 128^2 + 96^2$

$25{,}600 = 25{,}600$

The Pythagorean triple is 96, 128, 160.

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15. 11 and 9

$(11^2 + 9^2)^2 = (11^2 - 9^2)^2 + (2(11)(9)^2$

$(121 + 81)^2 = (121 - 81)^2 + (2(99))^2$

$202^2 = 40^2 + 198^2$

$40,804 = 40,804$

The Pythagorean triple is 40, 198, 202.

16. 7 and 13

$(13^2 + 7^2)^2 = (13^2 - 7^2)^2 + (2(13)(7)^2$

$(169 + 49)^2 = (169 - 49)^2 + (14(13))^2$

$218^2 = 120^2 + 182^2$

$47,524 = 47,524$

The Pythagorean triple is 120, 182, 218.

17. 50 and 60

$(60^2 + 50^2)^2 = (60^2 - 50^2)^2 + (2(60)(50))^2$

$(3600 + 2500)^2 = (3600 - 2500)^2 + (2(3000))^2$

$6100^2 = 1100^2 + 6000^2$

$37,210,000 = 37,210,000$

The Pythagorean triple is 1100, 6000, 6100.

18. 25 and 100

$(100^2 + 25^2)^2 = (100^2 - 25^2)^2 + (2(100)(25))^2$

$(10,000 + 625)^2 = (10,000 - 625)^2 + (2(2500))^2$

$10,625^2 = 9375^2 + 5000^2$

$112,890,625 = 112,890,625$

The Pythagorean triple is 5000, 9375, 10,625.

Verify each algebraic statement by transforming one side of the equation to show that it is equivalent to the other side of the equation.

19. $g^6 - h^6 = (g^2 - h^2)(g^2 - gh + h^2)(g^2 + gh + h^2)$

Method 1:

$g^6 - h^6 \stackrel{?}{=} (g^2 - h^2)(g^2 - gh + h^2)(g^2 + gh + h^2)$

$\stackrel{?}{=} (g^4 - g^3h + g^2h^2 - g^2h^2 + gh^3 - h^4)(g^2 + gh + h^2)$

$\stackrel{?}{=} g^6 + g^5h + g^4h^2 - g^5h - g^4h^2 - g^3h^3 + g^4h^2 + g^3h^3 + g^2h^4 - g^4h^2 - g^3h^3 - g^2h^4 + g^3h^3 + g^2h^4 - gh^5 - g^2h^4 - gh^5 - h^6$

$= g^6 - h^6$

Method 2:

$g^6 - h^6 \stackrel{?}{=} (g^2 - h^2)(g^2 - gh + h^2)(g^2 + gh + h^2)$

$(g^3 + h^3)(g^3 - h^3) \stackrel{?}{=}$

$(g + h)(g^2 - gh + h^2)(g - h)(g^2 + gh + h^2) \stackrel{?}{=}$

$(g^2 - h^2)(g^2 - gh + h^2)(g^2 + gh + h^2) =$

20. $(m^2 + n^2)^3 = (m^2 + n^2)(m^4 + 2m^2n^2 + n^4)$

Method 1:

$(m^2 + n^2)^3 \stackrel{?}{=} (m^2 + n^2)(m^4 + 2m^2n^3 + n^4)$

$\stackrel{?}{=} (m^2 + n^2)(m^2 + n^2)(m^2 + n^2)$

$= (m^2 + n^2)^3$

Method 2:

$(m^2 + n^2)^3 \stackrel{?}{=} (m^2 + n^2)(m^4 + 2m^2n^2 + n^4)$

$(m^2 + n^2)((m^2)^2 + 2m^2n^2 + (n^2)^2) \stackrel{?}{=}$

$(m^2 + n^2)(m^4 + 2m^2n^2 + n^4) =$

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21. $p^8 - q^8 = (p - q)(p + q)(p^2 + q^2)(p^4 + q^4)$

Method 1:

$$p^8 - q^8 \stackrel{?}{=} (p - q)(p + q)(p^2 + q^2)(p^4 + q^4)$$
$$\stackrel{?}{=} (p^2 - q^2)(p^2 + q^2)(p^4 + q^4)$$
$$\stackrel{?}{=} (p^4 - q^4)(p^4 + q^4)$$
$$= p^8 - q^8$$

Method 2:

$$p^8 - q^8 \stackrel{?}{=} (p - q)(p + q)(p^2 + q^2)(q^4 + p^4)$$
$$(p^4 - q^4)(p^4 + q^4) \stackrel{?}{=}$$
$$(p^2 - q^2)(p^2 + q^2)(p^4 + q^4) \stackrel{?}{=}$$
$$(p - q)(p + q)(p^2 + q^2)(p^4 + q^4) =$$

22. $r^4 - s^4 = (r^2 + s^2)(r + s)(r - s)$

Method 1:

$$r^4 - s^4 \stackrel{?}{=} (r^2 + s^2)(r + s)(r - s)$$
$$\stackrel{?}{=} (r^2 + s^2)(r^2 - s^2)$$
$$\stackrel{?}{=} r^4 - s^4$$

Method 2:

$$r^4 - s^4 \stackrel{?}{=} (r^2 + s^2)(r + s)(r - s)$$
$$(r^2)^2 - (s^2)^2 \stackrel{?}{=}$$
$$(r^2 + s^2)(r^2 - s^2) \stackrel{?}{=}$$
$$(r^2 + s^2)(r + s)(r - s) =$$

23. $a^{15} + b^{15} = (a^5 + b^5)(a^{10} - a^5b^5 + b^{10})$

Method 1:

$$a^{15} + b^{15} \stackrel{?}{=} (a^5 + b^5)(a^{10} - a^5b^5 + b^{10})$$
$$\stackrel{?}{=} (a^5 + b^5)((a^5)^2 - a^5b^5 + (b^5)^2)$$
$$\stackrel{?}{=} (a^5)^3 + (b^5)^3$$
$$= a^{15} + b^{15}$$

Method 2:

$$a^{15} + b^{15} \stackrel{?}{=} (a^5 + b^5)(a^{10} - a^5b^5 + b^{10})$$
$$(a^5)^3 + (b^5)^3 \stackrel{?}{=}$$
$$(a^5 + b^5)((a^5)^2 - a^5b^5 + (b^5)^2) \stackrel{?}{=}$$
$$(a^5 + b^5)(a^{10} - a^5b^5 + b^{10}) =$$

24. $(v^6 + w^6)^2 = (v^6 - w^6)^2 + (2v^3w^3)^2$

Method 1:

$$(v^6 + w^6)^2 \stackrel{?}{=} (v^6 - w^6)^2 + (2v^3w^3)^2$$
$$\stackrel{?}{=} v^{12} - 2v^6w^6 + w^{12} + 4v^6w^6$$
$$\stackrel{?}{=} v^{12} + 2v^6w^6 + w^{12}$$
$$= (v^6 + w^6)^2$$

Method 2:

$$(v^6 + w^6)^2 \stackrel{?}{=} (v^6 - w^6)^2 + (2v^3w^3)^2$$
$$v^{12} + 2v^6w^6 + w^{12} \stackrel{?}{=}$$
$$v^{12} + 2v^6w^6 + w^{12} + 2v^6w^6 - 2v^6w^6 \stackrel{?}{=}$$
$$v^{12} - 2v^6w^6 + w^{12} + 2v^6w^6 + 2v^6w^6 \stackrel{?}{=}$$
$$(v^6 - w^6)^2 + 4v^6w^6 \stackrel{?}{=}$$
$$(v^6 - w^6)^2 + (2v^3w^3)^2 =$$

6

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The Curious Case of Pascal's Triangle
Pascal's Triangle and the Binomial Theorem

Vocabulary

Write a definition for the term in your own words.

1. Binomial Theorem

   The Binomial Theorem states that it is possible to extend any power of $(a + b)$ into a sum of terms in the form:

   $(a + b)^n = \binom{n}{0}a^nb^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}a^1b^{n-1} + \binom{n}{n}a^0b^n$

Problem Set

Use Pascal's Triangle to expand each binomial.

1. $(a + b)^4 =$

   $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

2. $(a + b)^7 =$

   $(a + b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$

3. $(a + b)^8 =$

   $(a + b)^8 = a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$

4. $(a + b)^9 =$

   $(a + b)^9 = a^9 + 9a^8b + 36a^7b^2 + 84a^6b^3 + 126a^5b^4 + 126a^4b^5 + 84a^3b^6 + 36a^2b^7 + 9ab^8 + b^9$

5. $(a + b)^{10} =$

$(a + b)^{10} = a^{10} + 10a^9b + 45a^8b^2 + 120a^7b^3 + 210a^6b^4 + 252a^5b^5 + 210a^4b^6 + 120a^3b^7 + 45a^2b^8 + 10ab^9 + b^{10}$

6. $(a + b)^{13} =$

$(a + b)^{13} = a^{13} + 13a^{12}b + 78a^{11}b^2 + 286a^{10}b^3 + 715a^9b^4 + 1287a^8b^5 + 1716a^7b^6 + 1716a^6b^7 + 1287a^5b^8 + 715a^4b^9 + 286a^3b^{10} + 78a^2b^{11} + 13ab^{12} + b^{13}$

Perform each calculation and simplify.

7. $7! =$

$7! = (7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)$
$= 5040$

8. $12! =$

$12! = (12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)$
$= 479{,}001{,}600$

9. $3!4! =$

$3!4! = (3 \cdot 2 \cdot 1)(4 \cdot 3 \cdot 2 \cdot 1)$
$= (6)(24)$
$= 144$

6

10. $5!8! =$

$5!8! = (5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)(8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)$
$= (120)(40{,}320)$
$= 4{,}838{,}400$

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11. $\frac{6!}{4!} =$

$\frac{6!}{4!} = \frac{6 \cdot 5 \cdot \cancel{4 \cdot 3 \cdot 2 \cdot 1}}{\cancel{4 \cdot 3 \cdot 2 \cdot 1}}$

$= \frac{30}{1}$

$= 30$

12. $\frac{17!}{14!3!} =$

$\frac{17!}{14!3!} = \frac{17 \cdot 16 \cdot 15 \cdot \cancel{14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}{(\cancel{14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1})(3 \cdot 2 \cdot 1)}$

$= \frac{17 \cdot 16 \cdot 15}{3 \cdot 2 \cdot 1}$

$= \frac{4080}{6}$

$= 680$

Perform each calculation and simplify.

13. $\binom{5}{3} =$

$\binom{5}{3} = \frac{5!}{3!(5-3)!}$

$= \frac{5 \cdot 4 \cdot \cancel{3 \cdot 2 \cdot 1}}{(\cancel{3 \cdot 2 \cdot 1})(2 \cdot 1)}$

$= \frac{20}{2}$

$= 10$

14. ${}_6C_3 =$

${}_6C_3 = \frac{6!}{3!(6-3)!}$

$= \frac{6 \cdot 5 \cdot 4 \cdot \cancel{3 \cdot 2 \cdot 1}}{(3 \cdot 2 \cdot 1)(\cancel{3 \cdot 2 \cdot 1})}$

$= \frac{120}{6}$

$= 20$

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15. $\binom{10}{4} =$

$$\binom{10}{4} = \frac{10!}{4!(10-4)!}$$

$$= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot \cancel{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}{(4 \cdot 3 \cdot 2 \cdot 1)(\cancel{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1})}$$

$$= \frac{5040}{24}$$

$$= 210$$

16. ${}_{11}C_5 =$

$${}_{11}C_5 = \frac{11!}{5!(11-5)!}$$

$$= \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot \cancel{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}{(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)(\cancel{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1})}$$

$$= \frac{55{,}440}{120} = 462$$

17. $\binom{16}{2} =$

$$\binom{16}{2} = \frac{16!}{2!(16-2)!}$$

$$= \frac{16 \cdot 15 \cdot \cancel{14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}{(2 \cdot 1)(\cancel{14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1})}$$

$$= \frac{240}{2}$$

$$= 120$$

18. ${}_{13}C_4 =$

$${}_{13}C_4 = \frac{13!}{4!(13-4)!}$$

$$= \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot \cancel{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}{(4 \cdot 3 \cdot 2 \cdot 1)(\cancel{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1})}$$

$$= \frac{17{,}160}{24}$$

$$= 715$$

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Use the Binomial Theorem and substitution to expand each binomial.

19. $(x - 2y)^6 =$

$(a+b)^6 = \binom{6}{0}a^6b^0 + \binom{6}{1}a^5b^1 + \binom{6}{2}a^4b^2 + \binom{6}{3}a^3b^3 + \binom{6}{4}a^2b^4 + \binom{6}{5}a^1b^5 + \binom{6}{6}a^0b^6$

$= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

Let $a = x$ and let $b = -2y$.

$(x - 2y)^6 = x^6 + 6x^5(-2y) + 15x^4(-2y)^2 + 20x^3(-2y)^3 + 15x^2(-2y)^4 + 6x(-2y)^5 + (-2y)^6$

$= x^6 - 12x^5y + 15x^4(4y^2) - 20x^3(8y^3) + 15x^2(16y^4) - 6x(32y^5) + 64y^6$

$= x^6 - 12x^5y + 60x^4y^2 - 160x^3y^3 + 240x^2y^4 - 192xy^5 + 64y^6$

20. $(x + 3y)^5 =$

$(a+b)^5 = \binom{5}{0}a^5b^0 + \binom{5}{1}a^4b^1 + \binom{5}{2}a^3b^2 + \binom{5}{3}a^2b^3 + \binom{5}{4}a^1b^4 + \binom{5}{5}a^0b^5$

$= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

Let $a = x$ and let $b = 3y$.

$(x + 3y)^5 = x^5 + 5x^4(3y) + 10x^3(3y)^2 + 10x^2(3y)^3 + 5x(3y)^4 + (3y)^5$

$= x^5 + 15x^4y + 10x^3(9y^2) + 10x^2(27y^3) + 5x(81y^4) + 243y^5$

$= x^5 + 15x^4y + 90x^3y^2 + 270x^2y^3 + 405xy^4 + 243y^5$

21. $(4x - y)^7 =$

$(a + b)^7 = \binom{7}{0}a^7b^0 + \binom{7}{1}a^6b^1 + \binom{7}{2}a^5b^2 + \binom{7}{3}a^4b^3 + \binom{7}{4}a^3b^4 + \binom{7}{5}a^2b^5 + \binom{7}{6}a^1b^6 + \binom{7}{7}a^0b^7$

$= a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7a^1b^6 + b^7$

Let $a = 4x$ and let $b = -y$.

$(4x - y)^7 = (4x)^7 + 7(4x)^6(-y) + 21(4x)^5(-y)^2 + 35(4x)^4(-y)^3 + 35(4x)^3(-y)^4 +$
$21(4x)^2(-y)^5 + 7(4x)(-y)^6 + (-y)^7$

$= 16{,}384x^7 - 28{,}672x^6y + 21{,}504x^5y^2 - 8960x^4y^3 + 2240x^3y^4 - 336x^2y^5 + 28xy^6 - y^7$

22. $(3x + 2y)^4 =$

$(a + b)^4 = \binom{4}{0}a^4b^0 + \binom{4}{1}a^3b^1 + \binom{4}{2}a^2b^2 + \binom{4}{3}a^1b^3 + \binom{4}{4}a^0b^4$

$= a^4 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + b^4$

Let $a = 3x$ and let $b = 2y$.

$(3x + 2y)^4 = (3x)^4 + 4(3x)^3(2y) + 6(3x)^2(2y)^2 + 4(3x)(2y)^3 + (2y)^4$

$= 81x^4 + 4(27x^3)(2y) + 6(9x^2)(4y^2) + 4(3x)(8y^3) + 16y^4$

$= 81x^4 + 216x^3y + 216x^2y^2 + 96xy^3 + 16y^4$

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23. $(-x + 5y)^8 =$

$$(a+b)^8 = \binom{8}{0}a^8b^0 + \binom{8}{1}a^7b^1 + \binom{8}{2}a^6b^2 + \binom{8}{3}a^5b^3 + \binom{8}{4}a^4b^4 + \binom{8}{5}a^3b^5 + \binom{8}{6}a^2b^6 + \binom{8}{7}a^1b^7 + \binom{8}{8}a^0b^8$$

$$= a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$$

Let $a = -x$ and let $b = 5y$.

$$(-x+5y)^8 = (-x)^8 + 8(-x)^7(5y) + 28(-x)^6(5y)^2 + 56(-x)^5(5y)^3 + 70(-x)^4(5y)^4 + 56(-x)^3(5y)^5 + 28(-x)^2(5y)^6 + 8(-x)(5y)^7 + (5y)^8$$

$$= x^8 - 40x^7y + 28x^6(25y^2) - 56x^5(125y^3) + 70x^4(625y^4) - 56x^3(3125y^5) + 28x^2(15{,}625y^6) - 8x(78{,}125y^7) + 390{,}625y^8$$

$$= x^8 - 40x^7y + 700x^6y^2 - 7000x^5y^3 + 43{,}750x^4y^4 - 175{,}000x^3y^5 + 437{,}500x^2y^6 - 625{,}000xy^7 + 390{,}625y^8$$

24. $(2x - 3)^6 =$

$$(a+b)^6 = \binom{6}{0}a^6b^0 + \binom{6}{1}a^5b^1 + \binom{6}{2}a^4b^2 + \binom{6}{3}a^3b^3 + \binom{6}{4}a^2b^4 + \binom{6}{5}a^1b^5 + \binom{6}{6}a^0b^6$$

$$= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

Let $a = 2x$ and let $b = -3$.

$$(2x-3)^6 = (2x)^6 + 6(2x)^5(-3) + 15(2x)^4(-3)^2 + 20(2x)^3(-3)^3 + 15(2x)^2(-3)^4 + 6(2x)(-3)^5 + (-3)^6$$

$$= 64x^6 - 18(32x^5) + 135(16x^4) - 540(8x^3) + 1215(4x^2) - 1458(2x) + 729$$

$$= 64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729$$

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